09/23/15

Lecture 10 - Translating $\hat{x}$ to $\hat{p}$

1. Homework 3, Problem 3&4
2. Commutation Relations

1. Homework 3

Problem 3:

Diagonalizing 2×2 matrices from Homework 2

Problem 4:

Simultaneous diagonalization of commuting operators for a 3×3 matrix

2. Commutation Relations

Translation operator:

$$\hat{T}(\Delta x')|x'\rangle = |x'+\Delta x'\rangle$$

→ forms an Abelian group

→ Unitary

$$T(\Delta x') \lim_{\Delta x' \to 0} \simeq 1 - i\hat{K}\cdot\Delta x' + \Theta(\Delta x^2) \quad ①$$

$\hat{K}$ = Hermitian (related to momentum)

What is the relationship between $\hat{K}$ and $\hat{x}$?

→ Start with the relationship between T & X

$$\hat{T}(\Delta x')\underbrace{\hat{x}_j|x'\rangle}_{x'} = \hat{T}(\Delta x')\, x'|x'\rangle$$

$|x'\rangle = |x'_x, x'_y, x'_z\rangle$

$$= x'\hat{T}(\Delta x')|x'\rangle \quad ②$$

$$\hat{x}_j\hat{T}(\Delta x')|x'\rangle = \hat{x}_j|x'+\Delta x'\rangle$$

$$= (x'+\Delta x')|x'+\Delta x'\rangle \quad ③$$

eq. ② − eq. ③

$$[\hat{T}(\Delta x')\hat{x}_j - \hat{x}_j\hat{T}(\Delta x')]|x'\rangle = -\Delta x'|x'+\Delta x'\rangle$$

$$= -\Delta x'\hat{T}(\Delta x')|x'\rangle$$

$$\rightarrow [\hat{T}(\Delta x')\hat{x}_j - \hat{x}_j\hat{T}(\Delta x') + \Delta x'\hat{T}(\Delta x')]|x'\rangle = 0$$

must hold for any $|x'\rangle$

$$\Rightarrow \hat{T}(\Delta x')\hat{x}_j - \hat{x}_j\hat{T}(\Delta x') + \Delta x'\hat{T}(\Delta x') = 0$$

Use eq. ① to write T in terms of K →

$$(\cancel{1}-i\hat{K}\cdot\Delta x')\hat{x}_j-\hat{x}_j(\cancel{1}-i\hat{K}\cdot\Delta x')+\Delta x'(1-i\hat{K}\cdot\Delta x')+\Theta(\Delta x')^2=0$$

$\hat{K} = (\hat{K}_x, \hat{K}_y, \hat{K}_z)$

– linear order $(\Delta x')$

$$-iK\cdot\Delta x'\hat{x}_j+i\hat{x}_j K\cdot\Delta x'+\Delta x'=0$$

→ put back in vector indices

$$\{-i\hat{K}_j\cdot\Delta x_i'\hat{x}_j+i\hat{x}_j\hat{K}_i\cdot\Delta x_i'\}+\underbrace{\Delta x_j'}_{\sum_i \delta_{ij}\Delta x_j'}=0$$

$$=\sum_i\{-i(\hat{K}_i\hat{x}_j-\hat{x}_j\hat{K}_i)+\delta_{ij}\}\Delta x_i'=0$$

$\forall_i \Delta x_i'$

$$\hookrightarrow -i[\hat{K}_i,\hat{x}_j]+\delta_{ij}=0$$

generator of translation

$$\Rightarrow \underbrace{[\hat{x}_j,\hat{K}_i]=i\delta_{ij}}_{\text{commutation relation}}$$

Hamilton–Jacobi (from classical)

$p_i,\ q_j \sim x_j$

$q \mapsto q+\Delta q = q'$ } ~~general translation~~

$p \mapsto p'$

→ **Generating functions**

$$F(p,q)=p'q+p\Delta q$$

↑ p is the generator of translation

Classical "commutator"

$\{A(p,q),\ B(p,q)\}$ ← Poisson bracket

$$=\sum_{ij}\frac{\partial A}{\partial p}\frac{\partial B}{\partial q}-\frac{\partial A}{\partial q}\frac{\partial B}{\partial p}$$

$$\Rightarrow \underbrace{\{p_i,q_j\}}_{\text{has all the properties of a commutator}}=\delta_{ij}$$

Aside: **commutator identities**

$$[A,B]=AB-BA=-[B,A]$$

– Jacobi Identity:

$$[A,[B,C]]+[B,[C,A]]+[C,[B,A]]=0$$

So what's wrong with calling K momentum?
(K & p both generators of translation)
$K\Delta X' \sim 1$ (dimensionless)
$\Rightarrow \hat{K} \sim$ wave number (units: $1/\lambda$)

$\omega \xrightarrow{\hbar} E$ Planck

$K \xrightarrow{\hbar} p$ DeBroglie

similar relation holds

Heisenberg realized you need matrices. Born-Jordan recognized the relationship:

$\hat{K} = 1/\hbar \, \hat{p}$ $[\hat{p}_i, \hat{x}_j] = -i\hbar\delta_{ij}$

Dirac took this further to a general prescription between commutators and poisson brackets

$[A,B] \longrightarrow -i\hbar\{A,B\}$

handles p·q

for units