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Lecture 14 – Product & Entangled States

1. HW4, Problem 1b
2. Tensor Product Space
3. Interaction & Entanglement
4. EPR: Einstein-Podolsky-Rosen

1. HW4 – Problem 1b

$\hat{x} \to \hat{x}' = \hat{x} + \alpha$

$= U_x \hat{x} U_x^\dagger \longrightarrow \underbrace{U_x = e^{i\hat{p}\alpha/\hbar}}$

Assume this is true and work backwards

$U_x' U_x^\dagger = \Lambda$

$$\left.\begin{array}{l}\Lambda \hat{x} \Lambda^\dagger = \hat{x} \\ \Lambda \hat{p} \Lambda^\dagger = \hat{p}\end{array}\right\} \Lambda = e^{i\lambda}$$

Composite Systems

$\hat{\mathcal{H}}^{(A)}$ $\qquad$ $\hat{\mathcal{H}}^{(B)}$

$|\psi^{(A)}\rangle$ $\qquad$ $|\psi^{(B)}\rangle$

$|\langle a^{(A)}|a'^{(A)}\rangle|^2$ $\qquad$ $|\langle b^{(B)}|b'^{(B)}\rangle|^2$

$\Rightarrow \langle a^{(A)} b^{(B)} | a'^{(A)} b'^{(B)}\rangle = \langle a^{(A)}|a'^{(A)}\rangle\langle b^{(B)}|b'^{(B)}\rangle \quad (*)$

└ inner product on the space

2. Leads to Tensor Product Space

$\mathcal{H}^{(A)} \otimes \mathcal{H}^{(B)}$

└ inner product

($\mathcal{H}^{(A)}$ and $\mathcal{H}^{(B)}$ are NOT sub-spaces of $\mathcal{H}^{(A)} \otimes \mathcal{H}^{(B)}$!

$|a^{(A)}, b^{(B)}\rangle \equiv |a^{(A)}\rangle \otimes |b^{(B)}\rangle \in \mathcal{H}^{(A)} \otimes \mathcal{H}^{(B)}$

Inner product given by (*)

$\to c_1|a_1^{(A)}\rangle \otimes |b_1^{(B)}\rangle + c_2|a_2^{(A)}\rangle \otimes |b_2^{(B)}\rangle \in \mathcal{H}^{(A)} \otimes \mathcal{H}^{(B)}$

Use linearity of inner products for general vectors

$|a^{(A)}\rangle \otimes (c_1|b_1^{(B)}\rangle + c_2|b_2^{(B)}\rangle)$

$\quad = c_1|a^{(A)}\rangle \otimes |b_1^{(B)}\rangle + c_2|a^{(A)}\rangle \otimes |b_2^{(B)}\rangle$

$|\psi\rangle = c_1|a_1^{(A)}\rangle \otimes |b_1^{(B)}\rangle + c_2|a_2^{(A)}\rangle \otimes |b_2^{(B)}\rangle$

Use anti-linearity (bras)

$(\langle a_3^{(A)}| \otimes \langle b_3^{(B)}|)|\psi\rangle$

$\quad = c_1\langle a_3^{(A)}|a_1^{(A)}\rangle\langle b_3^{(B)}|b_1^{(B)}\rangle + c_2\langle a_3^{(A)}|a_2^{(A)}\rangle\langle b_3^{(B)}|b_2^{(B)}\rangle$

*Note: it is not meaninful to consider $\langle a| \otimes |b\rangle$; you only tensor product two like-things from the same space (kets & kets, etc.)

Tensor Products - Operators

$\mathcal{H}^{(A)}$ $\mathcal{H}^{(B)}$

$(\hat{F} \otimes \hat{G})(\underbrace{|\psi^{(A)}\rangle \otimes |\varphi^{(B)}\rangle})$

Product states - where non product states (sums) are the exciting ones that lead to entangled states

$= (\hat{F}|\psi^{(A)}\rangle) \otimes (\hat{G}|\varphi^{(B)}\rangle)$

→ linearity also applies here

Basis for Tensor Product Space

$\{|a_j\rangle\} \rightarrow$ basis for $\mathcal{H}^{(A)}$
$\{|b_k\rangle\} \rightarrow$ basis for $\mathcal{H}^{(B)}$
} these do not have to be the same dimension

$$|\psi^{(A)}\rangle = \sum_{j=1}^{N_A} c_j^{(A)} |a_j\rangle \; ; \quad |\varphi^{(B)}\rangle = \sum_{k=1}^{N_B} c_k^{(B)} |b_k\rangle$$

$\rightarrow \sum_{j,k} \Lambda_{j,k} |a_j\rangle \otimes |b_k\rangle = |\Psi\rangle$

└ the general state

Product vs. Entangled States

$$|\Psi\rangle = \sum_{j=1}^{N_A} |a_j\rangle \otimes \underbrace{\left(\sum_{k=1}^{N_B} \Lambda_{jk} |b_k\rangle\right)}_{|\Psi_j^{(B)}\rangle}$$

IFF $|\Psi_j^{(B)}\rangle = \lambda_j |\Psi_1^{(B)}\rangle$

↳ then ↓

$$|\Psi_j^{(B)}\rangle = \lambda_j |\Psi_1^{(B)}\rangle = \left(\sum^{N_A} \lambda_j |a_j\rangle\right) \otimes |\Psi_1^{(B)}\rangle$$

{ If something is a product state in one basis, then it is also a product state in another basis }

⇒ If you can collect all of your coefficients and they live in the same state (they are proportional to some state), then it is a product state (no sum), not an entangled state

3. ## Interaction & Entanglement

Assume we start entirely with product states → time-evolve this into an entangled state

Bring in the unitary time-evolution operator

$U(t) = e^{iHt/\hbar}$
$\quad\quad\quad\; \llcorner H \in \mathcal{H}^{(A)} \otimes \mathcal{H}^{(B)}$

Non-interacting H:

$H = \underbrace{H_A \otimes \mathbb{I} + \mathbb{I} \otimes H_B}_{\text{commutes}}$ ← H_A completely separate from H_B

Where $(\hat{F} \otimes \mathbb{I})|\psi^{(A)}\rangle \otimes |\varphi^{(B)}\rangle = (\hat{F}|\psi^{(A)}\rangle) \otimes |\varphi^{(B)}\rangle$

*Tensor operator commutation:

$(F_1 \otimes G_1)(F_2 \otimes G_2) = (F_1 F_2) \otimes (G_1 G_2)$

→ The unitary time evolution operator is a product operator...

$U(t) = \exp[i\underbrace{(H_A \otimes \mathbb{I} + \mathbb{I} \otimes H_B)}_{e^{A+B} = e^A e^B} t/\hbar]$

$= (U_A \otimes \mathbb{I})(\mathbb{I} \otimes U_B) = U_A \otimes U_B$

...Where the definition of $U_A \otimes \mathbb{I}$:

Taylor expansion

$\exp[i(H_A \otimes \mathbb{I})t/\hbar] = \sum_n \frac{1}{n!} (it/\hbar)^n \underbrace{(H_A \otimes \mathbb{I})^n}_{(H_A)^n \otimes \mathbb{I}}$

$= (\sum_n \frac{1}{n!} (it/\hbar)^n H_A^n) \otimes \mathbb{I} = U_A \otimes \mathbb{I}$

$U_A = e^{itH_A/\hbar}$

$\frac{1}{n!}$ does not distribute to the identity because linearity requires no double sum

for a non-interacting Hamiltonian:

$U(t)(|\psi^{(A)}\rangle \otimes |\varphi^{(B)}\rangle)$ — U on a product state

$= (U_A \otimes U_B)(|\psi^{(A)}\rangle \otimes |\varphi^{(B)}\rangle)$

$= (U_A|\psi^{(A)}\rangle) \otimes (U_B|\varphi^{(B)}\rangle)$

⇒ A non-interacting Hamiltonian (where $_A$ can only act on $^{(A)}$, etc.) can only produce a product state from a product state!

(but an interacting Hamiltonian CAN produce entanglement

Interacting Hamiltonian:

$\mathcal{H}^{(A)}, \mathcal{H}^{(B)}$ → 2-level systems

$H = -\varepsilon Z \otimes Z$ (spins want to point in the same direction here)

$\llcorner$ in $|\pm; z\rangle$ → $Z_{pauli} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Natural basis:

$|\pm;z\rangle \otimes |\pm;z\rangle$ (4-dimensional)

- Choose our state

$|\psi(0)\rangle = |+;x\rangle \otimes |+;x\rangle$

$= 1/\sqrt{2}\,(|+;z\rangle + |-;z\rangle) \otimes 1/\sqrt{2}\,(|+;z\rangle + |-;z\rangle)$

$= \frac{1}{2}(|+;z\rangle\otimes|+;z\rangle + |-;z\rangle\otimes|+;z\rangle + |+;z\rangle\otimes|-;z\rangle + |-;z\rangle\otimes|-;z\rangle)$

- Bring in time evolution

$|\psi(t)\rangle = U(t)|\psi(0)\rangle, \quad U(t) = e^{iHt/\hbar}$

$\hat{H}|\sigma_1;z\rangle\otimes|\sigma_2;z\rangle = -\varepsilon\sigma_1\sigma_2 \underbrace{|\sigma_1;z\rangle\otimes|\sigma_2;z\rangle}_{\text{neither of the states are changed!}}$

$e^{iHt/\hbar}|\sigma_1;z\rangle\otimes|\sigma_2;z\rangle = e^{-\varepsilon\sigma_1\sigma_2 it/\hbar}|\sigma_1;z\rangle\otimes|\sigma_2;z\rangle$

└ if any operator acts on an eigenstate, you can always replace it with the eigenvalue

$|\psi(t)\rangle = \frac{1}{2}[e^{-i\varepsilon t/\hbar}(|+;z\rangle\otimes|+;z\rangle + |-;z\rangle\otimes|-;z\rangle)$

$\quad + e^{i\varepsilon t/\hbar}(|+;z\rangle\otimes|+;z\rangle + |-;z\rangle\otimes|-;z\rangle)]$

→ Choose $t = \hbar/\varepsilon\,(\pi/4)$

$= \frac{1}{2}\underbrace{e^{-i\pi/4}}_{\text{overall phase factor}}[(|+;z\rangle\otimes|+;z\rangle + |-;z\rangle\otimes|-;z\rangle)$

$\quad + i(|+;z\rangle\otimes|-;z\rangle + |-;z\rangle\otimes|+;z\rangle)]$

Claim: this is NOT a product state

$= \frac{1}{2}e^{-i\pi/4}[|+;z\rangle\otimes\underbrace{(|+;z\rangle + i|-;z\rangle)}_{\sqrt{2}\,|+;y\rangle} + |-;z\rangle\otimes\underbrace{(i|+;z\rangle + |-;z\rangle)}_{i\sqrt{2}\,|-;y\rangle}]$

$|\psi(t)\rangle = 1/\sqrt{2}\,e^{-i\pi/4}[|+;z\rangle\otimes|+;y\rangle + i|-;z\rangle\otimes|-;y\rangle]$

*Note: our chosen time produces the <u>maximally</u> entangled state, but all time choices produce entangled states — there is no special product state time

4. <u>EPR – Einstein-Podolsky-Rosen</u>

$|\psi\rangle = 1/\sqrt{2}\,(|+;z\rangle\otimes|+;y\rangle + i|-;z\rangle\otimes|-;y\rangle)$

└ What's so strange about this state? Why is it entangled?

If you measure $|\psi\rangle$ in the $(S_z^{(A)}\otimes\mathbb{I}),(\mathbb{I}\otimes S_y^{(B)})$ basis:

Separate, commuting operators

$S_z^{(A)}\otimes\mathbb{I}$ → $+\hbar$ wavefunction collapsed to $(|+;z\rangle\otimes|+;y\rangle)$ (i)

→ $-\hbar$ wavefunction collapsed to $(|-;z\rangle\otimes|-;y\rangle)$ (ii) because they commute, we can simultaneously measure S_y

$$\mathbb{I} \otimes S_y^{(B)} \begin{cases} \nearrow +\hbar \ (i) \\ \searrow -\hbar \ (ii) \end{cases}$$

So these are a superposition of states (i) and (ii) and when you measure the wavefunction, it collapses into one or the other.

$\left.\begin{array}{l} (i)\ S_z^{(A)} = +\hbar\ ;\ S_y^{(B)} = +\hbar \\ (ii)\ S_z^{(A)} = -\hbar\ ;\ S_y^{(B)} = -\hbar \end{array}\right\}$ they're "Entangled"

$|\Psi_{EPR}\rangle = 1/\sqrt{2}\,(|+;z\rangle \otimes |-;z\rangle - |-;z\rangle \otimes |+;z\rangle)$

└ if you measure this, you get (+) on one end and (−) on the other or vice-versa (anti-correlated)

$|\Psi_{EPR}\rangle_{x\text{-basis}} = 1/\sqrt{2}\,(|+;x\rangle \otimes |-;x\rangle - |-;x\rangle \otimes |+;x\rangle)$

Choice: measure $S_{z,x}^{(A,B)}$

* measuring $S^{(A)}$ does not perturb measurements in (B) because they are far away in space and they commute

measure $S_z^{(A)} \longrightarrow$ know $S_z^{(B)}$ by anti-correlation

measure $S_x^{(B)}$

└ doesn't erase $S_z^{(A)}$ measurement → far away

We now "know" $S_x^{(B)}$ and $S_z^{(B)}$ – a paradox!

→ Violates the uncertainty principle!

→ This hints at no local "hidden variable" – the uncertainty principle doesn't just say you can't know both $S_x^{(B)}$ and $S_z^{(B)}$, it says they (the knowledge/info on the state) don't exist simultaneously!

* local hidden variable theory requires that distant events be independent, ruling out instantaneous interaction between separate events

This paradox persisted until John Stewart Bell "fixed" it → Bell showed that the information of x^B and z^B does not exist simultaneously, not just that you can't measure them simultaneously i.e., you can only measure $S_x^{(B)}$, but the hidden variable, the information on $S_z^{(B)}$, doesn't even exist – can't exist – when you're measuring in x.

Bell's Theorem: "No physical theory of local hidden variables can ever reproduce all of the predictions of Quantum Mechanics."

GHZ: Greenberger-Horne-Zeilinger Variant (no probability)

A paradox that needs no probability to state

$|\Psi_{GHZ}\rangle = 1/\sqrt{2}\,(|0\otimes 0\otimes 0\rangle - |1\otimes 1\otimes 1\rangle)$

└ just a fancier version of the Bell state

$|0\rangle \equiv |+;z\rangle,\ |1\rangle \equiv |-;z\rangle$

→ Introduce operators (which commute on different systems)

$X^{(A,B,C)}$ on $Y^{(A,B,C)}$

$X^{(A)}\otimes Y^{(B)}\otimes Y^{(C)},\ Y^{(A)}\otimes X^{(B)}\otimes Y^{(C)},\ Y^{(A)}\otimes Y^{(B)}\otimes X^{(C)}$

$(X^{(A)}\otimes Y^{(B)}\otimes Y^{(C)})|\Psi_{GHZ}\rangle = 1/\sqrt{2}\,[(X^{(A)}|0\rangle\otimes Y^{(B)}|0\rangle)\otimes Y^{(C)}|0\rangle$
$\quad - (X^{(A)}|1\rangle\otimes Y^{(B)}|1\rangle)\otimes Y^{(C)}|1\rangle]$

└ where $X^{(A)}, Y^{(B)}$ commute, etc.

$X|0\rangle = |1\rangle;\ X|1\rangle = |0\rangle$

$Y|0\rangle = i|1\rangle;\ Y|1\rangle = -i|0\rangle$

$X^{(A)}\otimes Y^{(B)}\otimes Y^{(C)}|\Psi_{GHZ}\rangle = \underbrace{1/\sqrt{2}\,[-|1\otimes 1\otimes 1\rangle + |0\otimes 0\otimes 0\rangle]}_{\text{wavefunction unchanged}} = |\Psi_{GHZ}\rangle$

⇒ eigenstate with eigenvalue = 1

According to Hidden Variable Theory (HVT): (everything commutes)

$$\overbrace{(X^{(A)}Y^{(B)}Y^{(C)})}^{\otimes\text{ implied}}\ \underbrace{}_{1}\ (Y^{(A)}X^{(B)}Y^{(C)})\ \underbrace{}_{1}\ (Y^{(A)}Y^{(B)}X^{(C)})\ \underbrace{}_{1} = X^{(A)}X^{(B)}X^{(C)}$$

└ only true for hidden variables!

If there are hidden variables in this system, according to HVT, applying $X^{(A)}\otimes X^{(B)}\otimes X^{(C)}$ should also leave the wavefunction unchanged

$(X^{(A)}\otimes X^{(B)}\otimes X^{(C)})|\Psi_{GHZ}\rangle = -|\Psi_{GHZ}\rangle$

└ A contradiction on hidden variable theory

GHZ shows that there cannot be any hidden variables (HVT does not hold in Quantum Mechanics!) This means there cannot be information on spin in x while there is information on spin in y.

→ It's not just that you can't measure both at once – they can't exist simultaneously!