Physics 606: Homework #6

Due April 19, 2016

Jackson: Problems 7.19b, 7.22, 7.26, 7.28, 8.2, 8.4

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Homework #6

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1. Jackson Problem 7.19(b)

(b) $f(x) = Ne^{-\alpha^2 x^2/4}$

– Calculate the wave-number spectrum $|A(k)|^2$ of the packet

From Jackson eq. (7.81)

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0) e^{-ikx} dx$$

where $u(x,0) = f(x) e^{ik_0 x}$
$= Ne^{-\alpha^2 x^2/4} e^{ik_0 x}$

$$= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{e^{i(k_0 - k)x - \alpha^2 x^2/4}} dx$$

recognize the cosine function:
$e^{i(k_0-k)x} = \cos((k_0 - k)x)$

$$= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos[(k_0 - k)x]\, e^{-\alpha^2 x^2/4} dx$$

↓ plugging into Mathematica ↓

$$A(k) = N\sqrt{\frac{2}{\alpha^2}}\, e^{-2(k-k_0)^2/\alpha^2}$$

Yielding a wave-number spectrum

$$|A(k)|^2 = \frac{2N^2}{\alpha^2} e^{-2(k-k_0)^2/\alpha^2}$$

– Calculate explicitly the rms deviations from the means Δx & Δk

By definition, the rms for a wave packet is

$$\Delta x = \sqrt{\frac{\int_{-\infty}^{\infty} x^2 [f(x)]^2 dx}{\int_{-\infty}^{\infty} [f(x)]^2 dx}}$$

$$\Delta x = \sqrt{\frac{\cancel{N} \int_{-\infty}^{\infty} x^2 e^{-\alpha^2 x^2/2} dx}{\cancel{N} \int_{-\infty}^{\infty} e^{-\alpha^2 x^2/2} dx}}$$

$$\Delta x = \frac{1}{\alpha}$$

Then, for the wave number

$$\Delta k = \sqrt{\frac{\int_{-\infty}^{\infty} k^2 [f(k)]^2 dk}{\int_{-\infty}^{\infty} [f(k)]^2 dk}}$$

→

where $f(k) = Ne^{-k^2/\alpha^2}$

$$\Delta k = \sqrt{\frac{\cancel{N} \int_{-\infty}^{\infty} k^2 e^{-2k^2/\alpha^2} dk}{\cancel{N} \int_{-\infty}^{\infty} e^{-2k^2/\alpha^2} dk}} \checkmark$$

$$\Delta k = \frac{\alpha}{2}$$

- Test the inequality $\Delta x \Delta k \geq \frac{1}{2}$ ✓

$$\Delta x \Delta k = \left(\frac{1}{\alpha}\right)\left(\frac{\alpha}{2}\right)$$

$= \frac{1}{2}$ ✓ the inequality is satisfied.

2. Jackson Problem 7.22

The Kramers-Krönig relation (7.120):

$$\mathrm{Re}\left\{\frac{\varepsilon(\omega)}{\varepsilon_0}\right\} = 1 + \frac{2}{\pi} P \int_0^\infty \frac{\omega' \,\mathrm{Im}\{\varepsilon(\omega')/\varepsilon_0\}}{\omega'^2 - \omega^2}\, d\omega'$$

$$\mathrm{Im}\left\{\frac{\varepsilon(\omega)}{\varepsilon_0}\right\} = -\frac{2\omega}{\pi} P \int_0^\infty \frac{[\mathrm{Re}\{\varepsilon(\omega')/\varepsilon_0\} - 1]}{\omega'^2 - \omega^2}$$

$$\frac{1}{\omega' - \omega - i\delta} = P\left(\frac{1}{\omega' - \omega}\right) + \pi i \delta(\omega' - \omega)$$

(a) $\mathrm{Im}\{\varepsilon/\varepsilon_0\} = \lambda[\theta(\omega - \omega_1) - \theta(\omega - \omega_2)], \quad \omega_2 > \omega_1 > 0$

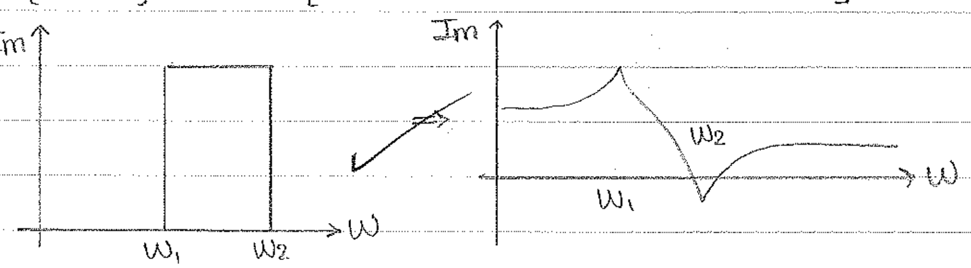
$$\mathrm{Re}\left\{\frac{\varepsilon(\omega)}{\varepsilon_0}\right\} = 1 + \frac{2}{\pi} P \int_0^\infty \frac{\omega' \lambda[\theta(\omega' - \omega_1) - \theta(\omega' - \omega_2)]}{\omega'^2 - \omega^2}$$

↓ applying step function ↓

$$= 1 + \frac{2\lambda}{\pi} P \int_{\omega_1}^\infty \frac{\omega'}{\omega'^2 - \omega^2}\, d\omega' - \frac{2\lambda}{\pi} P \int_{\omega_2}^\infty \frac{\omega'}{\omega'^2 - \omega^2}\, d\omega'$$

$$= 1 + \frac{2\lambda}{\pi}\left[\frac{1}{2}\ln(|\omega'^2 - \omega^2|)\right]_{\omega_1}^\infty - \frac{2\lambda}{\pi}\left[\frac{1}{2}\ln(|\omega'^2 - \omega^2|)\right]_{\omega_2}^\infty$$

$$\mathrm{Re}\left\{\frac{\varepsilon(\omega)}{\varepsilon_0}\right\} = 1 + \frac{\lambda}{\pi}\left[\ln(|\omega_2^2 - \omega^2|) - \ln(|\omega_1^2 - \omega^2|)\right]$$



(b) See following rough copy...

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2. Jackson Problem 7.22

Use the Kramers-Kronig relation (7.120) to calculate the real part of $\varepsilon(\omega)$, given the imaginary part of $\varepsilon(\omega)$ for positive ω as

(a) $\text{Im}\{\varepsilon/\varepsilon_0\} = \lambda[\theta(\omega-\omega_1) - \theta(\omega-\omega_2)], \quad \omega_2 > \omega_1 > 0$

(b) $\text{Im}\{\varepsilon/\varepsilon_0\} = \dfrac{\lambda\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$

In each case, sketch the behavior of $\text{Im}\{\varepsilon(\omega)\}$ and the result for $\text{Re}\{\varepsilon(\omega)\}$ as functions of ω. Comment on the reasons for similarities or differences of your results as compared with the curves in Fig. 7.8. The step function is $\theta(x)=0,\ x<1$, and $\theta(x)=1,\ x>0$.

The Kramers-Kronig relation (7.120):

$$\text{Re}\left\{\frac{\varepsilon(\omega)}{\varepsilon_0}\right\} = 1 + \frac{2}{\pi} P \int_0^\infty \frac{\omega' \,\text{Im}\{\varepsilon(\omega')/\varepsilon_0\}}{\omega'^2-\omega^2}\, d\omega'$$

$$\text{Im}\left\{\frac{\varepsilon(\omega)}{\varepsilon_0}\right\} = -\frac{2\omega}{\pi} P \int_0^\infty \frac{[\text{Re}\{\varepsilon(\omega')/\varepsilon_0\} - 1]}{\omega'^2-\omega^2}\, d\omega'$$

$$\frac{1}{\omega'-\omega-i\delta} = P\left(\frac{1}{\omega'-\omega}\right) + \pi i \delta(\omega'-\omega)$$

(a) $\text{Im}\{\varepsilon/\varepsilon_0\} = \lambda[\theta(\omega-\omega_1) - \theta(\omega-\omega_2)], \quad \omega_2 > \omega_1 > 0$

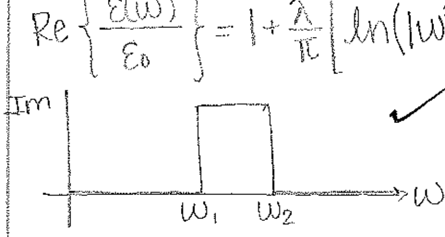
$$\text{Re}\left\{\frac{\varepsilon(\omega)}{\varepsilon_0}\right\} = 1 + \frac{2}{\pi} P \int_0^\infty \frac{\omega' \lambda[\theta(\omega'-\omega_1) - \theta(\omega'-\omega_2)]}{\omega'^2-\omega^2}\, d\omega'$$

↓ applying step functions ↓

$$= 1 + \frac{2\lambda}{\pi} P \int_{\omega_1}^\infty \frac{\omega'}{\omega'^2-\omega^2}\, d\omega' - \frac{2\lambda}{\pi} P \int_{\omega_2}^\infty \frac{\omega'}{\omega'^2-\omega^2}\, d\omega'$$

$$= 1 + \frac{2\lambda}{\pi}\left[\frac{1}{2}\ln(|\omega'^2-\omega^2|)\right]_{\omega_1}^\infty - \frac{2\lambda}{\pi}\left[\frac{1}{2}\ln(|\omega'^2-\omega^2|)\right]_{\omega_2}^\infty$$

$$\text{Re}\left\{\frac{\varepsilon(\omega)}{\varepsilon_0}\right\} = 1 + \frac{\lambda}{\pi}\left[\ln(|\omega_2^2-\omega^2|) - \ln(|\omega_1^2-\omega^2|)\right]$$



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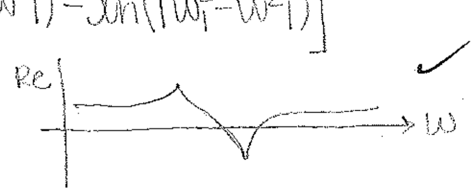

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(b) $\operatorname{Im}\left\{\frac{\varepsilon}{\varepsilon_0}\right\} = \frac{\lambda\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$

$$\operatorname{Re}\left\{\frac{\varepsilon(\omega)}{\varepsilon_0}\right\} = 1+\frac{2\lambda\gamma}{\pi}P\int_0^\infty \underbrace{\frac{\omega'^2}{((\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2)(\omega'^2-\omega^2)}}\,d\omega'$$

use Partial Fractions to simplify here

$$\frac{\omega'^2}{((\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2)(\omega'^2-\omega^2)} = \frac{A}{((\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2)}+\frac{B}{(\omega'^2-\omega^2)}$$

$$A\underbrace{(\omega'^2-\omega^2)}-\omega'^2+B((\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2)=0$$

since this must hold for all ω', we may choose the value $\omega'=\omega \to (\omega'^2-\omega^2)=0$

$$B((\omega_0^2-\omega^2)^2+\gamma^2\omega^2)=\omega^2$$

$$B=\frac{\omega^2}{((\omega_0^2-\omega^2)^2+\gamma^2\omega^2)}$$

$$A(\omega'^2-\omega^2)-\omega'^2+\frac{\omega^2((\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2)}{((\omega_0^2-\omega^2)^2+\gamma^2\omega^2)}=0$$

$$\to A=\frac{\omega_0^4-\omega'^2\omega^2}{((\omega_0^2-\omega^2)+\gamma^2\omega^2)}$$

$$\frac{\omega'^2}{((\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2)(\omega'^2-\omega^2)} = \frac{1}{((\omega_0^2-\omega^2)^2+\gamma^2\omega^2)}\left[\underbrace{\frac{\omega_0^4-\omega'^2\omega^2}{((\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2)}}+\frac{\omega^2}{(\omega'^2-\omega^2)}\right]$$

this can be similarly decomposed

$$\frac{\omega_0^4-\omega'^2\omega^2}{((\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2)} = \frac{A}{(\omega_0^2-\omega'^2+i\gamma\omega')}+\frac{B}{(\omega_0^2-\omega'^2-i\gamma\omega')}$$

$$A(\omega_0^2-\omega'^2-i\gamma\omega')+B(\omega_0^2-\omega'^2+i\gamma\omega')=\omega_0^4-\omega'^2\omega^2$$

↑ By inspection, we may choose here to write $B=\omega_0^2-A$

$$A(\omega_0^2-\omega'^2-i\gamma\omega')+(\omega_0^2-A)(\omega_0^2-\omega'^2+i\gamma\omega')=\cancel{\omega_0^4}-\omega'^2\omega^2$$

$$\cancel{A\omega_0^2}-\cancel{A\omega'^2}-Ai\gamma\omega'+\omega_0^4-\omega_0^2\omega'^2+i\gamma\omega'\omega_0^2-\cancel{A\omega_0^2}+\cancel{A\omega'^2}-Ai\gamma\omega'=-\omega'^2\omega^2$$

$$-2Ai\gamma\omega'-\omega_0^2\omega'^2+i\gamma\omega'\omega_0^2=-\omega'^2\omega^2$$

$$2Ai\gamma\omega'=\omega'^2\omega^2+i\gamma\omega'\omega_0^2-\omega_0^2\omega'^2$$

$$A=\frac{\cancel{i\gamma\omega'}\omega_0^2}{2\cancel{i\gamma\omega'}}-\frac{\omega'^{\cancel{2}}(\omega_0^2-\omega^2)}{2i\gamma\cancel{\omega'}}=\frac{\omega_0^2}{2}-\frac{\omega'(\omega_0^2-\omega^2)}{2i\gamma}$$

$$B=\omega_0^2-A=\frac{\omega_0^2}{2}+\frac{\omega'(\omega_0^2-\omega^2)}{2i\gamma}$$

$$\frac{\omega_0^4-\omega'^2\omega^2}{((\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2)(\omega'^2-\omega^2)}=\left[\frac{\omega_0^2}{2}-\frac{\omega'(\omega_0^2-\omega^2)}{2i\gamma}\right]\frac{1}{(\omega_0^2-\omega'^2+i\gamma\omega')}$$
$$+\left[\frac{\omega_0^2}{2}+\frac{\omega'(\omega_0^2-\omega^2)}{2i\gamma}\right]\frac{1}{(\omega_0^2-\omega'^2-i\gamma\omega')}$$

↓ Plugging this all back in to attain a form that can be solved analytically (or w/ Mathematica ☺) ↓

$$\mathrm{Re}\left\{\frac{\varepsilon(\omega)}{\varepsilon_0}\right\}=1+\frac{2\lambda\gamma}{\pi}P\int_0^\infty\frac{d\omega'}{((\omega_0^2-\omega'^2)^2+\gamma^2\omega^2)}\left[\frac{\omega_0^4-\omega'^2\omega^2}{((\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2)}+\frac{\omega^2}{(\omega'^2-\omega^2)}\right]$$

$$=1+\frac{2\lambda\gamma}{\pi}P\int_0^\infty\frac{d\omega'}{((\omega_0^2-\omega'^2)^2+\gamma^2\omega^2)}\left\{\left[\frac{\omega_0^2}{2}-\frac{\omega'(\omega_0^2-\omega^2)}{2i\gamma}\right]\frac{1}{(\omega_0^2+\omega'^2+i\gamma\omega')}\right.$$
$$\left.+\left[\frac{\omega_0^2}{2}+\frac{\omega'(\omega_0^2-\omega^2)}{2i\gamma}\right]\frac{1}{(\omega_0^2-\omega'^2-i\gamma\omega')}+\frac{\omega^2}{(\omega'^2-\omega^2)}\right\}$$

$$=1+\frac{2\lambda\gamma}{\pi}\frac{1}{((\omega_0^2-\omega^2)^2+\gamma^2\omega^2)}\left\{-\frac{\omega_0^2}{2}P\int_0^\infty\frac{d\omega'}{(\omega'^2-\omega_0^2-i\gamma\omega')}-\frac{\omega_0^2}{2}P\int_0^\infty\frac{d\omega'}{(\omega'^2-\omega_0^2+i\gamma\omega')}\right.$$
$$\left.+\frac{(\omega_0^2-\omega^2)}{2i\gamma}P\int_0^\infty\frac{\omega'\,d\omega'}{(\omega'^2-\omega_0^2-i\gamma\omega')}-\frac{(\omega_0^2-\omega^2)}{2i\gamma}P\int_0^\infty\frac{\omega'\,d\omega'}{(\omega'^2-\omega_0^2+i\gamma\omega')}+\omega^2P\int_0^\infty\frac{d\omega}{(\omega'^2-\omega^2)}\right\}$$

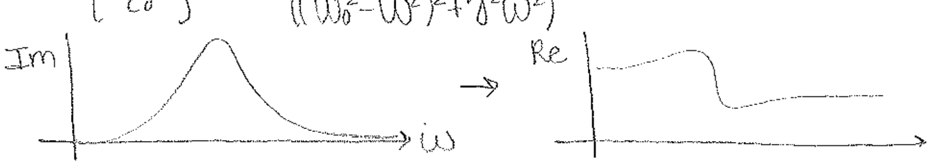
↓ plugging into Mathematica ↓

$$=1+\frac{2\lambda\gamma}{\pi}\frac{1}{((\omega_0^2-\omega^2)^2+\gamma^2\omega^2)}\left\{-\frac{\omega_0^2}{2}\left[\frac{1}{2\sqrt{\omega_0^2+i\gamma\omega'}}\ln\left(\frac{\omega'-\sqrt{\omega_0^2+i\gamma\omega'}}{\omega'+\sqrt{\omega_0^2+i\gamma\omega'}}\right)\right.\right.$$
$$\left.+\frac{1}{2\sqrt{\omega_0^2-i\gamma\omega'}}\ln\left(\frac{\omega'-\sqrt{\omega_0^2-i\gamma\omega'}}{\omega'+\sqrt{\omega_0^2-i\gamma\omega'}}\right)\right]_0^\infty$$
$$\left.+\frac{(\omega_0^2-\omega^2)}{2i\gamma}\left(\frac{1}{2}\right)\left[\ln(\omega'^2-\omega_0^2-i\gamma\omega')-\ln(\omega'^2-\omega_0^2+i\gamma\omega')\right]_0^\infty\right\}$$

recall: $\ln(z)=\ln(|z|)+i\,\mathrm{Arg}\{z\}$

$$=1+\frac{2\lambda\gamma}{\pi}\frac{1}{((\omega_0^2-\omega^2)^2+\gamma^2\omega^2)}\frac{(\omega_0^2-\omega^2)}{2i\gamma}\frac{1}{2}\left[i\left\{\pi+\arctan\left(\frac{-\gamma\omega'}{\omega'^2-\omega_0^2}\right)\right\}\right.$$
$$\left.-i\left\{-\pi+\arctan\left(\frac{\gamma\omega'}{\omega'^2-\omega_0^2}\right)\right\}\right]_0^\infty$$

$$\Rightarrow\mathrm{Re}\left\{\frac{\varepsilon(\omega)}{\varepsilon_0}\right\}=1+\lambda\frac{(\omega_0^2-\omega^2)}{((\omega_0^2-\omega^2)^2+\gamma^2\omega^2)}$$



3. Jackson Problem 7.26

(a). $\rho(\vec{x},t) = Q\delta(\vec{x}-(\vec{x}_0+\vec{v}t)) = \int \frac{d^3k\,d\omega}{(2\pi)^4} e^{i\vec{k}\cdot\vec{x}-i\omega t}\,\bar{\rho}(\vec{k},\omega)$

Where $Q = ze$ as defined & Jackson uses $\vec{q}$ here

$\bar{\rho}(\vec{q},\omega) = \frac{1}{(2\pi)^4}\int d^3x \int dt\, \exp[-i\vec{q}\cdot\vec{x} + i\omega t]\, ze\,\delta(\vec{x}-(\vec{x}_0-\vec{v}t))$

↓ apply δ-function → replace $\vec{x}$ with $(\vec{x}_0-\vec{v}t)$ ↓

$= \frac{1}{(2\pi)^4}\int d^3x \int dt\, \exp[-i\vec{q}\cdot(\vec{x}_0-\vec{v}t)+i\omega t]\, ze$

evaluating the $\int d^3x$ ↓

$\bar{\rho}(\vec{q},\omega) = \frac{1}{(2\pi)^4}\int dt\, \exp[-i\vec{q}\cdot\vec{x} - i(\vec{q}\cdot\vec{v}-\omega)t)\, ze$

$= \frac{e^{i\vec{q}\cdot\vec{x}_0}}{(2\pi)^4}\int dt\, e^{-i(\omega-\vec{q}\cdot\vec{v})t}\, ze$

$= \frac{ze}{(2\pi)^{\cancel{4}\,3}} e^{i\vec{q}\cdot\vec{x}_0}\cancel{(2\pi)}\,\delta(\omega-\vec{q}\cdot\vec{v})$

then, assuming $\vec{x}_0 = 0$, this yields the expected solution ↓

$\bar{\rho}(\vec{q},\omega) = \frac{ze}{(2\pi)^3}\,\delta(\omega-\vec{q}\cdot\vec{v})$ ✓

(b) If you assume there is a scalar potential...

→ start with Ampère's Law

$\nabla\times\vec{H} = \vec{J} + \frac{\partial}{\partial t}\vec{D}$

↓ taking the divergence ↓

$0 = \nabla\cdot\vec{J} + \frac{\partial}{\partial t}\nabla\cdot\vec{D}$

free current density

By the continuity equation

$\nabla\cdot\vec{J} = -\frac{\partial}{\partial t}\rho$

free charge density

$\rho = \bar{\rho}(\vec{q},\omega)$ where you may say in Fourier space ↓

$\bar{D}(\vec{q},\omega) = \varepsilon(\vec{q},\omega)\bar{E}(\vec{q},\omega)$

→ take the Fourier transform ↓

$\frac{\partial}{\partial t} \rightarrow i\omega$

$i\omega[\bar{\rho}(\vec{q},\omega) - i\vec{q}\cdot\bar{D}(\vec{q},\omega)] = 0$ ✓

→

$$i\omega[\bar{\rho}(\vec{q},\omega) - i\vec{q}\cdot\bar{D}(\vec{q},\omega)] = 0$$

plug in for $\bar{D}$, assuming quasistatic conditions:

$$\bar{E}(\vec{q},\omega) = -i\vec{q}\phi(\vec{q},\omega)$$

$$\bar{\rho}(\vec{q},\omega) = i\vec{q}\cdot\varepsilon(\vec{q},\omega)\bar{E}(\vec{q},\omega)$$
$$= i\vec{q}\cdot\varepsilon(\vec{q},\omega)[-i\vec{q}\phi(\vec{q},\omega)]$$
$$= q^2\cdot\varepsilon(\vec{q},\omega)\phi(\vec{q},\omega)$$

↓ solve for ϕ ↓

$$\phi(\vec{q},\omega) = \frac{\bar{\rho}(\vec{q},\omega)}{q^2\varepsilon(\vec{q},\omega)}$$

(c) Applicable equations:

$\bar{J}(\vec{q},\omega) = \sigma(\vec{q},\omega)\bar{E}(\vec{q},\omega)$ (Ohm's Law still valid under transform)

$\hookrightarrow \sigma(\vec{q},\omega) = i\omega[\varepsilon_0 - \varepsilon(\vec{q},\omega)]$

and from above

$\bar{E}(\vec{q},\omega) = -i\vec{q}\phi(\vec{q},\omega)$

$\hookrightarrow \phi(\vec{q},\omega) = \dfrac{\bar{\rho}(\vec{q},\omega)}{q^2\varepsilon(\vec{q},\omega)}$ $\qquad \bar{\rho}(\vec{q},\omega) = \dfrac{Ze}{(2\pi)^3}\delta(\omega - \vec{q}\cdot\vec{v})$

$\int d^3x \int dt \rightarrow 2\pi \int d^3q \; 2\pi \int d\omega$

Parcival's Theorem: the sum (or integral) of the square of a function is equal to the sum (or integral) of the square of its transformation.

$$\frac{dW}{dt} = \int \vec{J}\cdot\vec{E}\, d^3x$$

$\hookrightarrow \vec{J}(\vec{x},t) = \sigma(\vec{x},t)\vec{E}(\vec{x},t)$

$$= \int [\sigma(\vec{x},t)\vec{E}(\vec{x},t)]\cdot\vec{E}(\vec{x},t)\, d^3x$$
$$= \int \sigma(\vec{x},t)\underbrace{E^2(\vec{x},t)}\, d^3x$$

We know from Parcival's Theorem that it is equivalent to say ↴

implied from dt on left hand side

$$= (2\pi)^2 \iint \sigma(\vec{q},\omega)\bar{E}^2(\vec{q},\omega)\, d^3q\, d\omega$$
$$= (2\pi)^2 \iint (i\omega[\varepsilon_0 - \varepsilon(\vec{q},\omega)])[-i\vec{q}\phi(\vec{q},\omega)]^2\, d^3q\, dt$$
$$= (2\pi)^2 \iint (i\omega[\varepsilon_0 - \varepsilon(\vec{q},\omega)])[-q^2\phi^2(\vec{q},\omega)]\, d^3q\, dt$$

→

$$-\frac{dW}{dt} = (2\pi)^2 \iint \left(i\omega[\varepsilon_0 - \varepsilon(\vec{q},t)]\right) q^2 \left(\frac{\rho(\vec{q},\omega)}{q^2\,\varepsilon(\vec{q},\omega)}\right)^2 d^3q\, d\omega$$

$$= (2\pi)^2 \iint \left(i\omega[\varepsilon_0 - \varepsilon(\vec{q},t)]\right) \cancel{q^2} \frac{1}{q^{\cancel{4}\,2}\,\varepsilon^2(\vec{q},\omega)} \left[\frac{Ze}{(2\pi)^3}\,\delta(\omega - \vec{q}\cdot\vec{v})\right]^2 d^3q\, d\omega$$

$$= \frac{Z^2e^2}{(2\pi)^3} \int \frac{d^3q}{q^2} \int_{-\infty}^{\infty} d\omega\, \frac{i\omega[\varepsilon_0 - \varepsilon(\vec{q},t)]}{\varepsilon^2(\vec{q},\omega)} \left(\delta(\omega - \vec{q}\cdot\vec{v})\right)^2$$

$$= \frac{Z^2e^2}{8\pi^3} \int \frac{d^3q}{q^2} \int_{-\infty}^{\infty} d\omega\, \omega\, \underbrace{\frac{i[\varepsilon_0 - \varepsilon(\vec{q},t)]}{\varepsilon^2(\vec{q},\omega)}\, \delta(\omega - \vec{q}\cdot\vec{v})}\, \delta(\omega - \vec{q}\cdot\vec{v})$$

dividing by ε^2 and applying the delta function leaves only the imaginary component $\mathrm{Im}\{(\varepsilon(\vec{q},\omega))^{-1}\}$

$$= \frac{Z^2e^2}{8\pi^3} \int \frac{d^3q}{q^2} \underbrace{\int_{-\infty}^{\infty}} d\omega\, \omega\, \mathrm{Im}\left\{\frac{1}{\varepsilon(\vec{q},\omega)}\right\} \delta(\omega - \vec{q}\cdot\vec{v})$$

this integral is symmetric about ω, so we may convert $\int_{-\infty}^{\infty} \to 2\int_0^{\infty}$

$$-\frac{dW}{dt} = \frac{Z^2e^2}{4\pi^3} \int \frac{d^3q}{q^2} \int_0^{\infty} d\omega\, \omega\, \mathrm{Im}\left\{\frac{1}{\varepsilon(\vec{q},\omega)}\right\} \delta(\omega - \vec{q}\cdot\vec{v}) \quad \checkmark$$

4. Jackson Problem 7.28

Finite extent in x & y ⇒ the wave is not really a plane wave

→ Building off of Jackson eq. (7.21), assuming that the non-plane nature of the wave will leave some small longitudinal portion in the polarization of $\vec{E}$ (and as we will see, $\vec{B}$) ↓

$$\vec{E}(x,y,z,t) = [\underbrace{E_0(x,y)(\hat{e}_1 \pm i\hat{e}_2)}_{\text{circular polarization component}} + \overbrace{L(x,y)\hat{e}_3}^{\text{longitudinal polarization component}}]e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

$\vec{k}\cdot\vec{x} = k\hat{z}\cdot\vec{x} = kz$

Even though this field is not perfectly circularly polarized, it must still satisfy Maxwell's equations

– find $L(x,y)$ value such that $\nabla\cdot\vec{E}=0$ is satisfied

$$\nabla\cdot\vec{E}(x,y,z,t) = 0 = \left[\frac{\partial E_0(x,y)}{\partial x} \pm i\frac{\partial E_0(x,y)}{\partial y} + L(x,y)(ik)\right]e^{i(kz-\omega t)}$$

$$-ikL(x,y) = \frac{\partial E_0(x,y)}{\partial x} \pm i\frac{\partial E_0(x,y)}{\partial y}$$

$$L(x,y) = \frac{i}{k}\left[\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y}\right]$$

↓ plugging this back in yields the expected expression for $\vec{E}$ ↓

$$\vec{E}(x,y,z,t) = \left[E_0(x,y)(\hat{e}_1 \pm i\hat{e}_2) + \frac{i}{k}\left(\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y}\right)\hat{e}_3\right]e^{i(kz-\omega t)}$$

You can apply another of Maxwell's laws to determine $\vec{B}$:

$$-\frac{\partial\vec{B}}{\partial t} = \nabla\times\vec{E} = \nabla\times\left[E_0(x,y)(\hat{e}_1 \pm i\hat{e}_2) + \frac{i}{k}\left(\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y}\right)\hat{e}_3\right]e^{i(kz-\omega t)}$$

*GIVEN: the amplitude modulation is slowly varying

⇒ second derivative terms, such as $\partial^2 E_0/\partial x^2$, are sufficiently small that they may be ignored

$$-\frac{\partial\vec{B}}{\partial t} \simeq \nabla\times\left[E_0(x,y)(\hat{e}_1 \pm i\hat{e}_2)e^{i(kz-\omega t)}\right]$$

↓ performing the cross product & integrating the exponential ↓

(where $\vec{E}_1 = \vec{\varepsilon}_1 E_1 e^{ikz-i\omega t}$, $\vec{E}_2 = \vec{\varepsilon}_2 E_2 e^{ikz-i\omega t}$)

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$$\vec{B} \simeq \frac{-i}{\omega}\left[\underbrace{\frac{-\partial E_2}{\partial z}}_{}\hat{e}_1 + \frac{\partial E_1}{\partial z}\hat{e}_2 + \left(\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y}\right)\hat{e}_3\right]$$

replace $\frac{\partial}{\partial z} \to ik$

$$\simeq \frac{-i}{\omega}\left[\mp i E_0(x,y)(ik)\hat{e}_1 + E_0(x,y)(ik)\hat{e}_2 + \left(\pm i\frac{\partial E_0}{\partial x} - \frac{\partial E_0}{\partial y}\right)\hat{e}_3\right]$$

$$\simeq \frac{-i}{\omega}(\pm k)\left[\underbrace{E_0(x,y)(\hat{e}_1 \pm i\hat{e}_2) + \frac{i}{k}\left(\frac{\partial E_0}{\partial x} \pm i\frac{\partial E_0}{\partial y}\right)\hat{e}_3}_{\vec{E}(x,y,z,t)}\right]$$

$$\simeq \mp i\frac{k}{\omega}\vec{E}(x,y,z,t)$$

$k^2 = \omega^2\mu\varepsilon \to k = \omega\sqrt{\mu\varepsilon}$

$\vec{B} \simeq \mp i\sqrt{\mu\varepsilon}\,\vec{E}$, as expected ✓

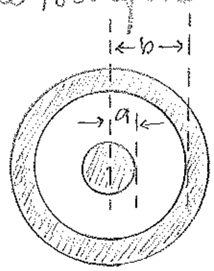
5. Jackson Problem 8.2

A transmission line consisting of two concentric circular cylinders of metal with conductivity σ and skin depth δ, as shown, is filled with uniform lossless dielectric (μ, ε). A TEM mode is propagated along this line. Section 8.1 applies.

(a) Show that the time-averaged power flow along the line is

$$P = \sqrt{\frac{\mu}{\varepsilon}}\, \pi a^2 |H_0|^2 \ln\left(\frac{b}{a}\right)$$

where H_0 is the peak value of the azimuthal magnetic field at the surface of the inner conductor.



(b) Show that the transmitted power is attenuated along the line as

$$P(z) = P_0 e^{-2\gamma z}$$

where

$$\gamma = \frac{1}{2\sigma\delta}\sqrt{\frac{\varepsilon}{\mu}}\,\frac{\left(\frac{1}{a}+\frac{1}{b}\right)}{\ln(b/a)}$$

(c) The characteristic impedance Z_0 of the line is defined as the ratio of the voltage between the cylinders to the axial current flowing in one of them at any position z. Show that for this line

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}\ln\left(\frac{b}{a}\right)$$

(d) Show that the series resistance and inductance per unit length of the line are

$$R = \frac{1}{2\pi\sigma\delta}\left(\frac{1}{a}+\frac{1}{b}\right)$$

$$L = \left\{\frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right) + \frac{\mu_c\delta}{4\pi}\left(\frac{1}{a}+\frac{1}{b}\right)\right\}$$

where μ_c is the permeability of the conductor. The correction

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to the inductance comes from the penetration of the flux into the conductors by a distance of order δ.

Engel-14

(a) ~~Power is given by the time average of the Poynting flux~~

~~$\langle S \rangle_{\hat{z}} = P$~~

~~For a plane polarization~~ moved to later in problem

~~$\langle S \rangle_{\hat{z}} = \frac{1}{2} \mathrm{Re}\{ \frac{1}{Z_0} |E_0|^2 \}$~~

~~$Z_0 = \sqrt{\mu/\varepsilon}$~~

* By definition, a TEM mode is a single frequency wave component w/ both the electric and magnetic fields transverse to the direction of propagation, which is the waveguide's axis

→ problem is 2D and pseudostatic

→ $\vec{B} = \sqrt{\mu\varepsilon}\, \hat{z} \times \vec{E}$ still holds

→ Begin by treating the inner cylinder as a line charge w/ charge/unit length λ

Gauss' law about λ gives us

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon\rho}\hat{\rho}$$

$$\vec{B} = \sqrt{\mu\varepsilon}\, \hat{z} \times \vec{E}$$

$$= \sqrt{\mu\varepsilon}\frac{\lambda}{2\pi\varepsilon\rho}\hat{\varphi} = \sqrt{\frac{\mu}{\varepsilon}}\frac{\lambda}{2\pi\rho}\hat{\varphi}$$

GIVEN

→ ~~[illegible]~~ : There is a peak magnetic field @ the surface of the inner conductor w/ defined value H_0

$$H(\rho = a) = \frac{1}{\sqrt{\mu\varepsilon}}\frac{\lambda}{2\pi a}$$

$$= \frac{1}{\mu}\vec{B}\cdot\hat{\varphi}$$

$$\lambda = 2\pi a\sqrt{\mu\varepsilon}\, H_0$$

↓ plugging in ↓

$$\vec{E} = \frac{2\pi a\sqrt{\mu\varepsilon}\, H_0}{2\pi\varepsilon\rho}\hat{\rho} = \sqrt{\frac{\mu}{\varepsilon}}\frac{aH_0}{\rho}\hat{\rho}$$

$$\vec{B} = \sqrt{\frac{\mu}{\varepsilon}}\frac{2\pi a\sqrt{\mu\varepsilon}\, H_0}{2\pi\rho}\hat{\varphi} = \frac{\mu a H_0}{\rho}\hat{\varphi}$$

} the static solution

add exponential time-dependence

→

direction of prop.

$$\vec{E} = \sqrt{\frac{\mu}{\varepsilon}}\, H_0(z)\frac{a}{\rho}\hat{\rho}\, e^{ikz-i\omega t}$$

$$\vec{B} = \mu H_0(z)\frac{a}{\rho}\hat{\varphi}\, e^{ikz-i\omega t}$$

→ Power is given by the time-average of the Poynting Flux over all space

Poynting Flux:

$$\vec{S} = \vec{E}\times\vec{H}$$

$$= \cancel{\frac{1}{\mu}}\left[\sqrt{\frac{\mu}{\varepsilon}}|H_0(z)|\frac{a}{\rho}\hat{\rho}\cos(kz-\omega t+Arg\{H_0\})\right]$$

$$\times\left[\cancel{\mu}|H_0(z)|\frac{a}{\rho}\hat{\varphi}\cos(kz-\omega t+Arg\{H_0\})\right]$$

$$= \sqrt{\frac{\mu}{\varepsilon}}|H_0(z)|^2\frac{a^2}{\rho^2}\cos^2(kz-\omega t+Arg\{H_0\})\hat{z}$$

Time-Averaged Poynting Flux:

$$\langle S\rangle = \left\langle\sqrt{\frac{\mu}{\varepsilon}}|H_0(z)|^2\frac{a^2}{\rho^2}\cos^2(kz-\omega t+Arg\{H_0\})\hat{z}\right\rangle$$

$$= \sqrt{\frac{\mu}{\varepsilon}}|H_0(z)|^2\frac{a^2}{\rho^2}\underbrace{\langle\cos^2(kz-\omega t+Arg\{H_0\})\rangle}_{=1/2}\hat{z}$$

$$\langle S\rangle = \sqrt{\frac{\mu}{\varepsilon}}|H_0(z)|^2\frac{a^2}{2\rho^2}\hat{z}$$

And finally, the time-averaged power flow along the line:

$$P = \int_A \hat{z}\cdot\langle S\rangle da$$

$$= \int_A \hat{z}\cdot\sqrt{\frac{\mu}{\varepsilon}}|H_0(z)|^2\frac{a^2}{2\rho^2}\hat{z}\,dA = \int_A\sqrt{\frac{\mu}{\varepsilon}}|H_0(z)|^2\frac{a^2}{2\rho^2}dA$$

$$\int_A dA = \int_0^{2\pi}\int_a^b \rho\, d\rho\, d\varphi$$

$$= \int_0^{2\pi}\int_a^b\frac{1}{2}\sqrt{\frac{\mu}{\varepsilon}}|H_0(z)|^2\frac{a^2}{\rho^{\cancel{2}}}\cancel{\rho}\,d\rho\, d\varphi$$

$$P = \sqrt{\frac{\mu}{\varepsilon}}\pi a^2|H_0|^2\ln\left(\frac{b}{a}\right)\ \checkmark$$

(b) Within the assumptions of a good conductor as given in section (8.1), we may begin with

$$\frac{dP_{loss}}{dA} = -\frac{1}{2}\mathrm{Re}\{\hat{n}\cdot\vec{E}\times\vec{H}^*\} = \frac{\mu_c \omega\delta}{4}|\vec{H}_{\parallel}|^2$$

where μ_c = permeability & δ = skin depth

→ focus on the azimuthal component

$$\frac{dP_{loss}}{dz} = \frac{\mu_c\omega\delta}{4}\underbrace{\int_0^{2\pi}|H_{\parallel}|^2\rho\, d\varphi}$$

must consider contributions from both boundary @ $\rho = a$ and @ $\rho = b$

$$= \frac{\pi\mu_c\omega\delta}{2}\left(|H_{\parallel}(a)|^2 a + |H_{\parallel}(b)|^2 b\right)$$

↓ plugging in the magnetic field from (a) ↓

$$\underbrace{\frac{dP_{loss}}{dz}} = \frac{\pi}{\sigma\delta}|H_0|^2 a\left(1+\frac{a}{b}\right)$$

the rate @ which energy is absorbed

rate @ which power decreases when prop. down

$$\frac{dP}{dz} = -\frac{dP_{loss}}{dz}$$ from Jackson eq. (8.58)

If we assume a linear relationship here, ?

$$-\frac{dP}{dz} = 2\gamma P$$

└ proportionality constant; chosen such that an acceptable solution for this eq. is the expected result:

$$\Rightarrow P(z) = P_0 e^{-2\gamma z}$$

But what is γ?

$$\gamma = -\frac{1}{2P}\frac{dP}{dz} = \frac{1}{2P}\frac{dP_{loss}}{dz}$$

$$= \left[2\sqrt{\frac{\mu}{\varepsilon}}\,\cancel{\pi} a^2 \cancel{|H_0|^2}\ln\left(\frac{b}{a}\right)\right]^{-1}\left[\frac{\cancel{\pi}}{\sigma\delta}\cancel{|H_0|^2}\cancel{a}\left(1+\frac{a}{b}\right)\right]$$

$$\gamma = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\frac{1}{\sigma\delta}\frac{\left(\frac{1}{a}+\frac{1}{b}\right)}{\ln(b/a)}$$ ✓

(c) $Z_0 = \frac{|\Delta\phi|}{I_a}$ $\frac{\text{potential difference}}{\text{current @ A}}$ = Ohm's law

$= \frac{{}_a\int^b \vec{E}\cdot\hat{\rho}\,d\rho}{I_a}$

$= \frac{1}{I_a}\sqrt{\frac{\mu}{\varepsilon}}H_0 a \,{}_a\int^b \frac{d\rho}{\rho}$

$Z_0 = \frac{a}{I_a}\sqrt{\frac{\mu}{\varepsilon}}H_0 \ln\left(\frac{b}{a}\right)$ ✓

→ Again from section (8.1), for a good conductor, the skin depth is so small that current flowing in the skin depth may be treated as an effective surface current, K_{eff} ↓

$K_{eff} = \hat{n}\times\vec{H}$ Jackson eq. (8.14)

$= \hat{\rho}\times\left[H_0\frac{a}{\rho}\hat{\varphi}\right]_{\rho=a}$

$= H_0\hat{z}$

following this,

$I_a = {}_0\int^{2\pi}{}_0\int^a K_{eff}\cdot\hat{z}\,\rho\,d\rho\,d\varphi$

$= 2\pi a H_0$

↓ plugging in ↓

$Z_0 = \frac{\cancel{aH_0}}{2\pi \cancel{aH_0}}\sqrt{\frac{\mu}{\varepsilon}}\ln\left(\frac{b}{a}\right) = \frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}\ln\left(\frac{b}{a}\right)$ ✓

(d) From Ohm's law, $V=IR \rightarrow P=IV \rightarrow P=I^2R$; we can say

$P_{loss} = I_{rms}^2 R$

where for a harmonic wave, $I_{rms} = I/\sqrt{2}$

$= \frac{1}{2}I^2 R$

→ must consider "per unit length" ↓

$dP_{loss} = \frac{1}{2}I^2 dR$

$\frac{dP_{loss}}{dz} = \frac{1}{2}I^2 \frac{dR}{dz}$ ← = resistance per unit length = R per problem definition

$R = \frac{2}{I^2}\frac{dP_{loss}}{dz}$

↓plugging in from (b) & (c)↓

$$R=\frac{2}{(2\pi a H_0)^2}\left[\frac{\pi}{\sigma\delta}|H_0|^2 a\left(1+\frac{a}{b}\right)\right]$$

$$\Rightarrow R=\frac{1}{2\pi\sigma\delta}\left(\frac{1}{a}+\frac{1}{b}\right)$$

Self-inductance: a device's ability to store energy in its magnetic field

$W=\frac{1}{2}LI_{rms}^2$ ← from Cheng's "Electromagnetics"

↑ total energy stored

$W=\frac{1}{4}LI^2$

$L=\frac{4}{I^2}W$ ← energy may be stored in any of the three regions $a>\rho$, $a<\rho<b$, $b<\rho$

$$L=\frac{4}{I^2}\left(W_{a>\rho}+W_{a<\rho<b}+W_{b<\rho}\right)$$

where $W=\frac{1}{4\mu}\int|\vec{B}|^2dA$ applies for all three regions

$$=\frac{4}{I^2}\left(\frac{1}{4\mu_c}\int|\vec{B}_{a>\rho}|^2dA+\frac{1}{4\mu}\int|\vec{B}_{a<\rho<b}|^2dA+\frac{1}{4\mu_c}\int|\vec{B}_{b<\rho}|^2dA\right)$$

↑ as in (a), $\int_A dA=\int_0^{2\pi}\int_a^b\rho\,d\rho\,d\varphi$

$$=\frac{4}{I^2}\left(\frac{1}{4\mu_c}\int_0^{2\pi}\int_0^a|\vec{B}_1|^2\rho\,d\rho\,d\varphi+\frac{1}{4\mu}\int_0^{2\pi}\int_a^b|\vec{B}_2|^2\rho\,d\rho\,d\varphi+\frac{1}{4\mu_c}\int_0^{2\pi}\int_b^\infty|\vec{B}_3|^2\rho\,d\rho\,d\varphi\right)$$

↓plugging in for $|\vec{B}|$'s↓

$$=\frac{2\pi}{I^2}|H_0|^2\left(\mu_0\int_0^a e^{-2(a-\rho)/\delta}\rho\,d\rho+\mu a^2\int_a^b\frac{1}{\rho}d\rho+\mu_c\frac{a^2}{b^2}\int_b^\infty e^{-2(\rho-b)/\delta}\rho\,d\rho\right)$$

$$=\frac{2\pi}{I^2}|H_0|^2\left(\mu_0 e^{-2a/\delta}\int_0^a e^{2\rho/\delta}\rho\,d\rho+\mu a^2\ln\left(\frac{b}{a}\right)+\mu_c\frac{a^2}{b^2}e^{2b/\delta}\int_b^\infty e^{-2\rho/\delta}\rho\,d\rho\right)$$

↓evaluating and plugging in for I↓

$$=\frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)+\frac{\mu_c\delta}{4\pi}\left[\left(\frac{1}{a}+\frac{1}{b}\right)+\frac{1}{a^2}e^{-2a/\delta}\frac{\delta}{2}\right]$$

↑ $a/\delta\gg1$; this term → 0

$$\Rightarrow L=\frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)+\frac{\mu_c\delta}{4\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$$

✓

6. Jackson Problem 8.4

Transverse electric and magnetic waves are propagated along a hollow, right circular cylinder with inner radius R and conductivity σ.

(a). Find the cutoff frequencies of the various TE and TM modes. Determine numerically the lowest cutoff frequency (the dominant mode) in terms of the tube radius and the ratio of cutoff frequencies of the next four higher modes to that of the dominant mode. For this part, assume that the conductivity of the cylinder is infinite.

(b) Calculate the attenuation constants of the waveguide as a function of frequency for the lowest two distinct modes and plot them as a function of frequency.

(a) For TE Modes:

$$\nabla^2 B_z - \frac{1}{c^2}\frac{\partial^2 B_z}{\partial t^2} = 0$$

→ the waves are free along the axis of propagation
↳ harmonic dependencies ↓

$$B_z(x,y,z,t) = B_z(x,y)e^{ikz-i\omega t}$$

↓ plugging into the wave eqn ↓

$$\nabla_t^2 B_z = -K^2 B_z$$

↳ $K^2 = \omega^2/c^2 - k^2$

solution of the form: (circular symmetry → polar coords)

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial B_z}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 B_z}{\partial\varphi^2} = -K^2 B_z$$

→ ansatz: $B_z = R(K\rho)e^{im\varphi}$, m = integer required for single-valued

$$\rho^2\frac{\partial^2 R(K\rho)}{\partial\rho^2} + \rho\frac{\partial R(K\rho)}{\partial\rho} + (K^2\rho^2 - m^2)R(K\rho) = 0$$

The Bessel function!

solutions → $R(x) = AJ_m(x) + BN_m(x)$

for $x \equiv K\rho$

*Note: because our cylinder contains the origin, our solution can not be dependent on $N_m(x)$ as these functions are not well-behaved as $x \to 0$

$\rightarrow B_z = A J_m(K\rho)\, e^{i(kz - \omega t + m\varphi)}$

Boundary condition:

$\left[\frac{\partial B_z}{\partial \rho}\right]_{\rho=R} = 0$ (walls of waveguide are perfect conductors)

$\hookrightarrow \left[\frac{\partial}{\partial \rho}\left(J_m(K\rho)\right)\right]_{\rho=R} = 0$

$\rightarrow K = \frac{X'_{mn}}{R} \leftarrow$ n^{th} zero of $J'_m(x)$

$$\Rightarrow B_z = A_{mn} J_m\left(X'_{mn}\frac{\rho}{R}\right) e^{i(kz - \omega t + m\varphi)}$$

$$k = \sqrt{\frac{\omega^2}{c^2} - \frac{X'^2_{mn}}{R^2}}$$

<u>For TM Modes:</u>

Similarly ↓

$E_z = A J_m(K\rho)\, e^{i(kz - \omega t + m\varphi)}$

$k = \sqrt{\frac{\omega^2}{c^2} - K^2}$

Boundary condition:

$E_z(\rho = R) = 0$

$\hookrightarrow J_m(KR) = 0$

$\rightarrow K = \frac{X_{mn}}{R} \leftarrow$ n^{th} zero of $J_{mn}(X)$

$$\Rightarrow E_z = A_{mn} J_m\left(X_{mn}\frac{\rho}{R}\right) e^{i(kz - \omega t + m\varphi)}$$

$$k = \sqrt{\frac{\omega^2}{c^2} - \frac{X^2_{mn}}{R^2}}$$

<u>For both modes:</u>

cutoff frequency, ω_c, corresponds w/ $k = 0$

<u>TE</u> $\omega_{mn} = c\frac{X'_{mn}}{R}$, <u>TM</u> $\omega_{mn} = c\frac{X_{mn}}{R}$

$\rightarrow$ looking up the Bessel function zeroes ↓

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TE mode

X'_{mn}	$m=0$	1	2
$n=1$	—	1.84	3.05
2	3.83	5.33	6.71

TM mode

X_{mn}	$m=0$	1	2
$n=1$	2.40	3.83	5.14
2	5.52	7.02	8.42

The dominant mode: The lowest zero crossing gives the dominant mode (lowest cutoff freq.). This corresponds to TE_{11}

mode	ω_c
* TE_{11}	$1.84\,c/R$
TM_{01}	$2.40\,c/R$
TE_{21}	$3.05\,c/R$
TE_{02}/TM_{11}	$3.83\,c/R$
TM_{21}	$5.14\,c/R$

mode	$\cancel{TE_{11}}$
TE_{11}	1
TM_{01}	1.30
TE_{21}	1.66
TE_{02}/TM_{11}	2.08
TM_{21}	2.79

(b) <u>For TM Modes:</u>

$$P=\frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\left(\frac{\omega}{\omega_{mn}}\right)^2\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{1/2}\underbrace{\int_A |J_m(k\rho)e^{im\varphi}|^2 dA}$$

$$\int_A |J_m(k\rho)e^{im\varphi}|^2 dA = 2\pi \int_0^R J_m(X_{mn}\,\rho/R)^2\rho\, d\rho$$

then, by the bessel function orthogonality relation

$$\int_0^a J_\nu(X_{\nu m}\,\rho/a)J_\nu(X_{\nu n}\,\rho/a)\rho\, d\rho = \frac{1}{2}a^2 J_{\nu+1}(X_{\nu m})^2\,\delta_{mn}$$

$$=2\pi\left[\frac{1}{2}R^2 J_{m+1}(X_{mn})^2\right.$$

$$=\pi R^2 J_{m+1}(X_{mn})^2$$

$$=\frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\left(\frac{\omega}{\omega_{mn}}\right)^2\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{1/2}\pi R^2 J_{m+1}(X_{mn})^2$$

with power loss being given by

$$-\frac{dP}{dz}=\frac{1}{2\sigma\delta}\frac{\varepsilon}{\mu}(2\pi R)J_{m+1}(X_{mn})^2$$

The attenuation coefficient β_{mn} can then be found to be

$$\beta_{mn}=-\frac{1}{2P}\frac{dP}{dz}$$

$$=\frac{1}{2\sigma\delta}\sqrt{\frac{\varepsilon}{\mu}}\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{-1/2}\frac{2\pi R}{\pi R^2}$$

$$\Rightarrow \beta_{mn} = \frac{1}{\sigma\delta}\sqrt{\frac{\varepsilon}{\mu}}\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{-1/2}\frac{1}{R}$$

For TE Modes:

Similar to the power expression for TM modes, the Bessel normalization integral may be computed separately

$$\int_0^R J_m(x'_{mn}\,\rho/R)^2\,\rho\,d\rho = \tfrac{1}{2}R^2\left(1-m^2/x'^2_{mn}\right)J_m(x'_{mn})^2$$

yielding

$$P = \frac{1}{2}\sqrt{\frac{\mu}{\varepsilon}}\left(\frac{\omega}{\omega_{mn}}\right)^2\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{1/2}\pi R^2\left(1-\frac{m^2}{x'^2_{mn}}\right)J_m(x'_{mn})^2$$

With a power loss expression is

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2\oint_c\left[\frac{1}{k_{mn}^2}\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)\left|\hat{n}\times\nabla_t\left(J_m(k\rho)e^{im\varphi}\right)\right|^2\right.$$

$$\left.+\underbrace{\frac{\omega_{mn}^2}{\omega^2}\left|J_m(k\rho)e^{im\varphi}\right|^2\right]dl}_{}$$

$$\oint_c\left|J_m(k\rho)e^{im\varphi}\right|^2 dl = (2\pi R)\,J_m(x'_{mn})^2$$

and

$$\oint_c\left|\hat{n}\times\nabla_t\left(J_m(k\rho)e^{im\varphi}\right)\right|^2 dl = \oint_c\left|-\hat{\rho}\times\nabla_t\left(J_m(k\rho)e^{im\varphi}\right)\right|^2 dl$$

$$=(2\pi R)\left|\frac{1}{\rho}\frac{\partial\left(J_m(k\rho)e^{im\varphi}\right)}{\partial\varphi}\right|^2 = (2\pi \cancel{R})\frac{m^2}{R^{\cancel{2}}}J_m(x'_{mn})^2$$

↓ combining terms ↓

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2(2\pi R)\left[\frac{m^2}{x'^2_{mn}}\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)+\frac{\omega_{mn}^2}{\omega^2}\right]J_m(x'_{mn})^2$$

The attenuation constant β_{mn} can then be found to be

$$\beta_{mn} = -\frac{1}{2P}\frac{dP}{dz}$$

$$= \frac{1}{\cancel{2}\sigma\delta}\sqrt{\frac{\varepsilon}{\mu}}\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{-1/2}\frac{\cancel{2\pi R}}{\cancel{\pi} R^{\cancel{2}}}\left[\frac{m^2}{x'^2_{mn}}\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)+\frac{\omega_{mn}^2}{\omega^2}\right]\left[1-\frac{m^2}{x'^2_{mn}}\right]$$

$$= \frac{1}{\sigma\delta}\sqrt{\frac{\varepsilon}{\mu}}\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{-1/2}\frac{1}{R}\left[\frac{m^2}{x'^2_{mn}-m^2}+\frac{\omega_{mn}^2}{\omega^2}\right]$$

Plot follows →

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