10/30/15

Lecture 26 - Electromagnetism in Quantum Mechanics

1. Homework 7
2. Electromagnetism in Quantum Mechanics
3. Hamiltonian/Schrödinger's Equation in EM Fields

1. Homework 7

#1. The no-node theorem

↳ where a node is where the wavefunction changes sign

$$\left[\frac{-\hbar^2}{2m}\partial_x^2 + V(x)\right]\psi(x) = E(x) \quad (1)$$

Examine the ground state wavefunction ψ_0

→ $\psi_0(x) \geq$ everywhere

↳ no phase changes, does not vanish / no nodes

Ground state is also non-degenerate

Proof: **Wronskian** (from PDEs)

$W(x) = \psi_1'(x)\psi_2(x) - \psi_1(x)\psi_2'(x)$

$(1) \Rightarrow dW/dx = 0$

for $V > 0$, $|x| \to \infty$ (bound states)

$\psi(x) \to 0 \Rightarrow W(x) = 0$

#2. WKB Approximation

a) Quantization conditions for classically-allowed states

$\oint \sqrt{2m(E_n - V(x))}\,dx = 2\pi(n+\tfrac{1}{2})\hbar$

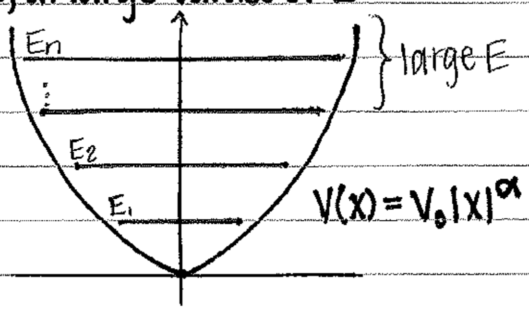
↳ for bound states

Estimate $\rho(E)$, density of states, at large values of E

$\rho(E)\Delta E = N(E, E+\Delta E)$

↳ At these energies, ρ scales as $E^{(1/\alpha - 1/2)}$

⇒ Allows you to more easily make statements about the properties of excited states



#3. Path Integral

- Calculate the path integral for the Harmonic Oscillator

$$\langle x_f, t_f | x_i, t_i \rangle = \int \mathcal{D}_x \exp(-i/\hbar\, S[x])$$

$S[x] = \int \frac{m}{2}(\dot{x}^2 - \omega^2 x^2)\, d\tau$; the action

→ **Write your path as an expansion about the classical path**

$$x(\tau) = \underset{\text{classical path}}{x_c(\tau)} + \overbrace{y(\tau)}^{\text{"small" in a Taylor expansion}} , \quad \mathcal{D}_x(\tau) = \mathcal{D}_y(\tau)$$

└ The product over all paths; a constant of the path for any path option

$$S[y] \sim \underbrace{(\ )x_c^2}_{S[x_c]} + \underbrace{(\ )x_c y}_{\cancel{\left(\frac{\delta S}{\delta y}\right)_y} \to 0} + (\ )y^2 + \boxed{\dots}$$

└ terms we ignore for the saddle-point approximation (stationary phase); $|y|$ small close to the classical path

$$\langle x_f t_f | x_i t_i \rangle = e^{iS[x_c]t} \underbrace{\int \mathcal{D}_y e^{i(\ )y^2}}_{\text{Gaussian Integral expansion}}$$

$$y(\tau) = \sum y_n \cos(\omega_n \tau) + \dots$$

$$\mathcal{D}_y(\tau) \propto \underbrace{\prod dy_n}_{\text{product of time slices}}$$

2. **Electromagnetism in Quantum Mechanics** (2.7 SAK)

First, think about electric fields

$$\vec{E} = \underbrace{-\vec{\nabla}_x V(\vec{x})}_{\text{the gradient of an electrostatic potential}}$$

$$H = \frac{-\hbar^2}{2m}\nabla^2 + V(x)$$

↳ we know how to approach $\vec{E}$ because of our experience with Hamiltonian

Classically, we can shift the potential

$V(x) \to V(x) + V_0$

with no physical consequence

"Baby Gauge Invariance"

* **But not all classical symmetries are necessarily also quantum mechanical symmetries** (either obviously or at all)

$$H \to \underbrace{H + V_0}_{\tilde{H}}$$

Where the purpose of the Hamiltonian was just to time-evolve the system

$e^{Ht/\hbar}|\alpha\rangle = |\alpha;t\rangle$

└ by above logic, we should be able to substitute $\tilde{H}$ with no consequence

$e^{\tilde{H}t/\hbar}|\alpha\rangle = e^{-i(H+V_0)t/\hbar}|\alpha\rangle$

$= \underbrace{e^{-iV_0t/\hbar}}|\alpha;t\rangle$

constant V_0 term can be pulled out

← causes phase shift on the ket

define $|\tilde{\alpha};t\rangle \equiv e^{-iV_0t/\hbar}|\alpha;t\rangle$ such that

$e^{-i\tilde{H}t/\hbar}|\alpha\rangle = |\tilde{\alpha};t\rangle$

└ we have absorbed the phase shift into the time-evolution of the ket here

We can do this because, *for the most part*, the phase shift is not an observable

$\langle\tilde{\alpha};t|\hat{A}|\tilde{\alpha};t\rangle = \cancelto{1}{e^{-iV_0t/\hbar}e^{iV_0t/\hbar}}\langle\alpha;t|\hat{A}|\alpha;t\rangle$

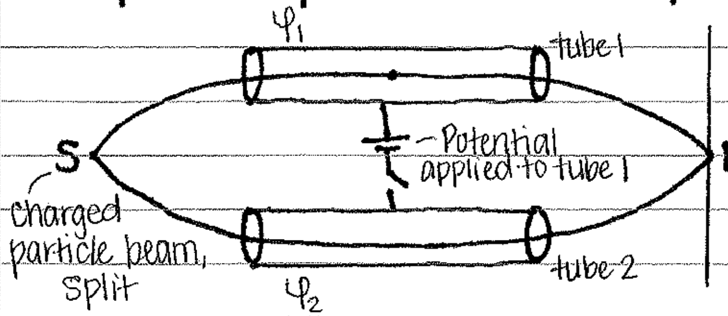
└ phase shift is cancelled out for the expectation value of a measurement on the shifted ket $|\tilde{\alpha};t\rangle$

$\rho(x;t) = |\langle x|\alpha;t\rangle|^2 = |\langle x|\tilde{\alpha};t\rangle|^2$

← shift causes no change in density here

<u>BUT</u> in the case of interferometry, phase is observable!

→ express the potential shift as a phase plate



Potential applied only while wave packet is inside the tube (the packet does not, however, feel an electric field)

Because of the existence of the potential difference, each beam component suffers a phase change with observable interference in the beam intensity $\varphi_1 - \varphi_2 = (1/\hbar)\int dt[V_2(t) - V_1(t)]$

$\Psi_{final} = \Psi_1 e^{i\varphi_1} + \Psi_2 e^{i\varphi_2}$

$P = |\Psi_1 e^{i\varphi_1} + \Psi_2 e^{i\varphi_2}|^2 \propto \cos(\varphi_1 - \varphi_2)$ ← Probability amplitude proportional to the phase shift

$H(t) = H + V_0^a(t)$, a = tube 1, tube 2

└ Hamiltonian describes the time evolution of the wave packet regardless of which leg the packet has gone through, even though the potential has only been applied to tube 1

$V_0^1(t) = 0$ — V_0 a constant

$V_0^2(t) = V_0(t)$

$$e^{-i/\hbar \int_0^t H(\tau)}|\alpha\rangle = e^{-i/\hbar \int_0^t V_0^a(\tau)d\tau} \overbrace{e^{-iHt/\hbar}|\alpha\rangle}^{|\alpha;t\rangle}$$

└ Assume the Hamiltonians @ different times commute

$$= |\alpha^2;t\rangle$$

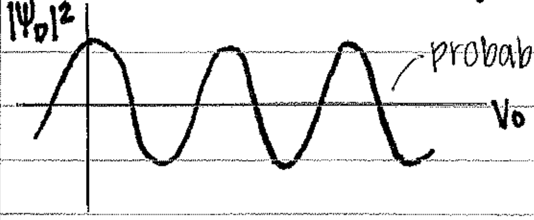
└ Product of the shifts in all time slices

$$\Psi_{final} = \underbrace{\psi_1 e^{i\varphi_1}}_{\langle 1|\alpha;t\rangle} + \underbrace{\psi_2 e^{i\varphi_2} e^{-i/\hbar \int_0^t V_0(\tau)d\tau}}_{\text{phase shift only in one branch}} = |\Psi(D)|^2$$

$$|\Psi_{final}^{(D)}|^2 = |\psi_1|^2 + |\psi_2|^2 + 2\mathrm{Re}\{\psi_1^* \psi_2 e^{i(\varphi_1 - \varphi_2)} \exp[-i/\hbar \int V_0(\tau)d\tau]\}$$

↑ interference term due to the phase shift caused by the applied potential

$$= (\quad) + (\quad) \cos(i/\hbar \int_0^t V_0(\tau)d\tau + \varphi_{shift})$$



→ **If we changed the potential, we would see interference fringes @ D**

We can also see this effect with Gravity in quantum systems!

Gravity: $V(x) = mgh_a$

└ Set up interferometer such that tubes are at different heights so the difference in GPE causes a phase shift

Newton: (classical)

$$\overset{\text{inertial mass}}{m}\ddot{h} = -\underset{\text{gravitational mass}}{m}g$$

Classically, these masses are equivalent

$\ddot{h} = -g \rightarrow$ Einstein: "gravity = geometry"

But in quantum, this equivalence/symmetry does not hold!

$\Rightarrow \cos[(mg/\hbar)\underbrace{h_{II-I}}\cdot t + (\ )]$

difference in height and the explicit relation of m and ħ in the quantum approach causes a phase shift not present in the classical realm!

Classically, you can formulate your Equation of Motion purely in terms of the electric & magnetic fields

$$m\frac{d^2x}{dt^2} = e(\vec{E} + \vec{v} \times \vec{B})$$

└ fully sufficient, no vector potential dependence

However, in Quantum Mechanics the A, φ is key to describing the time-evolution of the system (V(x) is highly impactful, as seen above)

- φ – electrostatic potential
- $\vec{A}$ – vector potential

3. Hamiltonian/Schrödinger's Equation in EM Fields

$$\vec{E} = \overbrace{-\vec{\nabla}\varphi}^{\text{gradient of potential}} \longleftrightarrow \vec{\nabla} \times \vec{E} = 0 \text{ (for the static case)}$$

$$\vec{B} = \underbrace{\vec{\nabla} \times \vec{A}}_{\text{curl of the potential}} \longleftrightarrow \vec{\nabla} \cdot \vec{B} = 0$$

Start by formulating your Hamiltonian

$$\mathcal{H} = \frac{1}{2m}\left(p - \frac{eA}{c^2}\right)^2 + e\varphi(x,t) \text{ (from classical)}$$

but! $A \equiv A(\hat{x}) \Rightarrow A$ and $\hat{p}$ will not commute

└ A is actually a function of potential, which can cause difficulties with commutators and the required Hermitian quality of the Hamiltonian

$$\left(\hat{p} - \frac{e}{c}A(\hat{x})\right)^2 \neq \hat{p}^2 - 2\frac{e}{c}\hat{p}\cdot\hat{A}(\hat{x})$$

$$\text{instead } = \left(\hat{p} - \frac{e}{c}\hat{A}(\hat{x})\right)\left(\hat{p} - \frac{e}{c}\hat{A}(\hat{x})\right) + \frac{\text{[illegible]}}{c^2}\hat{A}(\hat{x})^2$$

$$\mathcal{H}_{new} = \frac{p^2}{2m} - \frac{e}{c}\underbrace{(\hat{p}\cdot\hat{A}(\hat{x}) + \hat{A}(\hat{x})\cdot\hat{p})} + \frac{e^2}{c^2}\hat{A}(\hat{x})^2$$

Separate treatment of A & p is what allows $\mathcal{H}$ to be Hermitian

Consider the Heisenberg EOMs

- first in x

$$\frac{d\hat{x}_\alpha}{dt} = \frac{i}{\hbar}[\mathcal{H}, \hat{x}_\alpha], \quad \alpha = x, y, z$$

└ suppresed for simplicity

$$= \frac{i}{\hbar}\left[\frac{1}{2m}\left(\hat{p} - \frac{e\hat{A}}{c}\right)^2, \hat{x}\right]$$

$$= \frac{i}{2m\hbar}\left\{\left(p - \frac{eA}{c}\right)\underbrace{\left[\hat{p} - \frac{eA}{c}, \hat{x}\right]}_{-i\hbar} + \underbrace{\left[\hat{p} - \frac{eA}{c}, \hat{x}\right]}_{-i\hbar}\left(p - \frac{eA}{c}\right)\right\}$$

$$= \frac{1}{2m}\cdot 2\left(p - \frac{eA}{c}\right) \Rightarrow 1$$

* Why? This isn't in the book.

$\nabla_x S$, the canonical momentum

$[p, x] = -i\hbar$

$$\underbrace{m\frac{dx}{dt}} = \left(p - \frac{eA}{c}\right)$$

$= \Pi$, the kinematic/mechanical momentum

$$[p_\alpha, p_\beta] = 0$$

but the analogous commutator does not vanish for kinematic mom.

$$[\Pi_\alpha, \Pi_\beta] = \frac{i\hbar}{c}\underbrace{\varepsilon_{\alpha\beta\gamma}}_{\text{epsilon tensor}} B_\gamma$$

← canonical momentum commutes with itself, but kinematic momentum has a remainder

$$[\Pi_\alpha, X_\beta] = -i\hbar\delta_{\alpha\beta}$$

– the commutation of kinetic momentum with position is, however, the same as canonical momentum with position

$$[\Pi^2, \Pi_\beta] = \sum_\alpha [\Pi_\alpha^2, \Pi_\beta]$$

$$= \sum_\alpha \Pi_\alpha[\Pi_\alpha, \Pi_\beta] + [\Pi_\alpha, \Pi_\beta]\Pi_\beta$$

$$= \frac{i\hbar e}{c}\sum_\gamma \varepsilon_{\alpha\beta\gamma}(\Pi_\alpha B_\gamma + B_\gamma\Pi_\alpha)$$

Now, consider the Heisenberg EOMp

$$\frac{dp}{dt} = \frac{i}{\hbar}[\mathcal{H}, \hat{p}]$$

$$\rightarrow \frac{d\Pi}{dt} = \frac{i}{\hbar}\left[\mathcal{H}, \hat{\Pi}\right]$$

$\mathcal{H} = \frac{\Pi^2}{2m} + e\varphi(x,t)$

$$= \frac{i}{\hbar}\frac{1}{2m}\left[\Pi^2, \Pi\right] + \frac{ei}{\hbar}\left[\varphi, \hat{\Pi}\right]$$

$= \hat{p}$ – both $\hat{A}$ & φ are functions of $\hat{x}$, so they commute the same with $\hat{p}$ and $\hat{\Pi}$

$$= -e\nabla\varphi$$

$$= e\left[\vec{E} + \frac{1}{2c}\left(\frac{d\hat{x}}{dt}\times\vec{B} - \vec{B}\times\frac{d\hat{x}}{dt}\right)\right]$$

$$\frac{d\Pi}{dt} = e\vec{E} + e\left(\frac{\Pi}{m}\right)\times\vec{B}$$

$$= e(\vec{E} + \underset{= d\vec{x}/dt}{\vec{v}}\times\vec{B})$$

$$= m\frac{d^2\vec{x}}{dt^2}$$, as expected