12/09/15

Lecture 42 - Wigner-Eckart Theorem

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Review: **General Relation for Spherical Tensors**

spherical tensor

$$\mathcal{D}^\dagger(R)\,T_q^{(k)}\,\mathcal{D}(R) = \underbrace{\sum_{q'=-k}^{k} \mathcal{D}_{qq'}^{(k)*}\,T_{q'}^{(k)}} \qquad \text{eq. 3.11.22.a}$$

[linear] transformation of the tensor upon rotation

This equation is diagonal in $k \rightarrow$ different k's do not mix, they form their own subspace and do not mix upon rotation

equivalently,

$$\mathcal{D}(R)\,T_q^{(k)}\,\mathcal{D}^\dagger(R) = \sum_{q'=-k}^{k} T_{q'}^{(k)}\,\underbrace{\mathcal{D}_{qq'}^{(k)}}(R) \qquad \text{eq. 3.11.22.b}$$

generator of rotation $\langle kq'|(\ )|kq\rangle$

Spherical tensors transform like spherical functions

$T_q^{(k)} \sim Y_{\ell=k}^{m=q}$, where the spherical tensor representation is irreducible

the spherical wavefunction is the basis here, where different q's/m's can mix under rotation (unlike the k/ℓ values)

In the case of infinitesimal rotation:

infinitesimal rotation ε about $\hat{n}$ with angular momentum $\vec{J}$

$$\mathcal{D}(R) = \left(1 + \frac{i\vec{J}\cdot\hat{n}\,\varepsilon}{\hbar}\right)$$

matrix representation of infinitesimal rotation

- under ε rotation, we can make statements about commutation

where J is an operator

$$[\vec{J}\cdot\hat{n},\, T_q^{(k)}] = \sum_{q'} T_{q'}^{(k)} \langle kq'|\vec{J}\cdot\hat{n}|kq\rangle \qquad \text{eq. 3.11.24}$$

$\rightarrow$ choose $\hat{n} \parallel \hat{z}$

matrix elements of J_z are produced

$$[J_z,\, T_q^{(k)}] = \langle kq'|\vec{J}_z\cdot\hat{z}|kq\rangle$$
$$= \hbar q\,T_q^{(k)}$$

$\rightarrow$ choose $\hat{n} \parallel (x+iy)$ (n_+, n_- applicable)

$$[J_\pm,\, T_q^{(k)}] = \hbar\sqrt{(k\mp q)(k\pm q+1)}\,T_{q\pm1}^{(k)}$$

$\longrightarrow$

Proving the Wigner-Eckart Theorem:

$$\langle \alpha', j', m' | T_q^{(k)} | \alpha, j, m \rangle \propto \overbrace{\langle j, k; m q | j, k; m' q' \rangle}^{\text{the Clebsch-Gordan coefficients}} \qquad \text{eq. 3.11.31}$$

dependence on m & m' determined by the C-G Coefficients

state with angular momentum j and projected angular momentum m

$$= \langle j, k; m q | j, k; m' q' \rangle f(\underbrace{j, j', \alpha, \alpha'}_{\text{no dependence on } m\text{'s} \,\&\, q\text{'s}})$$

Where f is the reduced matrix element

$$f = \frac{\langle \alpha' j' \| T^{k} \| \alpha j \rangle}{\sqrt{2j+1}}$$

double-bar indicates "reduced"

- **convert the operator relation to a recursive relation to obtain the matrix elements**

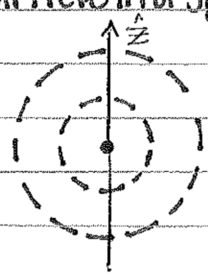
sandwich the commutator on both sides

$$[J_\pm, T_q^{(k)}] \rightarrow \langle \alpha', j', m' | J^\pm T_q^k | \alpha, j, m \rangle - \langle \alpha', j', m' | T_q^k J^\pm | \alpha, j, m \rangle$$
$$= \hbar \sqrt{(k \mp q)(k \pm q + 1)} \langle \alpha', j', m' | T_{q\pm 1}^k | \alpha, j, m \rangle$$

$\Rightarrow$ **Recursive relation to find the matrix elements of spherical operator T is the exact same as the recursive relation to find the Clebsch-Gordan coefficients!** (they must have the same solutions up to a constant that is not a function of m, m', and q (but can be a function of j, k, α, etc.))

ex: **F'13 Qualifier problem on Wigner-Eckart** (Aug. 13, II-4)

Particle in a spherically-symmetric potential *Problem 4 is always on symmetry



$\mathcal{H} = \mathcal{H}_0$

eigenstates of the Hamiltonian can be chosen to be eigenstates of angular momentum

$|n, j, m\rangle$; $E^{(0)}_{n,j,\cancel{m}}$

before perturbation, $E^{(0)}$ is independent of / degenerate in m

Now, add perturbation

$\mathcal{H} = \mathcal{H}_0 + V$

$$V(r,\theta,\varphi) = \underbrace{\lambda e^{-r^2/R^2} r^{-k}}_{\text{not important here}} Y_k^0(\theta,\varphi)$$

not actually dependent on φ

Spherical harmonic; projection 0, generic K

$\rightarrow$ We expect this perturbation to lift the degeneracy of E in m

(a) Is m a good Quantum Number?

└ "good" implies that $|n,j,m\rangle$ is still an eigenstate of the Hamiltonian after perturbation

Yes! $Y_\ell^m \sim e^{im\varphi} P_\ell(\theta)$

$e^{im\varphi} = 1 \; (m=0)$

$m=0 \rightarrow$ The potential is cylindrically symmetric around z, so we can label energies by value 'm' since eigenstates with respect to m are still eigenstates

(d) Now, let $K=6$

Given $E^0_{n,j}$, what is the size of the shift in the energy level?

$$\Delta E_{n,j,m} = E_{j,n,m} - E^{(0)}_{n,j} = \begin{cases} \lambda \text{ (corrections} \rightarrow 1^{st} \text{ order perturbation theory)} \\ \lambda^2 \text{ (}2^{nd}\text{ order perturbation theory)} \end{cases}$$

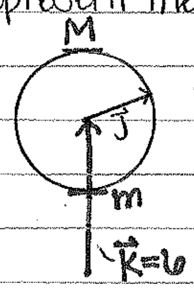
$\rightarrow$ Calculate the expectation value of $V \rightarrow$

$$\Delta E_{n,j,m} = \langle n,j,m|V|n,j,m\rangle \begin{cases} \neq 0 \sim \lambda \\ = 0 \sim \lambda^2 \end{cases}$$

└ if 0, must keep higher order terms

$$= \langle n,j,m|Y^0_{K=6}|n,j,m\rangle$$

Represent the addition of $\vec{K}+\vec{J}$:



Possible values of $\vec{K}+\vec{J}$ lie on the circle, as $\vec{K}$ and $\vec{J}$ have arbitrary direction with respect to each other.

$\min = \vec{K}+(-\vec{J})$; $\text{Max} = \vec{K}+\vec{J}$

$$\begin{array}{ll} j=0 & \langle j,m|Y^0_6|j,m\rangle = 0 \\ j=1 & = 0 \\ j=2 & = 0 \\ j=3 & \neq 0 \rightarrow \Delta E \sim \lambda \end{array}$$

($j=0,1,2$: $\Delta E \sim \lambda^2$)

(e) Let $K=2$, $j=1$ — because $j\neq 0$, your expectation value now represents energy shift with dependence on m

$$\Delta E_m = \frac{\langle 1,m|Y_2^0|1,m\rangle}{\langle 1,m'|Y_2^0|1,m'\rangle} = \frac{\text{C-G}(m)}{\text{C-G}(m')}$$

└ $m' = 1, 0, -1$ but $\neq m$

The ratio of the energy shift for different values of m is just the ratio of the C-G coefficients for those m, m' values