Physics 606: Homework #5

Due April 5, 2016

Jackson: Problems 6.12, 6.14, 7.14, 7.16

Show for an isotropic medium with a local response (Viz. $\partial\varepsilon / \partial k = \partial\mu / \partial k = 0$) that the ratio of the time averaged Poynting flux to the time averaged energy density for a plane wave is the group velocity.

Download and familiarize yourself with the paper labeled "PhysRev.125.pdf" in HNDO. It was written by a young researcher with a promising career in plasma physics. But then he just dissapeared from the scene. Don't let that happen to you.

10 ☺

Homework #5

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1. Jackson Problem 6.12

(a) Starting with Jackson eq. (6.134):

$$\frac{1}{2}\int_V \vec{J}^*\cdot\vec{E}\,d^3x + 2i\omega\int_V (w_e - w_m)\,d^3x + \underbrace{\oint_S \vec{S}\cdot\hat{n}\,dA}_{\frac{1}{2}I_i^* V_i = -\oint_{S_i}\vec{S}\cdot\hat{n}\,dA} = 0$$

$$\frac{1}{2}I_i^* V_i = \frac{1}{2}\int_V \vec{J}^*\cdot\vec{E}\,d^3x + 2i\omega\int_V (w_e - w_m)\,d^3x + \oint_{S-S_i}\vec{S}\cdot\hat{n}\,dA$$

$I_i = YV_i \longrightarrow I_i^* = Y^* V_i^*$ (Ohm's Law)

$$Y^*|V_i|^2 = \int_V \vec{J}^*\cdot\vec{E}\,d^3x + 4i\omega\int_V (w_e - w_m)\,d^3x + 2\oint_{S-S_i}\vec{S}\cdot\hat{n}\,dA$$

$$Y^* = \frac{1}{|V_i|^2}\left(\int_V \vec{J}^*\cdot\vec{E}\,d^3x + 4i\omega\int_V (w_e - w_m)\,d^3x + 2\oint_{S-S_i}\vec{S}\cdot\hat{n}\,dA\right)$$

$\rightarrow G = \mathrm{Re}\{Y^*\}$, $B = \mathrm{Im}\{Y^*\}$

$$G = \frac{1}{|V_i|^2}\mathrm{Re}\left\{\int_V \vec{J}^*\cdot\vec{E}\,d^3x + 4i\omega\int_V (w_e - w_m)\,d^3x + 2\oint_{S-S_i}\vec{S}\cdot\hat{n}\,dA\right\}$$

$$= \frac{1}{|V_i|^2}\left[\mathrm{Re}\left\{\int_V \vec{J}^*\cdot\vec{E}\,d^3x + 2\oint_{S-S_i}\vec{S}\cdot\hat{n}\,dA\right\} - 4\omega\,\mathrm{Im}\left\{\int_V (w_e - w_m)\,d^3x\right\}\right]$$

$$B = \frac{1}{|V_i|^2}\mathrm{Im}\left\{\int_V \vec{J}^*\cdot\vec{E}\,d^3x + 4i\omega\int_V (w_e - w_m)\,d^3x + 2\oint_{S-S_i}\vec{S}\cdot\hat{n}\,dA\right\}$$

$$= \frac{1}{|V_i|^2}\left[\mathrm{Im}\left\{\int_V \vec{J}^*\cdot\vec{E}\,d^3x + 2\oint_{S-S_i}\vec{S}\cdot\hat{n}\,dA\right\} - 4\omega\,\mathrm{Re}\left\{\int_V (w_e - w_m)\,d^3x\right\}\right]$$

(b) At low frequencies:

$w_e = \frac{1}{4}(\vec{E}\cdot\vec{D}^*) =$ real, $w_m = \frac{1}{4}(\vec{B}\cdot\vec{H}^*) =$ real

$\vec{J} = \sigma\vec{E}$, σ real

$\rightarrow$ Ignoring the surface integral...

$$G = \frac{1}{|V_i|^2}\left[\mathrm{Re}\left\{\int_V (\sigma\vec{E})\cdot\vec{E}\,d^3x\right\} - 4\omega\,\cancel{\mathrm{Im}\left\{\int_V (w_e - w_m)\,d^3x\right\}}^{\rightarrow 0,\ \text{wholly real}}\right]$$

$$\simeq \frac{1}{|V_i|^2}\int_V \sigma|\vec{E}|^2\,d^3x$$

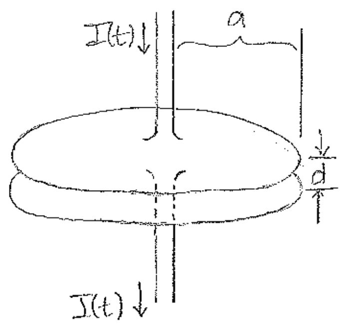
$$B = \frac{1}{|V_i|^2}\left[\cancel{\mathrm{Im}\left\{\int_V (\sigma\vec{E})\cdot\vec{E}\,d^3x\right\}}^{\rightarrow 0,\ \text{wholly real}} - 4\omega\,\mathrm{Re}\left\{\int_V (w_e - w_m)\,d^3x\right\}\right]$$

$$\simeq \frac{-4\omega}{|V_i|^2}\int_V (w_e - w_m)\,d^3x$$

✓

2. Jackson Problem 6.K1

An ideal circular parallel plate capacitor of radius a and plate separation $d \ll a$ is connected to a current source.



$$I(t) = I_0 \cos(\omega t) = I_0 e^{-i\omega t}$$

$$\nabla \cdot \vec{E} = \rho/\varepsilon \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = i\omega \vec{B} \qquad \nabla \times \vec{B} = \mu_0 \vec{J} - (i\omega/c^2)\vec{E}$$

(a) For $I(t) = I_0 e^{-i\omega t}$

$$q = \int I\,dt$$

$$= \frac{i}{\omega} I_0 \underbrace{e^{-i\omega t}}$$

for low frequency values, $\rightarrow 1$

$$q \approx iI_0/\omega$$

surface charge density $\sigma = q/A$

$$\sigma = \frac{q}{\pi a^2} = \frac{iI_0}{\omega \pi a^2}$$

lowest order term

$$E_z^{(0)} = \frac{-\sigma}{\varepsilon_0} = \frac{-iI_0}{\varepsilon_0 \pi a^2 \omega}$$

$$\nabla \times \vec{B}^{(1)} = \frac{-i\omega}{c^2} \vec{E}^{(0)}$$

$$= \frac{-\cancel{i}\cancel{\omega}}{c^2} \frac{-\cancel{i} I_0}{\cancel{\varepsilon_0} \pi a^2 \cancel{\omega}} \hat{z}$$

$$c = 1/\sqrt{\mu_0 \varepsilon_0} \rightarrow \frac{1}{c^2} = \mu_0 \varepsilon_0$$

$$= \frac{-\mu_0 I_0}{\pi a^2} \hat{z}$$

$$(\nabla \times \vec{B})_z^{(1)} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho B_\varphi^{(1)}$$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho B_\varphi^{(1)} = \frac{-\mu_0 I_0}{\pi a^2}$$

$$\rho B_\varphi^{(1)} = \int \frac{-\mu_0 I_0 \rho}{\pi a^2}\, d\rho$$

$\dashrightarrow$

$$\cancel{\rho} B_\varphi^{(1)} = \frac{-\mu_0 I_0 \rho^{\cancel{2}}}{2\pi a^2}$$

$$B_\varphi^{(1)} = \frac{-\mu_0 I_0}{2\pi a^2}$$

$$(\nabla \times \vec{E})_\varphi^{(2)} = i\omega B_\varphi^{(1)}$$

$$= \frac{-\mu_0 i\omega I_0 \rho}{2\pi a^2}$$

Use c^2 to convert back to ε_0 to match $\vec{E}$-field term

$$= \frac{-i\omega I_0 \rho}{2\pi a^2 \varepsilon_0 c^2}$$

$$(\nabla \times \vec{E})_\varphi^{(2)} = -\frac{\partial}{\partial \rho} E_z^{(2)}$$

$$E_z^{(2)} = \int \frac{i\omega I_0 \rho}{2\pi a^2 \varepsilon_0 c^2}\, d\rho$$

$$= \frac{i\omega I_0 \rho^2}{4\pi a^2 \varepsilon_0 c^2}$$

$$\nabla \times \vec{B}^{(3)} = \frac{-i\omega}{c^2} \vec{E}^{(2)}$$

$$= \frac{-\cancel{i}\omega}{\cancel{c^2}} \frac{\cancel{i}\omega I_0 \rho^2}{4\pi a^2 \cancel{\varepsilon_0} c^2} \mu_0$$

$$= \frac{\omega^2 \mu_0 I_0 \rho^2}{4\pi a^2 c^2} \hat{z}$$

$$(\nabla \times \vec{B})_z^{(3)} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho B_\varphi^{(3)}$$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho B_\varphi^{(3)} = \frac{\omega^2 \mu_0 I_0 \rho^2}{4\pi a^2 c^2}$$

$$\cancel{\rho} B_\varphi^{(3)} = \int \frac{\omega^2 \mu_0 I_0 \rho^3}{4\pi a^2 c^2}\, d\rho$$

$$= \frac{\omega^2 \mu_0 I_0 \rho^{\cancel{4}3}}{16\pi a^2 c^2}$$

$$B_\varphi^{(3)} = \frac{\omega^2 \mu_0 I_0 \rho^3}{16\pi a^2 c^2}$$

- Then, assembling terms...

$$E_z = \frac{-i I_0}{\varepsilon_0 \pi a^2 \omega} + \frac{i\omega I_0 \rho^2}{4\pi a^2 \varepsilon_0 c^2}$$

$$\vec{E} = \frac{-iI_0}{\varepsilon_0 \pi a^2 \omega}\left[1 - \frac{\rho^2\omega^2}{4c^2} + \dots\right]\hat{z}$$

$$B_\varphi = \frac{-\mu_0 I_0 \rho}{2\pi a^2} + \frac{\omega^2 \mu_0 I_0 \rho^3}{16\pi a^2 c^2}$$

$$\vec{B} = \frac{-\mu_0 I_0 \rho}{2\pi a^2}\left[1 - \frac{\omega^2\rho^2}{8c^2} + \dots\right]\hat{\varphi}$$

(b) $w_e = \frac{1}{4}\vec{E}\cdot\vec{D}^* = \frac{\varepsilon_0}{4}|\vec{E}|^2$

$$= \frac{|I_0|^2}{4\pi^2 a^4 \omega^2 \varepsilon_0}\left[1 - \frac{\rho^2\omega^2}{2c^2} + \dots\right]$$

$w_m = \frac{1}{4}\vec{B}\cdot\vec{H}^* = \frac{1}{4\mu_0}|\vec{B}|^2$

$$= \frac{\mu_0 |I_0|^2 \rho^2}{16\pi^2 a^4}\left[1 - \frac{\omega^2\rho^2}{4c^2} + \dots\right]$$

$$\int w_e\, d^3x = \int \frac{|I_0|^2}{4\pi^2 a^4 \omega^2 \varepsilon_0}\left(1 - \frac{\rho^2\omega^2}{2c^2} + \dots\right) d^3x$$

$$\int w_m\, d^3x = \int \frac{\mu_0 |I_0|^2 \rho^2}{16\pi^2 a^4}\left(1 - \frac{\omega^2\rho^2}{4c^2} + \dots\right) d^3x$$

Then, where the volume integral over the capacitor is

$$\int d^3x = 2\pi d \int_0^a \rho\, d\rho$$

$$\int w_e\, d^3x = \frac{\cancel{2\pi} d |I_0|^2}{{}_2\cancel{4}\pi^{\cancel{2}} a^4 \omega^2 \varepsilon_0}\int_0^a \rho\left(1 - \frac{\rho^2\omega^2}{2c^2} + \dots\right) d\rho$$

$$= \frac{|I_0|^2 d}{2\pi a^4 \omega^2}\,\frac{a^2}{2}\left(1 - \frac{a^2\omega^2}{4c^2} + \dots\right)$$

$$= \frac{|I_0|^2 d}{4\pi \varepsilon_0 a^2 \omega^2}\left(1 - \frac{a^2\omega^2}{4c^2} + \dots\right)$$

$$\int w_m\, d^3x = \frac{\mu_0 |I_0|^2}{{}_8\cancel{16}\pi^{\cancel{2}} a^4}\,\cancel{2\pi} d \int_0^a \rho^3\left(1 - \frac{\omega^2\rho^2}{4c^2} + \dots\right) d\rho$$

$$= \frac{\mu_0 |I_0|^2 d}{8\pi a^4}\,\frac{a^4}{4}\left(1 - \frac{a^2\omega^2}{16c^2} + \dots\right)$$

$$\simeq \frac{\mu_0 |I_0|^2 d}{32\pi}$$

✓

→ We must now obtain I_i with respect to I_0 ↴

$$I_i = -i\omega Q$$

$$Q = 2\pi \int_0^a \sigma(\rho)\rho\, d\rho$$

$$\sigma = \varepsilon_0(-E_z)$$

$$= \frac{iI_0}{\pi a^2\omega}\left(1 - \frac{\rho^2\omega^2}{4c^2} + \dots\right)$$

$$Q = 2\cancel{\pi}\int_0^a \frac{iI_0\rho}{\cancel{\pi} a^2\omega}\left(1-\frac{\rho^2\omega^2}{4c^2}+\dots\right)d\rho$$

$$= \frac{2iI_0}{a^2\omega}\int_0^a\left(\rho - \frac{\rho^3\omega^2}{4c^2}+\dots\right)d\rho$$

$$= \frac{\cancel{2}iI_0}{\cancel{a^2}\omega}\left(\frac{\cancel{a^2}}{\cancel{2}} - \frac{\cancel{a^4}^{\,2}\omega^2}{{}_8\cancel{16}c^2}+\dots\right)$$

$$= \frac{iI_0}{\omega}\left(1-\frac{a^2\omega^2}{8c^2}+\dots\right)$$

$$I_i = -i\omega Q = I_0\left(1-\frac{a^2\omega^2}{8c^2}+\dots\right)$$

$$|I_i|^2 = \underbrace{|I_0|^2\left(1-\frac{a^2\omega^2}{4c^2}+\dots\right)}_{\text{Substitute for this expression}}$$

$$\int w_e\, d^3x = \frac{|I_i|^2 d}{4\pi\varepsilon_0 a^2\omega^2}$$

$$\int w_m\, d^3x = \frac{\mu_0|I_i|^2 d}{32\pi}\left(1+\frac{\omega^2a^2}{12c^2}\right)$$

(c) From equation (6.138)

$$X \approx \frac{4\omega}{|I|^2}\int(w_m-w_e)\,d^3x = \frac{4\omega}{|I|^2}\left(\frac{\mu_0|I|^2d}{32\pi} - \frac{|I|^2 d}{4\pi\varepsilon_0a^2\omega^2}\right)$$

$$= \underbrace{\frac{\mu_0 d}{8\pi}\omega}_{X = L\omega} - \underbrace{\frac{d}{\pi\varepsilon_0a^2}\frac{1}{\omega}}_{X = 1/C\omega}$$

$$\Rightarrow L = \frac{\mu_0 d}{8\pi} \quad ; \quad C = \frac{\pi\varepsilon_0 a^2}{d}$$

The resonant frequency of an LC circuit:

$$\omega_{res} = \frac{1}{\sqrt{LC}}$$

$$= \left(\frac{\mu_0 \cancel{d}}{8\cancel{\pi}} \cdot \frac{\cancel{\pi}\varepsilon_0 a^2}{\cancel{d}} \right)^{-1/2}$$

$$= \sqrt{\frac{8}{\underbrace{\mu_0 \varepsilon_0}_{1/c^2} a^2}}$$

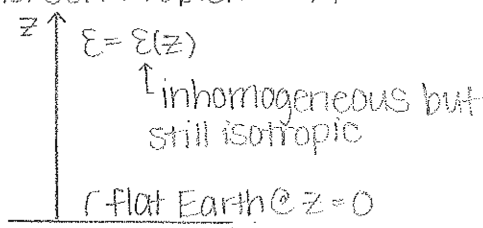
$$= \frac{2\sqrt{2}\,c}{a}$$

The expansion terms for $\vec{E}$ & $\vec{B}$ are actually $J_0(\omega\rho/c)$

$\rightarrow$ when $\rho = a$, $\vec{E}$ must vanish (no fringe effects)

$E(\rho = a) = 0$ for $\omega = 2.4$, which is slightly smaller than $\sqrt{8}$

3. Jackson Problem 7.14



* The fields are independent and can be written as functions of z times $e^{i(kx-\omega t)}$

$$\vec{E}(\vec{x},t) = f(z)\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$
$$\vec{H}(\vec{x},t) = f(z)\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

(a) $\nabla\times\hat{\vec{E}} = +i\omega\mu\hat{\vec{H}}$

$\nabla\times\hat{\vec{H}} = -i\omega\varepsilon\hat{\vec{E}}$

* For horizontal polarization:

- note → this method only works for this polarization direction

$$\frac{d^2F}{dz^2} + q^2F = 0 \qquad F = E_y$$
$$q^2(z) = \omega^2\mu_0\varepsilon(z) - k^2$$

$[\nabla^2 + \omega^2\mu_0\varepsilon(z)]\breve{\vec{E}} = 0$ · from Lecture 16 ·

$\nabla^2\breve{\vec{E}} + \omega^2\mu_0\varepsilon(z)\breve{\vec{E}} = 0$

$\breve{\vec{E}}(\vec{x}) = \vec{\mathcal{E}}\, e^{i\vec{k}\cdot\vec{x}}$

$$\nabla^2(\vec{\mathcal{E}}e^{i\vec{k}\cdot\vec{x}}) = \vec{\mathcal{E}}\underbrace{\nabla^2 e^{i\vec{k}\cdot\vec{x}}}_{=-k^2e^{i\vec{k}\cdot\vec{x}}} + e^{i\vec{k}\cdot\vec{x}}\nabla^2\vec{\mathcal{E}}$$

$$\cancel{e^{i\vec{k}\cdot\vec{x}}}\nabla^2\vec{\mathcal{E}} - k^2\cancel{e^{i\vec{k}\cdot\vec{x}}}\vec{\mathcal{E}} + \omega^2\mu_0\varepsilon(z)[\vec{\mathcal{E}}\cancel{e^{i\vec{k}\cdot\vec{x}}}] = 0$$

$\vec{\mathcal{E}} = E_y$

$$\frac{d^2E_y}{dz^2} + (\omega^2\mu_0\varepsilon(z) - k^2)E_y = 0 \checkmark$$

* For vertical polarization:

→ the more general method

$$\frac{d^2F}{dz^2} + q^2F = 0 \qquad F = \sqrt{\varepsilon/\varepsilon_0}\, E_z$$
$$q^2(z) = \omega^2\mu_0\varepsilon(z) + \frac{1}{2\varepsilon}\frac{d^2\varepsilon}{dz^2} - \frac{3}{4\varepsilon^2}\left(\frac{d\varepsilon}{dz}\right)^2 - k^2$$

- Start by writing out all components

$\nabla\times\vec{H} = \frac{\partial}{\partial t}\varepsilon\vec{E}$

$\nabla\times\vec{E} = -\frac{\partial}{\partial t}\mu\vec{H}$

where $\vec{H},\vec{E} = \mathrm{Re}\{\hat{\vec{H}},\hat{\vec{E}}(z)\,e^{iky-i\omega t}\}$ ✓

Engel-8

Ampere
$$\begin{cases} \cancel{\frac{\partial}{\partial x}}H_y^{\,0} - \frac{\partial}{\partial y}H_x = -i\omega\varepsilon E_z \\ \frac{\partial}{\partial y}H_z - \frac{\partial}{\partial z}H_y = -i\omega\varepsilon E_x \\ \frac{\partial}{\partial z}H_x - \cancel{\frac{\partial}{\partial x}}H_z^{\,0} = -i\omega\varepsilon E_y \end{cases}$$

Faraday
$$\begin{cases} -ikE_x = i\omega\mu H_z \\ ikE_z - \frac{\partial}{\partial z}E_y = i\omega\mu H_x \\ \frac{\partial}{\partial z}E_x = i\omega\mu H_y \end{cases}$$

$*\frac{\partial}{\partial y} \to ik$ — no dependence on $x \to \frac{\partial}{\partial x} = 0$

$\to$ Solve in terms of $\hat{H}_x$

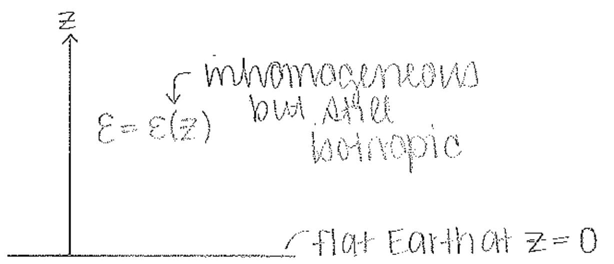
$$ik\hat{E}_z - \frac{\partial}{\partial z}\hat{E}_y = i\omega\mu\hat{H}_x$$

$$\hat{E}_y = \frac{i}{\omega\varepsilon(z)}\frac{\partial\hat{H}_x}{\partial z} \; ; \quad \hat{E}_z = \frac{k}{\omega\varepsilon(z)}\hat{H}_x$$

$$ik\left(\frac{k}{\omega\varepsilon(z)}\hat{H}_x\right) - \frac{\partial}{\partial z}\left(\frac{i}{\omega\varepsilon(z)}\frac{\partial\hat{H}_x}{\partial z}\right) = i\omega\mu\hat{H}_x$$

$$\frac{ik^2}{\omega\varepsilon(z)}\hat{H}_x - \frac{i}{\omega\varepsilon(z)}\frac{\partial^2\hat{H}_x}{\partial z^2} - \frac{\partial\hat{H}_x}{\partial z}\frac{i}{\omega}\frac{\partial}{\partial z}\left(\frac{1}{\varepsilon(z)}\right) = i\omega\mu_0\hat{H}_x$$

I ran out of time, I'm sorry :(I've marked where this picks up on my scratch work on the following page.



* The fields are independent and can be written as functions of z times $e^{i(kx-\omega t)}$

$\vec{E}(\vec{x},t) = f(\hat{z})\, e^{i(\vec{k}\vec{x}-\omega t)}$

$\vec{H}(\vec{x},t) = f(\hat{z})\, e^{i(\vec{k}\vec{x}-\omega t)}$

(a) * For horizontal polarization

$\frac{d^2E}{dz^2} + q^2 F = 0$ — $F = E_y$

$q^2(z) = \omega^2 \mu_0 \varepsilon(z) - k^2$

$\cancel{\vec{k}\cdot(\vec{k}\times\hat{H}) = 0 = -i\omega\varepsilon\, \vec{k}\cdot\hat{\vec{E}}}$

$\cancel{\vec{k}\times(\vec{k}\times\hat{\vec{E}}) = \omega\mu(\vec{k}\times\hat{H})}$

$\cancel{\vec{k}(\vec{k}\cdot\hat{\vec{E}}) - k^2\hat{\vec{E}} = -\omega^2\varepsilon\mu\hat{\vec{E}}} \leftarrow \cancel{\mu = \mu_0,\ \varepsilon = \varepsilon(z)}$

$\cancel{\vec{k}(\vec{k}\cdot\hat{\vec{E}}) + \hat{\vec{E}}\underbrace{(\omega^2\mu_0\varepsilon(z) - k^2)}_{q^2(z)} = 0}$

$\cancel{E_y}$

~~$\vec{k} = k\hat{n}$, where $\hat{n}$ is the direction of propagation~~

$\cancel{(k\hat{n})[(k\hat{n})\cdot E_y] + E_y(\omega^2\mu_0\varepsilon(z) - k^2) = 0}$

$[\nabla^2 + \omega^2\mu_0\varepsilon(z)]\,\vec{E} = 0$ · from Lecture 16 ·

$\nabla^2\vec{E} + \omega^2\mu_0\varepsilon(z)\vec{E} = 0$ $\quad \vec{E}(\vec{x}) = \vec{\mathcal{E}}\, e^{i\vec{k}\cdot\vec{x}}$, $\nabla^2(\vec{\mathcal{E}} e^{i\vec{k}\cdot\vec{x}}) = \vec{\mathcal{E}}\underbrace{\nabla^2 e^{i\vec{k}\cdot\vec{x}}}_{=-k^2 e^{i\vec{k}\cdot\vec{x}}} + e^{i\vec{k}\cdot\vec{x}}\nabla^2\vec{\mathcal{E}}$

$\cancel{e^{i\vec{k}\cdot\vec{x}}}\nabla^2\vec{\mathcal{E}} - k^2\cancel{e^{i\vec{k}\cdot\vec{x}}}\vec{\mathcal{E}} + \omega^2\mu_0\varepsilon(z)[\vec{\mathcal{E}}\cancel{e^{i\vec{k}\cdot\vec{x}}}] = 0$, $\vec{\mathcal{E}} = E_y$

$\frac{d^2E_y}{dz^2} + (\omega^2\mu_0\varepsilon(z) - k^2)E_y = 0$ ✓

* For vertical polarization

$\frac{d^2F}{dz^2} + q^2 F = 0$ — $F = \sqrt{\varepsilon/\varepsilon_0}\, E_z$

$q^2(z) = \omega^2\mu_0\varepsilon(z) + \underbrace{\frac{1}{2\varepsilon}\frac{d^2\varepsilon}{dz^2} - \frac{3}{4\varepsilon^2}\left(\frac{d\varepsilon}{dz}\right)^2}_{} - k^2$

☆ assuming you can use the same process as above, I don't see where this comes from

Engel-10

7.14 (a) Vertical Polarization

$\nabla \times \hat{E} = +i\omega\mu\hat{H}$ $\quad \frac{\partial}{\partial y}\hat{E}_z - \frac{\partial}{\partial z}\hat{E}_y = i\omega\mu H_x$ ← plug in

$\nabla \times \hat{H} = -i\omega\varepsilon\hat{E}$ $\quad \frac{\partial}{\partial z}\hat{E}_x - \frac{\partial}{\partial x}\hat{E}_z = i\omega\mu H_y \quad 0$

$\hat{E}(z) = \underbrace{e^{iky}}_{\frac{\partial}{\partial y} \to ik}$ $\quad \frac{\partial}{\partial x}\hat{E}_y - \frac{\partial}{\partial y}\hat{E}_x = i\omega\mu H_z \quad 0$

no x-dependence

$$\left.\begin{array}{l} \to \frac{\partial}{\partial z}H_x = -i\omega\varepsilon E_y \\ -ikH_x = -i\omega\varepsilon E_z \end{array}\right\}$$ combine for statement on H_x ~ same as choosing E_z!

$ik\hat{E}_z - \frac{\partial}{\partial z}\hat{E}_y = i\omega\mu\hat{H}_x$

$$\hat{E}_y = \frac{i}{\omega\varepsilon(z)}\frac{\partial\hat{H}_x}{\partial z}, \quad \hat{E}_z = \frac{k}{\omega\varepsilon(z)}\hat{H}_x$$

$ik\hat{E}_z - \frac{\partial}{\partial z}\hat{E}_y = i\omega\mu_0 H_x$

$$\left.\begin{array}{l} \frac{\partial}{\partial z}\hat{E}_x = i\omega\mu_0 H_y \\ -ik\hat{E}_x = i\omega\mu_0 H_z \end{array}\right\} 0$$

$$\left.\begin{array}{l} \to \frac{\partial}{\partial z}\hat{H}_x = -i\omega\varepsilon(z)\hat{E}_y \\ -ik\hat{H}_x = -i\omega\varepsilon(z)\hat{E}_z \end{array}\right\} \hat{E}_y = \frac{i}{\omega\varepsilon(z)}\frac{\partial\hat{H}_x}{\partial z}, \quad \hat{E}_z = \frac{k}{\omega\varepsilon(z)}\hat{H}_x$$

$$ik\left(\frac{k}{\omega\varepsilon(z)}\hat{H}_x\right) - \frac{\partial}{\partial z}\left(\frac{i}{\omega\varepsilon(z)}\frac{\partial\hat{H}_x}{\partial z}\right) = i\omega\mu\hat{H}_x$$

$$\frac{ik^2}{\omega\varepsilon(z)}\hat{H}_x - \frac{i}{\omega\varepsilon(z)}\frac{\partial^2\hat{H}_x}{\partial z^2} - \frac{\partial\hat{H}_x}{\partial z}\frac{i}{\omega}\frac{\partial}{\partial z}\left(\frac{1}{\varepsilon(z)}\right) = i\omega\mu_0\hat{H}_x$$

formal copy stops here!! →

$$k^2\hat{H}_x - \frac{\partial^2\hat{H}_x}{\partial z^2} - \frac{\partial\hat{H}_x}{\partial z}\varepsilon(z)\frac{\partial}{\partial z}\left(\frac{1}{\varepsilon(z)}\right) = \omega^2\mu_0\varepsilon(z)\hat{H}_x$$

$$\frac{\partial^2\hat{H}_x}{\partial z^2} + \frac{\partial\hat{H}_x}{\partial z}\varepsilon(z)\frac{\partial}{\partial z}\left(\frac{1}{\varepsilon(z)}\right) + \omega^2\mu_0\varepsilon(z)\hat{H}_x = 0$$

convert $\hat{H}_x \to \hat{E}_z$

$$\frac{\partial^2}{\partial z^2}\sqrt{\frac{\varepsilon}{\varepsilon_0}}E_z + \frac{1}{2\varepsilon}\frac{\partial^2\varepsilon(z)}{\partial z^2}\sqrt{\frac{\varepsilon}{\varepsilon_0}}E_z - \frac{3}{4\varepsilon^2}\left(\frac{\partial\varepsilon(z)}{\partial z}\right)^2\sqrt{\frac{\varepsilon}{\varepsilon_0}}E_z - k^2\sqrt{\frac{\varepsilon}{\varepsilon_0}}E_z = 0$$ ✓

(b) Assuming that the dielectric constant is given by

$\frac{\varepsilon(\omega)}{\varepsilon_0} \simeq 1 - \frac{\omega_p^2}{\omega^2}$ — plasma frequency   Jackson eq. 7.59

$\omega_p^2 = \frac{\overbrace{NZ}^{\text{total \# of electrons}} e^2}{\varepsilon_0 m}$ per unit volume (actually ω_p

→ electron density like that in Fig. 7.11 (p. 318)

Propagation eq. for vertical polarization

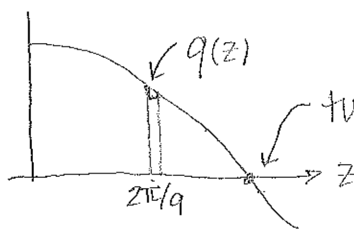
$-ikH_x = -i\omega\varepsilon(z)E_z$

└ if $k=0 \rightarrow E_z = 0$

$-\frac{\partial}{\partial z}E_y = i\omega\mu\hat{H}_x$

$\frac{\partial}{\partial z}\hat{H}_x = -i\omega\varepsilon(z)\hat{E}_y \rightarrow \frac{\partial^2}{\partial z^2}E_y = -\omega^2\varepsilon(z)\mu E_y$

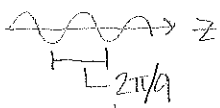
$$\frac{d^2F}{dz^2} + q^2(z)F = 0$$



turning pt (appx's break down here) ✓

if q were const.:

$F \sim e^{iqz}$ would be a soln. here



local wavelength is much shorter than the variation in the medium $\varepsilon'(z) < \frac{\partial}{\partial z}$ Field (must be)

→ allows us to treat as prop. in one direction, no complications

Express F(z) as z

$F(z) = A(z)\,e^{i\varphi(z)}$

↑ ampl. varies slowly

└ phase oscillates quickly

$\frac{dF}{dz} = A'e^{i\varphi} + i\varphi'Ae^{i\varphi}$

$\frac{d^2F}{dz^2} = A''e^{i\varphi}\, i\varphi''Ae^{i\varphi} + 2i\varphi'A'e^{i\varphi} - \varphi'^2Ae^{i\varphi} + q^2Ae^{i\varphi} = 0$

→ Scale the eqn. based on fast/slow assumptions ⇒ $(\varphi')^2$ is the largest term

order 0: $\varphi'^2 = q^2(z)$

$\frac{d\phi}{dz} = \pm q(z)$

$\varphi(z) = \int^z dz'\, q(z')$ ✓

Engel-12

$$\text{remaining} = \overbrace{A''}^{\text{smallest term here}} + i\varphi''A + 2i\varphi'A' = 0$$

$$\varphi''A + 2\varphi'A' = 0 \quad \text{div. by } \phi'A$$

$$\frac{\varphi''}{\varphi'} + 2\frac{A'}{A} = 0$$

$$\frac{d}{dz}\ln(\varphi') + \frac{d}{dz}\ln(A^2) = 0$$

$$\rightarrow \varphi'A^2 = \text{const.}$$

$$A = \frac{\text{const}}{\sqrt{q(z)}}$$

→ just plug in for $\varphi(z)$ and A for $F(z) = A(z)e^{i\varphi(z)}$ and check assumptions made in Section 7.6

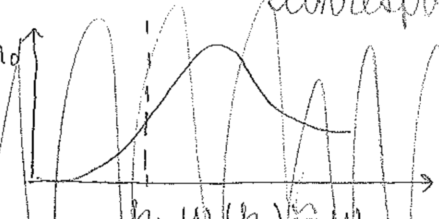
$$F(z) = A(z)e^{i\varphi(z)}$$

$$= \frac{\text{const.}}{\sqrt{q(z)}}e^{i\int^z dz' q(z')}$$

7.6 Assumptions:

- electronic motion is small
- collisions are neglected
- transverse wave is circularly polarized

We can see that, when plugged in for $q(z)$, our equation for $F(z)_{WKB}$ satisfies all of these assumptions w/ deviations for large $\underbrace{\omega \sim \omega_{p,max}}$ corresponding to magnetic precession.
corresponds to T.P. of $q(z)$ plot → WKB appx. breaks down here

(c) 

→ h' is the turning point of the $q^2(z)$ plot

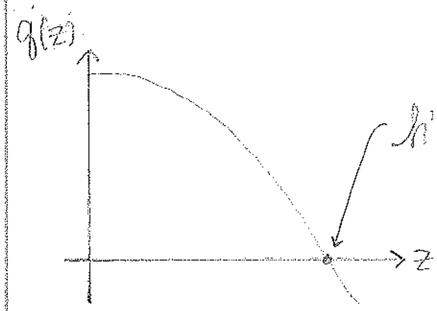
→ reflection occurs when $\varepsilon(z) = 0$

$$\varepsilon(z) = 1 - \frac{\omega_p(z)^2}{\omega^2}, \quad q^2 = \frac{\omega^2}{c^2}\varepsilon(z)$$

→ h' = the z-value that causes $\omega_p(z) = \omega \rightarrow \varepsilon = 0$

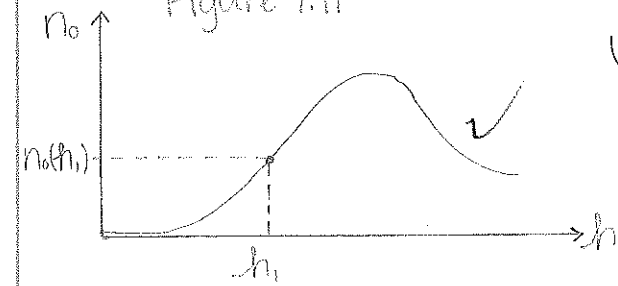
Engel-

(c)



h' corresponds to the turning point on the $g(z)$ plot. Our WKB approximation ✓ breaks down here

Figure 7.11



We can see from Fig. 7.11 that the turning point (here, h_1) corresponds to $\omega \simeq \omega_p(h_1)$ (where $n_0 = \rho_{e^-}$) ✓

From part (b), we know the dielectric constant is given by

$$\varepsilon(z) = 1 - \frac{\omega_p(z)^2}{\omega^2} \qquad \text{Jackson eq. 7.59}$$

$\Rightarrow$ Reflection occurs when $\varepsilon(z) = 0$ $(\omega = \omega_p)$

- this agrees with the turning point of the $g(z)$ plot as $g(z) \simeq \frac{\omega^2}{c^2}\varepsilon(z)$ under the assumptions made in (b)

- for $\mu = \mu_0$, $\varepsilon = \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)$

$$k(\omega) = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_0\varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)}$$

$\omega = \omega_p$ @ reflection

↑ Actual argument

$$= \omega_p\sqrt{\mu_0\varepsilon_0\left(1 - \cancel{\frac{\omega_p^2}{\omega_p^2}}^{\,1}\right)} = 0$$ in agreement w/ assumption in part (b)

Other crap I tried

"When the density n_0 [# of free electrons per unit volume] is large enough, ... $\omega_p(h_1) \simeq \omega$. Then the dielectric constants vanish and the pulse is reflected. The actual density where the reflection occurs is given by the roots of the right hand side of [eq.] (7.67)." – Jackson pg. 318

$$n_0 = \sqrt{1 - \frac{\omega_p^2}{\omega(\omega \mp \omega_B)}} = \sqrt{1 - \omega_p^2\left(\frac{1}{\omega\left(\omega \mp \frac{eB_0}{m}\right)}\right)} = \sqrt{1 - \left(\omega_p \mp \frac{eB_0}{m}\right)}$$

$\omega_B = \frac{eB_0}{m}$ $\qquad$ $\omega = \omega_p$

4. Jackson Problem 7.16

Plane waves propagate in a homogeneous, non-permeable, but anisotropic dielectric.

(a) $\nabla\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0$; $\frac{1}{\mu_0}\nabla\times\vec{B}-\underbrace{\varepsilon_0\frac{\partial\vec{E}}{\partial t}}_{\vec{D}}$

If $\vec{B}$ and $\vec{E}$ have solutions that are harmonic in time, then $\frac{\partial}{\partial t}\to -i\omega$

$$\nabla\times\vec{E}-i\omega\vec{B}=0\ ;\ \frac{1}{\mu_0}\nabla\times\vec{B}+i\omega\vec{D}=0$$

→ Then, take the Fourier Transform of these equations with respect to ω such that $\nabla\to i\vec{k}$

$$i\vec{k}\times\vec{E}-i\omega\vec{B}=0$$

$$\cancel{i}\omega\vec{B}=\cancel{i}\vec{k}\times\vec{E}$$

$$\vec{B}=\frac{\vec{k}\times\vec{E}}{\omega}$$

$$i\vec{k}\times\vec{B}+i\omega\mu_0\vec{D}=0$$

$$\cancel{i}\vec{k}\times\left(\frac{\vec{k}\times\vec{E}}{\cancel{\omega}}\right)+\cancel{i}\omega^{\cancel{2}}\mu_0\vec{D}=0$$

$$\vec{k}\times(\vec{k}\times\vec{E})+\omega^2\mu_0\vec{D}=0$$

(b) $\vec{k}\times(\vec{k}\times\vec{E})+\omega^2\mu_0\vec{D}=0$

$\quad\llcorner\ \vec{k}=k\hat{n}$

$$k\hat{n}\times(k\hat{n}\times\vec{E})+\omega^2\mu_0\vec{D}=0$$

$$k^2[\hat{n}(\hat{n}\cdot\vec{E})-\vec{E}\underbrace{(\hat{n}\cdot\hat{n})}_{1}]+\frac{\omega^2\mu_0\vec{D}}{k^2}=0$$

$$n_i(n_jE_j)-E_i+\underbrace{\frac{\omega^2}{k^2}}_{v^2}\mu_0\underbrace{D_i}_{\varepsilon_iE_i}=0$$

$$n_in_jE_j-E_i+v^2\underbrace{\mu_0\varepsilon_i}_{1/v_i^2}E_i=0$$

$$n_in_jE_j-E_i+(v^2/v_i^2)E_i=0$$

$i=1,2,3$ ↴

$$\begin{bmatrix} n_1(n_1E_1+n_2E_2+n_3E_3)-E_1\\ n_2(n_1E_1+n_2E_2+n_3E_3)-E_2\\ n_3(n_1E_1+n_2E_2+n_3E_3)-E_3\end{bmatrix}+v^2\begin{bmatrix}E_1/v_1^2\\ E_2/v_2^2\\ E_3/v_3^2\end{bmatrix}$$ ✓

↓ combine like terms in E ↓

$$\begin{bmatrix} E_1(n_1n_1-1)+n_1n_2E_2+n_1n_3E_3 \\ n_2n_1E_1+E_2(n_2n_2-1)+n_2n_3E_3 \\ n_3n_1E_1+n_3n_2E_2+E_3(n_3n_3-1) \end{bmatrix} + v^2 \begin{bmatrix} E_1/v_1^2 \\ E_2/v_2^2 \\ E_3/v_3^2 \end{bmatrix} = 0$$

$$\left(\underbrace{\begin{bmatrix} n_1^2-1 & n_1n_2 & n_1n_3 \\ n_2n_1 & n_2^2-1 & n_2n_3 \\ n_3n_1 & n_3n_2 & n_3^2-1 \end{bmatrix}}_{A} + v^2 \underbrace{\begin{bmatrix} 1/v_1^2 & 0 & 0 \\ 0 & 1/v_2^2 & 0 \\ 0 & 0 & 1/v_3^2 \end{bmatrix}}_{B} \right) \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = 0$$

We can solve for v^2 using $\mathrm{Det}\{A - v^2 B\} = 0$

↓ plugging into Mathematica, with $n_1^2+n_2^2+n_3^2=0$ ↓

$$\frac{v^2}{v_1v_2v_3}\left[n_1^2(v^4+v_2^2v_3^2-v^2v_2^2-v^2v_3^2)+n_2^2(v^4+v^2v_3^2+v_1^2v_3^2-v^2v_1^2)\right.$$
$$\left.+\,n_3^2(v^4+v^2v_2^2+v_1^2v_2^2-v^2v_1^2)\right]=0$$

↓ factor v terms ↓

$$\frac{v^2}{v_1v_2v_3}\left[n_1^2(v^2-v_2^2)(v^2-v_3^2)+n_2^2(v^2-v_1^2)(v^2-v_3^2)+n_3^2(v^2-v_1^2)(v^2-v_2^2)\right]=0$$

→ divide by $(v^2-v_1^2)(v^2-v_2^2)(v^2-v_3^2)$ to get terms only of a single integer ↓

$$\frac{v^2}{v_1v_2v_3}\left[\frac{n_1^2}{(v^2-v_1^2)}+\frac{n_2^2}{(v^2-v_2^2)}+\frac{n_3^2}{(v^2-v_3^2)}\right]=0$$

$$\underbrace{\frac{v^2}{v_1v_2v_3}}\sum_{i=1}^{3}\frac{n_i^2}{(v^2-v_i^2)}=0$$

$$\sum_{i=1}^{3}\frac{n_i^2}{(v^2-v_i^2)}=0$$

(c) Starting with

$$k^2[\hat{n}(\hat{n}\cdot\vec{E})]+\omega^2\mu_0\vec{D}=0$$

$$(\hat{n}\cdot\vec{E}_a)\hat{n}-\vec{E}_a=-\mu_0\varepsilon(\omega/k)^2\cdot\vec{E}_a$$
$$=-v_a^2\vec{D}_a$$

If $\vec{D}_b$ is another mode with the same $\hat{n}$

$$\vec{D}_b\cdot\vec{D}_a v_a^2=\vec{D}_b\cdot(\vec{E}_a-\underbrace{(\hat{n}\cdot\hat{E}_a)}_{0}\hat{n})$$

$\vec{D}_b \cdot \vec{D}_a v_a^2 = \vec{D}_b \cdot \vec{E}_a$

Likewise, if we begin with mode b,

$(\hat{n} \cdot \vec{E}_b)\hat{n} - \vec{E}_b = -\mu_0 \varepsilon (\omega/k)^2 \cdot E_b$

$= -v_b^2 \vec{D}_b$

$\vec{D}_a \cdot \vec{D}_b v_b^2 = \vec{D}_a \cdot (\vec{E}_b - (\hat{n} \cdot \vec{E}_b)\hat{n})$

$= \vec{D}_a \cdot \vec{E}_b$

↓ taking the difference ↓

$\vec{D}_b \cdot \vec{D}_a (v_a^2 - v_b^2) = \underbrace{\vec{D}_b \cdot \vec{E}_a - \vec{D}_a \cdot \vec{E}_b}_{\vec{D}_b \cdot \vec{E}_a = \vec{D}_a \cdot \vec{E}_b}$

$\underbrace{\vec{D}_b \cdot \vec{D}_a}(v_a^2 - v_b^2) = 0$

If the phase velocities are different, then $\vec{D}_a \cdot \vec{D}_b$ must equal zero for this statement to hold true.

5. Antonsen Problem

$$v_g = \frac{\partial \omega}{\partial k} \;,\quad \omega = \frac{k}{\sqrt{\mu \varepsilon}}$$

$\vec{S} = \frac{1}{2}\vec{E}\times\vec{H}^* = \frac{1}{2\mu}(\vec{E}\times\vec{B}^*)$ → want to find $\frac{\langle S \rangle}{\langle u \rangle}$

$u = \frac{1}{4}(\varepsilon|\vec{E}|^2 + \frac{1}{\mu}|\vec{B}|^2)$

where $\vec{B} = \frac{n}{c}\vec{k}\times\vec{E} \rightarrow |\vec{B}| = \frac{n}{c}|\vec{E}|$

$$\vec{S} = \frac{1}{2\mu}(|\vec{E}|\cos(\vec{k}\cdot\vec{x}-\omega t))(|\vec{B}|\cos(\vec{k}\cdot\vec{x}-\omega t))$$

$$= \frac{1}{2\mu}(|\vec{E}|\cos(\vec{k}\cdot\vec{x}-\omega t))(\tfrac{n}{c}|\vec{E}|\cos(\vec{k}\cdot\vec{x}-\omega t))$$

$$= \frac{n}{2\mu c}|\vec{E}|^2\cos^2(\vec{k}\cdot\vec{x}-\omega t)$$

$$= \frac{1}{2}\sqrt{\varepsilon/\mu}\,|\vec{E}|^2\cos^2(\vec{k}\cdot\vec{x}-\omega t)$$

$$\langle \vec{S} \rangle = \frac{1}{4}\sqrt{\varepsilon/\mu}\,|\vec{E}|^2$$

$$\langle u \rangle = \frac{1}{4}\varepsilon|\vec{E}|^2$$

$$\frac{\langle S \rangle}{\langle u \rangle} = \frac{\frac{1}{4}\sqrt{\varepsilon/\mu}\,|\vec{E}|^2}{\frac{1}{4}\varepsilon|\vec{E}|^2} = \frac{\sqrt{\varepsilon/\mu}}{\varepsilon} = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$v_g = \frac{\partial}{\partial k}\left(\frac{k}{\sqrt{\mu\varepsilon}}\right) = \frac{1}{\sqrt{\mu\varepsilon}}$$ same value ✓ ✓