09/25/15

Lecture 11 - Campbell-Baker-Hausdorff & Wavefunctions

1. Commutation Relations
2. Finite Translation Operators
3. Wavefunctions

1. Commutation Relations

p-p commutations

$T(\Delta x_j')p_i$ — we don't know how to effectively multiply this

↳ $T(\Delta x_j'')$, an Abelian group

Abelian Group properties:

$$T(\Delta x_j')T(\Delta x_j'') = T((\Delta x' + \Delta x'')_j)$$
$$= T(\Delta x_j'')T(\Delta x_j')$$ — members of an Abelian Group commute

$$\cancel{T(\Delta x) = 1 - \frac{i\hat{p}\cdot\Delta x}{\hbar} + \Theta(\Delta x^2)}$$

$$\Rightarrow \cancel{T(\Delta x_j')p_i = \left(1 - \underbrace{\frac{i\hat{p}\cdot\Delta x'}{\hbar}}_{\hat{p}\cdot\Delta x = \sum_j \hat{p}_j \Delta x_j}\right)\left(1 - \frac{i\hat{p}\cdot\Delta x''}{\hbar}\right) + \Theta(\Delta x'^2, \Delta x''^2) = 0}$$

don't keep higer-order terms

$$= -i\hat{p}/\hbar \cdot (\cancel{\Delta x'} + \cancel{\Delta x''} - \cancel{\Delta x'} - \cancel{\Delta x''}) - 1/\hbar^2(\hat{p}\cdot\Delta x')(\hat{p}\cdot\Delta x'') + 1/\hbar^2(\hat{p}\cdot\Delta x'')(\hat{p}\cdot\Delta x') = 0$$

$$\rightarrow (\hat{p}\cdot\Delta x')(\hat{p}\cdot\Delta x'') - (\hat{p}\cdot\Delta x'')(\hat{p}\cdot\Delta x') = 0$$

expand in components

$$= \sum_{j,k} \hat{p}_j \Delta x_j' \hat{p}_k \Delta x_k'' - \hat{p}_k \Delta x_k'' \hat{p}_j \Delta x_j' = 0$$
$$= \sum_{j,k} \underbrace{[\hat{p}_j, \hat{p}_k]}_{=0} \Delta x_j' \Delta x_k'' = 0$$

Yields: Cannonical Commutation Relations

$$[\hat{x}_j, \hat{x}_k] = [\hat{p}_j, \hat{p}_k] = 0$$
$$[\hat{x}_j, \hat{p}_k] = i\hbar\delta_{jk} \quad j,k = x,y,z$$

2. Finite Translation Operators

(vs. infinitesimal translation)

$T(\Delta x') = ?$ for $\Delta x' \not\to 0$

$$= \lim_{N\to\infty} T\underbrace{\left(\frac{\Delta x'}{N}\right)}_{\text{plug-in, neglecting higher-order terms}}^N$$ — break up your translation

→

$$= \lim_{N\to\infty}\left(1-\frac{i\hat{p}\cdot\Delta x'}{\hbar N}\right)^N$$

just the exponential! $\lim_{N\to\infty}(1+\frac{x}{N})^N = e^x$

~~HW4(P1)~~ $\Rightarrow T(\Delta x') = e^{\frac{-i\hat{p}\cdot\Delta x'}{\hbar}}$, ~~the general translation operator!~~

power series $e^A = \sum_{n=0}^{\infty}\frac{A^n}{n!}$ applies

Campbell-Baker-Hausdorff Identities

expansion of operators

$$e^B A e^{-B} = A + [B,A] + \frac{1}{2}[B,[B,A]] + \frac{1}{3!}[B[B[B,A]]] + \ldots$$

Use this to show that operators commute:

$$T^\dagger(\Delta x')\hat{x}_k T(\Delta x')|x'\rangle$$

$$= \underbrace{e^{i\hat{p}\cdot\Delta x'/\hbar}\hat{x}_k e^{-i\hat{p}\cdot\Delta x'/\hbar}}|x'\rangle$$

$$= \hat{x}_k + i[\hat{p}\cdot\Delta x'/\hbar, \hat{x}_k] + \frac{i}{2}[\hat{p}\cdot\Delta x'/\hbar[\hat{p}\cdot\Delta x'/\hbar, \hat{x}_k]] + \ldots$$

$$= \hat{x}_k + \underbrace{i\sum_j\left[\frac{\hat{p}_j\cdot\Delta x_j'}{\hbar}, \hat{x}_k\right]} + \ldots$$

$$= i\sum_j\frac{\Delta x_j'}{\hbar}\underbrace{[\hat{p}_j, \hat{x}_k]}_{i\hbar\delta_{jk}}$$

$$= \hat{x}_k + i\sum_j\frac{\Delta x_j'}{\cancel{\hbar}}\cdot(i\cancel{\hbar})\delta_{jk} + 0$$

$$= \hat{x}_k + \Delta x_k' \checkmark$$ Translation operator took the position operator and shifted it.

For momentum:

$$T^\dagger(\Delta x')\hat{p}_k T(\Delta x') = \hat{p}_k + 0$$

we do not expect momenta to change under translation

$$\rightarrow T^\dagger(\Delta x')\underbrace{e^{\frac{-i\hat{p}\cdot\Delta x''}{\hbar}}}_{T(\Delta x'')}T(\Delta x') = \underbrace{e^{\frac{-i\hat{p}\cdot\Delta x''}{\hbar}}}_{T(\Delta x'')}$$

$$T(\Delta x')T(\Delta x'') = T(\Delta x'')T(\Delta x')$$

All Abelian Group properties still apply

$\Rightarrow$ Translation works for finite Δx

3. Wavefunctions

Position wavefunctions

$\hat{x}|x'\rangle = x'|x'\rangle$

any state

$$|\alpha\rangle = \underbrace{\int dx' |x'\rangle}_{\mathbb{1}} \underbrace{\langle x'|\alpha\rangle}_{\psi_\alpha(x')}$$

$\psi_\alpha(x')$ is the position-space wavefunction

$$\langle\beta| = \Big(\int dx'' |x''\rangle \underbrace{\langle x''|\beta\rangle}_{\psi_\beta(x'')}\Big)^{\dagger} \text{ adjoint}$$

$$\langle\beta|\alpha\rangle = \int dx'' dx' \psi_\beta^*(x'') \underbrace{\langle x''|x'\rangle}_{\delta(x''-x'),\ \text{dirac delta}} \psi_\alpha(x')$$

$= 1$ if $x''=x'$; 0 otherwise

$$= \int dx' \psi_\beta^*(x')\psi_\alpha(x')$$

OLD definition of inner product

$\hat{A} \rightarrow$ operator ~ Hamiltonians (an observable)

$\hat{A}|a'\rangle = a'|a'\rangle$

In UG quantum mechanics:

$$|a'\rangle = \int dx' |x'\rangle \underbrace{\langle x'|a'\rangle}_{u_{a'x'}}$$

$u_{a'x'}$ - eigenstates [of the Hamiltonian]

In our new language:

$$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle$$

$$\underbrace{\langle x'|\alpha\rangle}_{\psi_\alpha(x')} = \sum_{a'} \underbrace{\langle x'|a'\rangle}_{u_{a'}(x')} \underbrace{\langle a'|\alpha\rangle}_{\text{coefficients, } c_{a'}}$$

Operators

$$\langle\beta|\hat{A}|\alpha\rangle = \int dx' dx'' \psi_\beta^*(x'')\psi_\alpha(x') \underbrace{\langle x''|\hat{A}|x'\rangle}_{\text{matrix elements of } \hat{A} \text{ in the } x\text{-basis}} \quad (1)$$

If $A = \hat{x}$:

$$\langle x''|\hat{x}|x'\rangle = x' \underbrace{\langle x''|x'\rangle}_{\delta(x'-x'')}$$

$\langle x''|\hat{x}^N|x'\rangle = x'^N \delta(x'-x'')$

$\langle\beta|\hat{x}^n|\alpha\rangle = \int dx'dx'' \psi_\beta^*(x'')\psi_\alpha(x') x'^n \delta(x'-x'')$

But what if it's NOT a delta function? (off-diagonal operator)

let $\hat{A} = \hat{T}(\Delta x')$

$\langle x''|\hat{T}(\Delta x')|x'\rangle = \langle x''|x'+\Delta x'\rangle$

$= \delta(x''-x'-\Delta x')$

→ plug into equation ①

$\langle\beta|\hat{T}(\Delta x')|\alpha\rangle = \int dx'dx'' \psi_\beta^*(x'')\psi_\alpha(x')\delta(x''-x'-\Delta x')$

└ take this integral first

$= \int dx'' \psi_\beta^*(x'')\psi_\alpha(x''-\Delta x')$

Recall: $T(\Delta x) = 1 - \frac{i}{\hbar}(\hat{p}\cdot\Delta x) + \mathcal{O}(\Delta x^2)$

→ expand both sides in a Taylor series

$\cancel{\langle\beta|\alpha\rangle} - (\frac{i}{\hbar})\langle\beta|\hat{p}|\alpha\rangle\cdot\Delta x' + \mathcal{O}(\Delta x^2) = \int dx'' \psi_\beta^*(x'')[\psi_\alpha(x'') - \Delta x'\cdot\vec{\nabla}\psi_\alpha(x'') + \ldots \mathcal{O}(\Delta x^2)]$

└ $\cancel{\langle\beta|\alpha\rangle}$

$-\frac{i}{\hbar}\langle\beta|\hat{p}|\alpha\rangle\cdot\cancel{\Delta x'} = -\cancel{\Delta x'}\cdot\int dx'' \psi_\beta^*(x'')\vec{\nabla}\psi_\alpha(x'')$

$\langle\beta|\hat{p}|\alpha\rangle = \int dx'' \psi_\beta^*(x'')\underbrace{(-i\hbar\vec{\nabla})}_{\hat{p}}\psi_\alpha(x'')$

$\Rightarrow \langle x'|\hat{p}|x''\rangle = -i\hbar\vec{\nabla}\delta(x'-x'')$

Momentum-Space Wavefunctions

$[\hat{p}_j, \hat{p}_k] = 0$

→ simultaneous eigenkets

$\hat{p}_j|p'\rangle = p_j'|p'\rangle$

└ exactly analogous to $\hat{x}_j|x'\rangle = x_j'|x'\rangle$ such that we can write relationships with $|\alpha\rangle$ & $|\beta\rangle$ in the same ways

$|\alpha\rangle = \int dp' |p'\rangle \underbrace{\psi_\alpha(p')}_{\langle p'|\alpha\rangle}$

$\langle p'|\alpha\rangle$, the momentum-space wavefunction

$\langle\beta|\alpha\rangle = \int dp' \psi_\beta^*(p')\psi_\alpha(p')$

$\langle p'|\hat{p}|p''\rangle = \delta(p'-p'')p''$

$\sim \langle x'|x''\rangle = \delta(x'-x'')$

When things get weird:

$\langle p'|\hat{x}|p''\rangle = +i\hbar\vec{\nabla}_{p'}\delta(p'-p'')$

* we can't just apply $\hat{T}$, it does not translate p

$\rightarrow$ change basis from $|p\rangle$ to $|x\rangle$

$$\langle p'|\hat{x}|p''\rangle = \int dx'dx'' \langle p'|x'\rangle\langle x'|\hat{x}|x''\rangle \underbrace{\langle x''|p''\rangle}_{\text{basis transformation matrix}}$$

In unitary language:

$$\hat{u} = \int d\lambda\, |x'=\lambda\rangle\langle p'=\lambda|$$

$$\hat{u}|\hat{p}=\lambda\rangle = |\hat{x}=\lambda\rangle$$

$\Rightarrow$ such that $\langle \hat{p}=p'|\hat{u}|\hat{p}=\lambda\rangle = \langle p'|\hat{x}=\lambda\rangle$

But instead of trying to compute this, we simply use something we already know:

$$\langle x'|p'\rangle \longrightarrow \langle x'|\hat{p}|p'\rangle = p'\langle x'|p'\rangle$$

$$= \int dx'' \underbrace{\langle x'|\hat{p}|x''\rangle}_{-i\hbar\nabla_{x'}\delta(x'-x'')}\langle x''|p'\rangle$$

differential eqs

$$= -i\hbar\nabla_{x'}\underbrace{\langle x'|p'\rangle} = p'\langle x'|p'\rangle$$

$$\langle x'|p'\rangle = U_{p'}(x') = N\exp\left(\frac{+ip'x'}{\hbar}\right)$$

still needs to be normalized!

$$\langle p'|p''\rangle = \int dx' \langle p'|x'\rangle\langle x'|p''\rangle = \delta(p'-p'')$$

$$= N^2 \underbrace{\int dx'\, e^{i(p''-p')\cdot x'/\hbar}}_{\text{Fourier Integrals} = (2\pi\hbar)^D\delta}$$

$\Rightarrow N = 1/(2\pi\hbar)^{D/2}$ (normalization constant)

$$\Rightarrow \underbrace{\langle x'|p'\rangle}_{\text{wavefunction of a momentum ket}} = \frac{1}{(2\pi\hbar)^{D/2}} e^{ip'x'/\hbar} \rightarrow \text{~~the free particle wavefunction~~}$$

momentum eigenkets $\rightarrow$ Hamiltonian eigenkets (energy) for the free particle

This is expected since $[\hat{p}, \mathcal{H}] = 0$

$$\psi_\alpha(p') = \langle p'|\alpha\rangle = \int dx' \langle p'|x'\rangle\langle x'|\alpha\rangle$$

$$= \frac{1}{(2\pi\hbar)^{D/2}} \int dx'\, e^{-ip'x'/\hbar}\psi_\alpha(x')$$

similarly,

$$\psi_\alpha(x') = \frac{1}{(2\pi\hbar)^{D/2}} \int dx'\, e^{ip'x'/\hbar}\psi_\alpha(p')$$

Uncertainty Principle

$\Delta x \Delta p \geq \hbar/2$

└ derived from $[\hat{x}, \hat{p}] = i\hbar$

$|p'\rangle \rightarrow \Delta p = 0$

$\rightarrow \Delta x = \infty$ – perfectly-known momentum yields position spread out over all space

$|x'\rangle \rightarrow \Delta x = 0$

$\rightarrow \Delta p = \infty$ – implies infinite energy

Solution: **Gaussian Wavepacket**

$$\langle x'|\alpha\rangle = \frac{1}{\pi^{1/4}\sqrt{d}} e^{ip'x'/\hbar} e^{-x'^2/2d}$$

└ for any finite d, it acts like a reasonable state

$d \rightarrow 0 \Rightarrow |x' = 0\rangle$

$d \rightarrow \infty \Rightarrow |p'\rangle$