Physics 606: Homework #4

Due: Tuesday March 8, 2016

Jackson: Problems 5.3, 5.4, 5.6

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Homework #4

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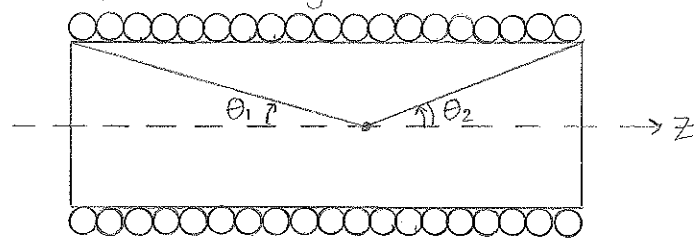
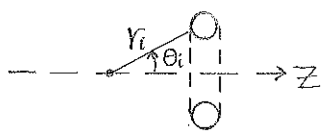
1. Jackson Problem 5.3

A right-circular solenoid of finite length L and radius a has N-turns per unit length and carries a current I.



*First examine the magnetic induction due to a single loop:



$$\underbrace{J_\varphi = I\sin(\theta')\delta(\cos(\theta'))}\frac{\delta(r'-a)}{a}$$ Jackson eq. (5.33)

our analysis is not over all θ, but just at $\theta = \theta_i \rightarrow$ substitute for a δ-function.

$$J_\varphi = I\delta(\theta-\theta_i)\frac{\delta(r-r_i)}{r}$$

$\rightarrow$ Plug into the Biot-Savart Law

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}) \times \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} d^3x'$$ Jackson eq. (5.14)

convert to spherical coordinates

$$\int d^3x' = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} r'^2\sin(\theta')dr'd\theta'd\varphi'$$

$$\vec{x}-\vec{x}' = (r\sin(\theta)\cos(\varphi) - r'\sin(\theta')\cos(\varphi'))\hat{\imath} + (r\sin(\theta)\sin(\varphi) - r'\sin(\theta')\sin(\varphi'))\hat{\jmath} + (r\cos(\theta) - r'\cos(\theta'))\hat{k}$$

$$\rightarrow |\vec{x}-\vec{x}'|^3 = [r^2 + r'^2 - 2rr'\{\cos(\theta)\cos(\theta') + \sin(\theta)\sin(\theta')\cos(\varphi-\varphi')\}]^{3/2}$$

$$\vec{J} = I\delta(\theta-\theta_i)\frac{\delta(r-r_i)}{r}\hat{\varphi}$$

$$\hat{\varphi} = -\sin(\varphi)\hat{\imath} + \cos(\varphi)\hat{\jmath}$$

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} r'^2\sin(\theta')dr'd\theta'd\varphi'\left[I\delta(\theta-\theta')\frac{\delta(r'-r_i)}{r'}(-\sin(\varphi')\hat{\imath} + \cos(\varphi')\hat{\jmath})\right] \times \ldots$$

$$\times \frac{(r\sin(\theta)\cos(\varphi) - r'\sin(\theta')\cos(\varphi'))\hat{\imath} + (r\sin(\theta)\sin(\varphi) - r'\sin(\theta')\sin(\varphi'))\hat{\jmath} + (r\cos(\theta) - r'\cos(\theta'))\hat{k}}{[r^2 + r'^2 + 2rr'\{\cos(\theta)\cos(\theta') + \sin(\theta)\sin(\theta')\cos(\varphi-\varphi')\}]^{3/2}}$$

$\rightarrow$ Enact the δ-functions

$\delta(r'-r_i) \rightarrow r' = r_i;\quad \int_0^{\infty} dr' = 1$

$\delta(\theta'-\theta_i) \rightarrow \theta' = \theta_i;\quad \int_0^{\pi} d\theta' = 1$

$$\vec{B}(\vec{x}) = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} r_i^2 \sin(\theta_i)\, d\varphi' \left[\frac{1}{r_i}\left(-\sin(\varphi')\hat{\imath} + \cos(\varphi')\hat{\jmath}\right)\right] \times \ldots$$

$$\times \frac{(r\sin(\theta)\cos(\varphi) - r_i\sin(\theta_i)\cos(\varphi'))\hat{\imath} + (r\sin(\theta)\sin(\varphi) - r_i\sin(\theta_i)\sin(\varphi'))\hat{\jmath} + (r\cos(\theta) - r_i\cos(\theta_i))\hat{k}}{[r^2 + r_i^2 - 2rr_i\{\cos(\theta)\cos(\theta_i) + \sin(\theta)\sin(\theta_i)\cos(\varphi-\varphi')\}]^{3/2}}$$

$$\vec{B}(\vec{x}) = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} r_i \sin(\theta_i)\, d\varphi' \cdot \ldots$$

$$\cdot \frac{(r\cos(\theta) - r_i\cos(\theta_i))(\cos(\varphi')\hat{\imath} + \sin(\varphi')\hat{\jmath}) + (r_i\sin(\theta_i) - r\sin(\theta)\cos(\varphi-\varphi'))\hat{k}}{[r^2 + r_i^2 - 2rr_i\{\cos(\theta)\cos(\theta_i) + \sin(\theta)\sin(\theta_i)\cos(\varphi-\varphi')\}]^{3/2}}$$

→ to simplify this integral, let us set our observation point, r, to be at the origin $_2$

$$\vec{B}(\vec{x}) = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \cancel{r_i} \sin(\theta_i)\, d\varphi' \frac{(-\cancel{r_i}\cos(\theta_i))(\cos(\varphi')\hat{\imath} + \sin(\varphi')\hat{\jmath}) + \cancel{r_i}\sin(\theta_i)\hat{k}}{(r_i^2)^{3/\cancel{2}}}$$

$$= \frac{\mu_0 I}{2\cancel{4\pi}} \frac{\sin(\theta_i)}{r_i} \int_0^{2\pi} d\varphi \left(-\cos(\theta_i)\right)\left[\underbrace{\cos(\varphi')\hat{\imath}}_{0} + \underbrace{\sin(\varphi')\hat{\jmath}}_{0}\right] + \underbrace{\sin(\theta_i)\hat{k}}_{2\pi\sin(\theta_i)\hat{k}}$$

$$= \frac{\mu_0 I}{2} \frac{\sin^2(\theta_i)}{r_i} \hat{k}$$

the loop's radius is fixed, so reverting back to the original notation, $r_i \sin(\theta_i) = a \rightarrow \sin^2(\theta_i) = a^2/r_i^2$

$$= \frac{\mu_0 I}{2} \frac{a^2}{r_i^3} \hat{k}$$ — where $r_i = \sqrt{a^2 + z_i^2}$

$$\vec{B}(\vec{z}) = \frac{\mu_0 I}{2} \frac{a^2}{(a^2 + z_i^2)^{3/2}} \hat{k}$$

* We can then use superposition to find the contribution for N loops:

→ If we place N loops of radius a within an infinitesimal length dz (corresponding to NL→∞)

$$\vec{B}(\vec{z}) = \int_{z_1}^{z_2} \frac{\mu_0 I}{2} \frac{a^2}{(a^2 + z_i^2)^{3/2}} \hat{k} \cdot N dz$$

└ a continuous linear superposition of contributions to $\vec{B}(\vec{z})$ by each of the N loops

$$= \frac{\mu_0 N I}{2} \hat{k} \int_{z_1}^{z_2} \frac{a^2\, dz}{(a^2 + z_i^2)^{3/2}}$$

$$= \frac{\mu_0 N I}{2} \hat{k} \left[\frac{z_i}{\sqrt{a^2 + z_i^2}} \right]_{z_1}^{z_2}$$

$$= \frac{\mu_0 N I}{2} \hat{k} \left(\underbrace{\frac{z_2}{\sqrt{a^2 + z_2^2}}}_{\cos(\theta_2)} - \underbrace{\frac{z_1}{\sqrt{a^2 + z_1^2}}}_{\cos(\theta_1)} \right)$$

$$= \frac{\mu_0 N I}{2} \hat{k} \left[\cos(\theta_2) - \cos(\theta_1) \right]$$

But! Our integration here assumed θ_1 and θ_2 were both measured with respect to the $+z$-axis. θ_1 actually $= \pi - \theta_{1\,\mathrm{wrt}\,+z}$

$$\vec{B}(\vec{z}) = \frac{\mu_0 N I}{2} \left[\cos(\theta_1) + \cos(\theta_2) \right]$$

✓

2. Jackson Problem 5.4

(a) Near the axis, $\rho \approx 0$

Per Jackson eq. (5.65), because ρ is small and we know the field is slowly varying, we can perform a Taylor expansion of the magnetic induction.

$$B_K(\vec{X}) = B_K(0) + \vec{X}\cdot\nabla B_K(0) + \ldots \qquad \text{Jackson eq. (5.65)}$$

$$\begin{cases} B_z(\rho,z) = a_0(z) + \rho a_1(z) + \frac{1}{2}\rho^2 a_2(z) + \ldots = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} a_n(z) \\ B_\rho(\rho,z) = c_0(z) + \rho c_1(z) + \frac{1}{2}\rho^2 c_2(z) + \ldots = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} c_n(z) \end{cases}$$

– no third component → cylindrical symmetry

$\vec{B}$ is in a current-free region

$\to \nabla\cdot\vec{B} = 0$

$\to \nabla\times\vec{B} = \mu J = 0$

$$\nabla\cdot\vec{B} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho B_\rho + \frac{\partial}{\partial z}B_z = 0$$

$$= \sum_n \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\frac{\rho^{n+1}}{n!}c_n(z)\right] + \frac{\partial}{\partial z}\left[\frac{\rho^n}{n!}a_n(z)\right]\right)$$

$$= \sum_n \left(\frac{1}{\rho}\frac{(n+1)\rho^n}{n!}c_n(z) + \frac{\rho^n}{n!}a_n'(z)\right) = 0$$

– Shift $n \to n+1$ for the c_n terms to allow for consolidation with the a_n' terms

$$0 = \frac{1}{\rho}c_0(z) + \sum_{n=0}^{\infty}\left[\frac{((n+1)+1)\rho^n}{(n+1)!}c_{n+1}(z) + \frac{\rho^n}{n!}a_n'(z)\right]$$

$$= \frac{1}{\rho}c_0(z) + \sum_n\left[\frac{(n+2)\rho^n}{(n+1)n!}c_{n+1}(z) + \frac{\rho^n}{n!}a_n'(z)\right]$$

$$0 = \frac{1}{\rho}c_0(z) + \sum_n \frac{\rho^n}{n!}\left[\frac{(n+2)}{(n+1)}c_{n+1}(z) + a_n'(z)\right]$$

each term must vanish individually in order for this statement to hold true for any value of ρ

$$\longrightarrow c_0(z) = 0 \;;\quad c_{n+1}(z) = -\frac{(n+1)}{(n+2)}a_n'(z)$$

→

$$[\nabla\times\vec{B}]_\varphi = \frac{\partial}{\partial z}B_\rho - \frac{\partial}{\partial\rho}B_z = 0$$

$$= \sum_n\left(\frac{\partial}{\partial z}\left[\frac{\rho^n}{n!}c_n(z)\right] - \frac{\partial}{\partial\rho}\left[\frac{\rho^n}{n!}a_n(z)\right]\right)$$

$$= \sum_n\left(\frac{\rho^n}{n!}c_n'(z) - \cancel{n}\frac{\rho^{n-1}}{\cancel{n}(n-1)!}a_n(z)\right) = 0$$

- Shift $n\to n+1$ for the a_n terms to allow for consolidation with the c_n' terms

$$0 = \sum_{n=0}^{\infty}\frac{\rho^n}{n!}\underbrace{\left(c_n'(z) - a_{n+1}(z)\right)}$$

$c_n'(z) = a_{n+1}(z)$ for this to hold true for all ρ

$$a_{n+1}(z) = c_n'(z) = -\frac{n}{(n+1)}a_{n-1}''(z)$$

$-\; c_0(z) = 0 \to c_0'(z) = 0 = \underbrace{a_1(z)}$

use the recursion relation to find $a_n(z)$ with respect to $a_0^{(n)}$ for even integers

$$a_n(z) = \frac{(-1)^{n/2}}{2^n}\frac{n!}{[(n/2)!]^2}a_0^{(n)}(z)\ ,\quad n\text{ even}$$

- If only $a_{n\text{-even}}$ terms are non-vanishing, where $c_{n+1}\propto a_{n\text{-even}}'$, the c_{n+1} is non-vanishing for odd integers.

$$c_{n+1}(z) = -\frac{(n+1)}{(n+2)}a_n'(z)$$

$$= \frac{(-1)^{n/2+1}}{2^n}\frac{(n+1)!}{(n+2)[(n/2)!]^2}a_0^{(n+1)}(z)\ ,\quad n\text{ odd}$$

$\to$ Let $m = n/2$ and plug in for B_z and B_ρ

$$B_z(\rho,z) = \sum_{n=0}^{\infty}\frac{\rho^n}{n!}a_n(z)$$

$$= \sum_{m=0}^{\infty}\frac{\rho^{2m}}{\cancel{(2m)!}}\frac{(-1)^m}{2^{2m}}\frac{\cancel{(2m)!}}{[m!]^2}a_0^{2m}(z)$$

recognize that $a_0(z) = B_z(0,z)$

$$= \sum_{m=0}^{\infty}\frac{(-1)^m}{4^m}\frac{\rho^{2m}}{(m!)^2}\left[\frac{\partial^{2m}B_z(0,z)}{\partial z^{2m}}\right]$$

✓

$$B_\rho(\rho,z) = \sum_{n=0}^{\infty}\frac{\rho^n}{n!}c_n(z)$$

$$= \sum_{m=0}^{\infty} \frac{\rho^{2m+1}}{(2m+1)!} \frac{(-1)^{m+1}}{4^m} \frac{\cancel{(2m+1)!}}{(2m+2)[m!]^2} a_0^{(2m+1)}(z)$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{4^m} \frac{\rho^{2m+1}}{(2m+2)[m!]^2} \left[\frac{\partial^{2m+1} B_z(0,z)}{\partial z^{2m+1}}\right]$$

Expanding the first few terms yields the desired results:

$$B_z(\rho,z) = B_z(0,z) - \frac{\rho^2}{4}\left[\frac{\partial^2 B_z(0,z)}{\partial z^2}\right] + \dots$$

$$B_\rho(\rho,z) = -\frac{\rho}{2}\left[\frac{\partial B_z(0,z)}{\partial z}\right] + \frac{\rho^3}{16}\left[\frac{\partial^3 B_z(0,z)}{\partial z^3}\right] + \dots$$

(b) Inspect the ratio of adjacent terms in $\vec{B}(\rho,z)$ (neglecting constants)

$$\frac{b_{n+2}}{b_n} \sim \rho^2 \frac{(\partial^{n+2} B_z(0,z)/\partial z^{n+2})}{(\partial^n \partial_z(0,z)/\partial z^n)}$$

this ratio must be < 1 to converge

$$\Rightarrow \rho \ll \sqrt{\frac{(\partial^n B_z(0,z)/\partial z^n)}{(\partial^{n+2} B_z(0,z)/\partial z^{n+2})}}$$, the criterion defining "near" the axis

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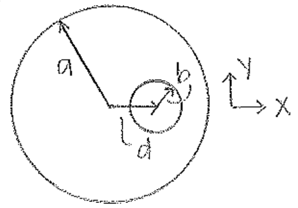
3. A cylindrical conductor of radius a has a hole of radius b bored parallel to, and centered a distance d from, the cylinder axis ($d+b<a$).



Ampere's Law:

$\oint_c \vec{B}\cdot d\vec{l} = \mu_0 \int_S \vec{J}\cdot\hat{n}\,dA$

Let there be a uniform current density J_0 flowing in the $+z$-direction for the $r=a$ cylindrical conductor and in the $-z$-direction for the $r=b$ hole.

→ for a closed contour inside $r=a$, but not including the hole

$B_\varphi \oint_c dl = \mu_0 J_0 \int_S dA$

magnetic field is constant along this contour

$B_\varphi 2\pi r = \mu_0 J_0 \pi r^2$

$\vec{B} = \frac{\mu_0}{2} J_0 r \hat{\varphi}$ $\quad \hat{\varphi} = -y\hat{\imath} + x\hat{\jmath}$

$= \frac{\mu_0}{2} J_0 r(-y\hat{\imath} + x\hat{\jmath})$

→ repeated for a contour inside b ($-J_0$)

$\vec{B} = -\frac{\mu_0}{2} J_0 r(-y\hat{\imath} + x\hat{\jmath})$

the hole is not at $x=0$ but $x=d$

$= -\frac{\mu_0}{2} J_0 r(-y\hat{\imath} + (x-d)\hat{\jmath})$

We can then simply add these two equations to get the result:

$\vec{B} = \frac{\mu_0}{2} J_0 r(-y\hat{\imath} + x\hat{\jmath}) + \frac{\mu_0}{2} J_0 r(y\hat{\imath} - x\hat{\jmath} + d\hat{\jmath})$

$\Rightarrow \vec{B} = \frac{\mu_0}{2} J_0 r d\hat{\jmath}$