Physics 606: Homework #3

Due: Tuesday March 1, 2016

Jackson: Problems 3.5, 4.8, 4.9, 4.10, 4.11,

A conducting sheet lies in the x-y infinite half plane (x>0). A current I is injected at the point $(x_0, 0)$. Find an expression for the potential and electric field in the conductor.

100

Homework #3

Kristi Engel
109898522

1. Jackson Problem 3.5

A hollow sphere of inner radius a has the potential specified on its surface to be $\Phi = V(\theta,\varphi)$.

→ First recognize that solution (a) is a Green's function solution and solution (b) is a Laplace's equation solution for the potential inside the sphere.

(a) From Jackson eq. 2.17 we know the Green's function in spherical coordinates to be:

$$G(\vec{x},\vec{x}') = \frac{1}{(x^2+x'^2-2xx'\cos(\gamma))^{1/2}} - \frac{1}{\left(\frac{x^2x'^2}{a^2}+a^2-2xx'\cos(\gamma)\right)^{1/2}}$$

where γ is the angle between $\vec{x}$ and $\vec{x}'$ such that

$\cos(\gamma) = \cos(\theta)\cos(\theta') + \sin(\theta)\sin(\theta')\cos(\varphi-\varphi')$

The general solution for the Green's Function method with Dirichlet Boundary Conditions is:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \underbrace{\rho(\vec{x}')}\, G_D(\vec{x},\vec{x}')\,d^3x' - \frac{1}{4\pi}\oint_S \Phi(\vec{x}')\frac{\partial G_D}{\partial n'}\,da' \quad \text{(Jackson eq. 1.44)}$$

$\rho(\vec{x}') = 0$ (hollow sphere)

$$= \frac{-1}{4\pi}\oint_S \Phi(\vec{x}')\frac{\partial G_D}{\partial n'}\,da' \qquad \oint_S da' = \int a^2\,d\Omega'$$

$$\left.\frac{\partial G}{\partial n'}\right|_{x'=a} = \frac{(x^2-a^2)}{a(x^2+a^2-2xa\cos(\gamma))^{3/2}}$$

à la Jackson eq. 2.18, but $\hat{n}$ points inwards

→ Substitute r for x where appropriate

$$\Phi(\vec{x}) = \frac{-1}{4\pi}\int V(\theta',\varphi')\frac{(r^2-a^2)}{\cancel{a}(r^2+a^2-2ra\cos(\gamma))^{3/2}}\,a^{\cancel{2}}\,d\Omega'$$

$$= \frac{-1}{4\pi}\int V(\theta',\varphi')\frac{-a(a^2-r^2)}{(r^2+a^2-2ra\cos(\gamma))^{3/2}}\,d\Omega'$$

$$\Phi(\vec{x}) = \frac{a(a^2-r^2)}{4\pi}\int \frac{V(\theta',\varphi')}{(r^2+a^2-2ra\cos(\gamma))^{3/2}}\,d\Omega'$$

→

(b) Returning to Jackson eq. 1.44

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \underbrace{\rho(\vec{x}')}_{\rho(\vec{x}')=0} G_D(\vec{x},\vec{x}')\,d^3x' - \frac{1}{4\pi}\oint_S \Phi(\vec{x}')\frac{\partial G_D}{\partial n'}\,da'$$

$\oint_S da' = \int a^2 d\Omega'$

However, now we will express the derivative of the Green's function using spherical harmonics (based on Jackson eq. 3.127)

$$\frac{\partial G}{\partial n'} = \left.\frac{\partial G}{\partial r'}\right|_{r'=a} = \frac{-4\pi}{a^2}\sum_{\ell,m}\left(\frac{r}{a}\right)^{\ell} Y^*_{\ell m}(\theta',\varphi')\,Y_{\ell m}(\theta,\varphi)$$

↓ Plugging in ↓

$$\Phi(\vec{x}) = \sum_{\ell,m}\left[\int V(\theta',\varphi')\,Y^*_{\ell m}(\theta',\varphi')\,d\Omega'\right]\left(\frac{r}{a}\right)^{\ell} Y_{\ell m}(\theta,\varphi)$$

Then, define $A_{\ell m} \equiv \int d\Omega'\, Y^*_{\ell m}(\theta',\varphi')\,V(\theta',\varphi')$ such that

$$\Phi(\vec{x}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_{\ell m}\left(\frac{r}{a}\right)^{\ell} Y_{\ell m}(\theta,\varphi)$$

boundaries based on definition of m

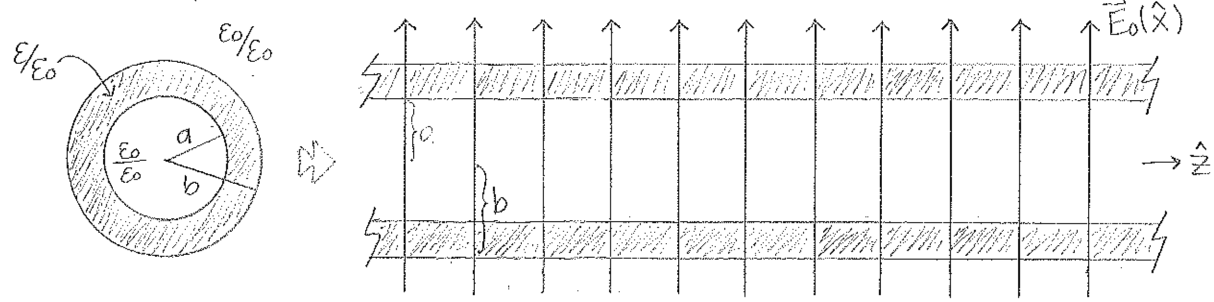
Where this may be seen to be equivalent to

$$\Phi(\vec{x}) = \frac{a(a^2-r^2)}{4\pi}\int \frac{V(\theta',\varphi')}{(r^2+a^2-2ra\cos(\gamma))^{3/2}}\,d\Omega'$$

as they are both solutions to the general equation for the potential inside the sphere

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')G_D(\vec{x},\vec{x}')\,d^3x' - \frac{1}{4\pi}\oint_S \Phi(\vec{x}')\frac{\partial G_D}{\partial n'}\,da'$$

2. Jackson Problem 4.8



(a) – The most intuitive choice is cylindrical coordinates

– We can neglect end effects ⇒ focus on r & φ

Begin by constructing the general solution for Laplace's equation in two-dimensional polar coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \varphi^2}=0$$

↳ Separation of Variables → $\Phi(r,\varphi)=R(r)\Psi(\varphi)$

– Choose positive constant for R because we do not expect oscillatory behavior in r

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)=n^2 \quad , \quad \frac{1}{\Psi}\frac{d^2\Psi}{d\varphi^2}=-n^2$$

↳ $R(r)=A_n r^n + B_n r^{-n}$ ↳ $\Psi(\varphi)=C_n\cos(n\varphi)+D_n\sin(n\varphi)$

such that

$$\Phi(r,\varphi)=\sum_n\{(A_n r^n+B_n r^{-n})[C_n\cos(n\varphi)+D_n\sin(n\varphi)]\}$$

Three regions of interest: $r<a$, $a<r<b$, $r>b$

• $r<a$

Because this region includes the origin, the coefficients for the r^{-n} term, the B_n's, must go to zero to keep the solution from blowing up.

$$\Phi(r<a,\varphi)=\sum_n r^n[C_n\cos(n\varphi)+D_n\sin(n\varphi)]$$

• $r>b$

Similarly, the coefficients for the r^n term, the A_n's, must go to zero in this region to keep the solution from blowing up at large r. A term must also be added here to account for the contribution from the external electric field – the uniform field we expect very far away from the cylinder: $-E_0 r\cos(\varphi)$ ($\vec{E}=-\nabla\Phi$)

$$\Phi(r>b,\varphi)=-E_0 r\cos(\varphi)+\sum_n r^{-n}[E_n\cos(n\varphi)+F_n\sin(n\varphi)]$$

- $a<r<b$

The best treatment here would be separate treatment of the r^n and r^{-n} terms to more easily see the impact of the boundary conditions on the constants of integration

$$\Phi(r,\varphi)=\begin{cases}\sum_n r^n[C_n\cos(n\varphi)+D_n\sin(n\varphi)] & ,\ r<a\\ \sum_n\{r^n[G_n\cos(n\varphi)+H_n\sin(n\varphi)]+r^{-n}[I_n\cos(n\varphi)+J_n\sin(n\varphi)]\} & ,\ a<r<b\\ -E_0 r\cos(\varphi)+\sum_n r^{-n}[E_n\cos(n\varphi)+F_n\sin(n\varphi)] & ,\ r>b\end{cases}$$

→ Use the boundary conditions at $r=a$ and $r=b$ to solve for the unknown constants of integration

- The normal boundary condition @ $r=a$

$$\left.\frac{\partial\Phi}{\partial r}\right|_{a-0}=\frac{\varepsilon}{\varepsilon_0}\left.\frac{\partial\Phi}{\partial r}\right|_{a+0}$$

$$\sum_n \cancel{n}\cancel{a^{n-1}}[C_n\cos(n\varphi)+D_n\sin(n\varphi)]=\frac{\varepsilon}{\varepsilon_0}\sum_n\{\cancel{n}\cancel{a^{n-1}}[G_n\cos(n\varphi)+H_n\sin(n\varphi)]$$
$$-\cancel{n}a^{\cancel{-n-1}\,-2n}[I_n\cos(n\varphi)+J_n\sin(n\varphi)]\}$$

$$\sum_n[C_n\cos(n\varphi)+D_n\sin(n\varphi)]=\frac{\varepsilon}{\varepsilon_0}\sum_n\{[G_n\cos(n\varphi)+H_n\sin(n\varphi)]-a^{-2n}[I_n\cos(n\varphi)+J_n\sin(n\varphi)]\}$$

↳ coefficients of sines & cosines must be treated independently

$$\frac{\varepsilon_0}{\varepsilon}C_n=G_n-I_n a^{-2n}$$
$$\frac{\varepsilon_0}{\varepsilon}D_n=H_n-J_n a^{-2n}$$

- The tangential boundary condition @ $r=a$

$$\left.\frac{\partial\Phi}{\partial\varphi}\right|_{a-0}=\left.\frac{\partial\Phi}{\partial\varphi}\right|_{a+0}$$

$$\sum_n \cancel{n}\cancel{a^{n}}[-C_n\sin(n\varphi)+D_n\cos(n\varphi)]=\sum_n\{\cancel{n}\cancel{a^{n}}[-G_n\sin(n\varphi)+H_n\cos(n\varphi)]$$
$$+\cancel{n}a^{\cancel{-n}\,-2n}[-I_n\sin(n\varphi)+J_n\cos(n\varphi)]\}$$

$$\sum_n[-C_n\sin(n\varphi)+D_n\cos(n\varphi)]=\sum_n\{[-G_n\sin(n\varphi)+H_n\cos(n\varphi)]+a^{-2n}[-I_n\sin(n\varphi)+J_n\cos(n\varphi)]\}$$

$$C_n=G_n+I_n a^{-2n}$$
$$D_n=H_n+J_n a^{-2n}$$

- The normal boundary condition @ $r=b$

$$\frac{\varepsilon}{\varepsilon_0}\left.\frac{\partial\Phi}{\partial r}\right|_{b-0}=\left.\frac{\partial\Phi}{\partial r}\right|_{b+0}$$

$$\frac{\varepsilon}{\varepsilon_0}\sum_n\{\cancel{n}b^{\cancel{n-1}\,2n}[G_n\cos(n\varphi)+H_n\sin(n\varphi)]-\cancel{n}\cancel{b^{-n-1}}[I_n\cos(n\varphi)+J_n\sin(n\varphi)]\}$$
$$=-{}_{\wedge}E_0\cos(\varphi)-\sum_n\cancel{n}\cancel{b^{-n-1}}[E_n\cos(n\varphi)+F_n\sin(n\varphi)]$$

$*\frac{1}{n}b^{2n}$ ← however, based on the origin of this term, it is only valid for $n=1$

$$\frac{\varepsilon}{\varepsilon_0}\sum_n\{b^{2n}[G_n\cos(n\varphi)+H_n\sin(n\varphi)]-[I_n\cos(n\varphi)+J_n\sin(n\varphi)]\}$$
$$=-E_0b^2\cos(\varphi)\delta_{n1}-\sum_n[E_n\cos(n\varphi)+F_n\sin(n\varphi)]$$

$$b^{2n}G_n-I_n=-\frac{\varepsilon_0}{\varepsilon}E_0b^2\delta_{n1}-\frac{\varepsilon_0}{\varepsilon}E_n$$
$$b^{2n}H_n-J_n=-\frac{\varepsilon_0}{\varepsilon}F_n$$

- The tangential boundary conditions @ $r=b$

$$\left.\frac{\partial\Phi}{\partial\varphi}\right|_{b-0}=\left.\frac{\partial\Phi}{\partial\varphi}\right|_{b+0}$$

$$\sum_n\{\cancel{n}b^{\cancel{n}\,2n}[-G_n\sin(n\varphi)+H_n\cos(n\varphi)]+\cancel{nb^{-n}}[-I_n\sin(n\varphi)+J_n\cos(n\varphi)]\}$$
$$=E_0b\sin(\varphi)+\sum_n\cancel{nb^{-n}}[-E_n\sin(n\varphi)+F_n\cos(n\varphi)]$$
$$*\frac{1}{n}b^n$$

$$\sum_n\{b^{2n}[-G_n\sin(n\varphi)+H_n\cos(n\varphi)]+[-I_n\sin(n\varphi)+J_n\cos(n\varphi)]\}$$
$$=E_0b^2\sin(\varphi)\delta_{n1}+\sum_n[-E_n\sin(n\varphi)+F_n\cos(n\varphi)]$$

$$b^{2n}G_n+I_n=-E_0b^2\delta_{n1}+E_n$$
$$b^{2n}H_n+J_n=F_n$$

→ This yields a total of eight equations for our eight unknown constants

(i) $\frac{\varepsilon_0}{\varepsilon}C_n=G_n-I_na^{-2n}$

(ii) $\frac{\varepsilon_0}{\varepsilon}D_n=H_n-J_na^{-2n}$

(iii) $C_n=G_n+I_na^{-2n}$

(iv) $D_n=H_n+J_na^{-2n}$

(v) $b^{2n}G_n-I_n=-\frac{\varepsilon_0}{\varepsilon}E_0b^2\delta_{n1}-\frac{\varepsilon_0}{\varepsilon}E_n$ $\Rightarrow$ (ix) $b^{2n}G_n-I_n=-\frac{\varepsilon_0}{\varepsilon}E_n,\ n\neq 1$

(vi) $b^{2n}H_n-J_n=-\frac{\varepsilon_0}{\varepsilon}F_n$

(vii) $b^{2n}G_n+I_n=-E_0b^2\delta_{n1}+E_n$ $\Rightarrow$ (x) $b^{2n}G_n+I_n=E_n,\ n\neq 1$

(viii) $b^{2n}H_n+J_n=F_n$

- For equations valid for all n (ii, iv, vi, and viii)

$$\frac{\varepsilon_0}{\varepsilon}D_n=H_n-J_na^{-2n}$$
$$D_n=H_n+J_na^{-2n}$$
$$b^{2n}H_n-J_n=-\frac{\varepsilon_0}{\varepsilon}F_n$$
$$b^{2n}H_n+J_n=F_n$$

By inspection, it may be seen here that these four equations can only be satisfied in the case that $D_n=H_n=J_n=F_n=0$ for all n. ✓

→

- For equations valid for $n \neq 1$ (i, iii, ix, and x)

$$\frac{\varepsilon_0}{\varepsilon} C_n = G_n - I_n a^{-2n}$$
$$C_n = G_n + I_n a^{-2n}$$
$$b^{2n} G_n - I_n = -\frac{\varepsilon_0}{\varepsilon} E_n$$
$$b^{2n} G_n + I_n = E_n$$

As above, it may be seen here that these four equations can only be satisfied in the case that $C_n = G_n = I_n = E_n = 0$ for $n \neq 1$

- For equations valid for $n = 1$ (i, iii, v, and vii)

$$\frac{\varepsilon_0}{\varepsilon} C_1 = G_1 - I_1 a^{-2}$$
$$C_1 = G_1 + I_1 a^{-2}$$
$$b^2 G_1 - I_1 = -\frac{\varepsilon_0}{\varepsilon} E_0 b^2 - \frac{\varepsilon_0}{\varepsilon} E_1$$
$$b^2 G_1 + I_1 = -E_0 b^2 + E_1$$

As this does NOT form a degenerate set for which the only solution is the null solution, we can use these equations to solve for our four nonzero constants.

$$C_1 = G_1 + I_1 a^{-2}$$
$$\rightarrow I_1 = a^2 (C_1 - G_1)$$
$$\frac{\varepsilon_0}{\varepsilon} C_1 = G_1 - \cancel{a^2}(C_1 - G_1)\cancel{a^{-2}}$$
$$= 2G_1 - C_1$$
$$2G_1 = C_1\left(1 + \frac{\varepsilon_0}{\varepsilon}\right)$$
$$\rightarrow G_1 = \frac{1}{2} C_1 \left(1 + \frac{\varepsilon_0}{\varepsilon}\right)$$
$$C_1 = \frac{1}{2} C_1 \left(1 + \frac{\varepsilon_0}{\varepsilon}\right) + I_1 a^{-2}$$
$$2C_1 = C_1 + \frac{\varepsilon_0}{\varepsilon} C_1 + 2 I_1 a^{-2}$$
$$C_1\left(1 - \frac{\varepsilon_0}{\varepsilon}\right) = 2 I_1 a^{-2}$$
$$\rightarrow I_1 = \frac{1}{2} C_1 a^2 \left(1 - \frac{\varepsilon_0}{\varepsilon}\right)$$
$$-E_1 = E_0 b^2 + \frac{\varepsilon}{\varepsilon_0} b^2 G_1 - \frac{\varepsilon}{\varepsilon_0} I_1$$
$$\underline{E_1 = E_0 b^2 + b^2 G_1 + I_1} \quad +$$
$$0 = 2E_0 b^2 + \underbrace{b^2 G_1 \left(1 + \frac{\varepsilon}{\varepsilon_0}\right) + I_1 \left(1 - \frac{\varepsilon}{\varepsilon_0}\right)}_{\text{substitute in favor of } C_1}$$
$$-\overset{4}{\cancel{2}} E_0 b^2 = b^2 \frac{1}{\cancel{2}} C_1 \left(1 + \frac{\varepsilon_0}{\varepsilon}\right)\left(1 + \frac{\varepsilon}{\varepsilon_0}\right) + \frac{1}{\cancel{2}} C_1 a^2 \left(1 - \frac{\varepsilon_0}{\varepsilon}\right)\left(1 - \frac{\varepsilon}{\varepsilon_0}\right)$$
$$-4E_0 b^2 = b^2 C_1 \left(2 + \frac{\varepsilon_0}{\varepsilon} + \frac{\varepsilon}{\varepsilon_0}\right) + a^2 C_1 \left(2 - \frac{\varepsilon_0}{\varepsilon} - \frac{\varepsilon}{\varepsilon_0}\right) * \frac{\varepsilon\varepsilon_0}{\varepsilon\varepsilon_0}$$
$$= \frac{1}{\varepsilon\varepsilon_0} C_1 \left[b^2 \varepsilon\varepsilon_0 \left(2 + \frac{\varepsilon_0}{\varepsilon} + \frac{\varepsilon}{\varepsilon_0}\right) + a^2 \varepsilon\varepsilon_0 \left(2 - \frac{\varepsilon_0}{\varepsilon} - \frac{\varepsilon}{\varepsilon_0}\right)\right]$$

$$-4\varepsilon\varepsilon_0 E_0 b^2 = C_1\Big[b^2\underbrace{(2\varepsilon\varepsilon_0+\varepsilon_0^2+\varepsilon^2)}_{(\varepsilon+\varepsilon_0)^2}+a^2\underbrace{(2\varepsilon\varepsilon_0-\varepsilon_0^2-\varepsilon^2)}_{-(\varepsilon-\varepsilon_0)^2}\Big]$$

$$\longrightarrow C_1=\frac{-4\varepsilon\varepsilon_0 E_0 b^2}{b^2(\varepsilon+\varepsilon_0)^2-a^2(\varepsilon-\varepsilon_0)^2}$$

$$I_1=\frac{\cancel{1}}{\cancel{2}}\,\frac{-\cancel{4}^{\,2}\varepsilon\varepsilon_0 E_0 b^2}{b^2(\varepsilon+\varepsilon_0)^2-a^2(\varepsilon-\varepsilon_0)^2}\,a^2\left(1-\frac{\varepsilon_0}{\varepsilon}\right)$$

$$=\frac{-2a^2b^2E_0(\varepsilon\varepsilon_0-\varepsilon_0^2)}{b^2(\varepsilon+\varepsilon_0)^2-a^2(\varepsilon-\varepsilon_0)^2}$$

$$\longrightarrow I_1=\frac{-2a^2b^2\varepsilon_0E_0(\varepsilon-\varepsilon_0)}{b^2(\varepsilon+\varepsilon_0)^2-a^2(\varepsilon-\varepsilon_0)^2}$$

$$G_1=\frac{\cancel{1}}{\cancel{2}}\,\frac{-\cancel{4}^{\,2}\varepsilon\varepsilon_0 E_0 b^2}{b^2(\varepsilon+\varepsilon_0)^2-a^2(\varepsilon-\varepsilon_0)^2}\left(1+\frac{\varepsilon_0}{\varepsilon}\right)$$

$$\longrightarrow G_1=\frac{-2b^2\varepsilon_0E_0(\varepsilon+\varepsilon_0)}{b^2(\varepsilon+\varepsilon_0)^2-a^2(\varepsilon-\varepsilon_0)^2}$$

$$E_1=E_0b^2+b^2\,\frac{-2b^2\varepsilon_0E_0(\varepsilon+\varepsilon_0)}{b^2(\varepsilon+\varepsilon_0)^2-a^2(\varepsilon-\varepsilon_0)^2}+\frac{-2a^2b^2\varepsilon_0E_0(\varepsilon-\varepsilon_0)}{b^2(\varepsilon+\varepsilon_0)^2-a^2(\varepsilon-\varepsilon_0)^2}$$

$$=E_0b^2\left[1+\frac{-2\varepsilon_0(\varepsilon+\varepsilon_0)-2a^2\varepsilon_0(\varepsilon-\varepsilon_0)}{b^2(\varepsilon+\varepsilon_0)^2-a^2(\varepsilon-\varepsilon_0)^2}\right]$$

$$=E_0b^2\left[\frac{b^2(\varepsilon+\varepsilon_0)^2-a^2(\varepsilon-\varepsilon_0)^2}{b^2(\varepsilon+\varepsilon_0)^2-a^2(\varepsilon-\varepsilon_0)^2}-\frac{2\varepsilon_0}{b^2(\varepsilon+\varepsilon_0)^2-a^2(\varepsilon-\varepsilon_0)^2}\big(b^2(\varepsilon+\varepsilon_0)+a^2(\varepsilon-\varepsilon_0)\big)\right]$$

$$[b^2(\varepsilon+\varepsilon_0)^2-a^2(\varepsilon-\varepsilon_0)^2]-2\varepsilon_0[b^2(\varepsilon+\varepsilon_0)+a^2(\varepsilon-\varepsilon_0)]$$

$$=b^2(\varepsilon^2+\varepsilon_0^2+2\varepsilon\varepsilon_0)-a^2(\varepsilon^2+\varepsilon_0^2-2\varepsilon\varepsilon_0)-2\varepsilon_0(b^2\varepsilon+b^2\varepsilon_0+a^2\varepsilon-a^2\varepsilon_0)$$

$$=b^2\varepsilon^2+\cancel{b^2\varepsilon_0^2}+\cancel{2\varepsilon\varepsilon_0b^2}-a^2\varepsilon^2-\cancel{a^2\varepsilon_0^2}+\cancel{2\varepsilon\varepsilon_0a^2}-\cancel{2\varepsilon\varepsilon_0b^2}-\cancel{2}\varepsilon_0^2b^2-\cancel{2\varepsilon\varepsilon_0a^2}+\cancel{2}a^2\varepsilon_0^2$$

$$=b^2\varepsilon^2-a^2\varepsilon^2-b^2\varepsilon_0^2+a^2\varepsilon_0^2=-(b^2\varepsilon_0^2-b^2\varepsilon^2-a^2\varepsilon_0^2+a^2\varepsilon^2)$$

$$=-(b^2-a^2)(\varepsilon_0^2-\varepsilon^2)$$

$$\longrightarrow E_1=\frac{-b^2E_0(b^2-a^2)(\varepsilon_0^2-\varepsilon^2)}{b^2(\varepsilon+\varepsilon_0)^2-a^2(\varepsilon-\varepsilon_0)^2}$$

Where $D_n=H_n=J_n=F_n=0$ for all n & $C_n=I_n=G_n=E_n=0$ for all $n\neq 1$

$\longrightarrow$

$\Rightarrow$ So the potential in the three regions of interest can be seen to be:

$$\Phi(r,\varphi) = E_0 \begin{cases} \dfrac{-4\varepsilon\varepsilon_0 b^2}{b^2(\varepsilon+\varepsilon_0)^2 - a^2(\varepsilon-\varepsilon_0)^2}\, r\cos(\varphi) & , r<a \\[2ex] \dfrac{-2\varepsilon_0 b^2(\varepsilon+\varepsilon_0)}{b^2(\varepsilon+\varepsilon_0)^2 - a^2(\varepsilon-\varepsilon_0)^2}\, r\cos(\varphi) - \dfrac{2a^2b^2\varepsilon_0(\varepsilon-\varepsilon_0)}{b^2(\varepsilon+\varepsilon_0)^2 - a^2(\varepsilon-\varepsilon_0)^2}\, \dfrac{1}{r}\cos(\varphi) & , a<r<b \\[2ex] \dfrac{-b^2(b^2-a^2)(\varepsilon_0^2-\varepsilon^2)}{b^2(\varepsilon+\varepsilon_0)^2 - a^2(\varepsilon-\varepsilon_0)^2}\, \dfrac{1}{r}\cos(\varphi) - r\cos(\varphi) & , r>b \end{cases}$$

$\Rightarrow$ The electric field may be found using $\vec{E} = -\nabla\Phi = -\left(\frac{\partial\Phi}{\partial r} + \frac{1}{r}\frac{\partial\Phi}{\partial\varphi}\right)$:

$$\vec{E}(r,\varphi) = E_0 \begin{cases} \dfrac{+4\varepsilon\varepsilon_0 b^2}{b^2(\varepsilon+\varepsilon_0)^2 - a^2(\varepsilon-\varepsilon_0)^2}\left(\cos(\varphi)\hat{r} - \sin(\varphi)\hat{\varphi}\right) & , r<a \\[2ex] \dfrac{+2\varepsilon_0 b^2(\varepsilon+\varepsilon_0)}{b^2(\varepsilon+\varepsilon_0)^2 - a^2(\varepsilon-\varepsilon_0)^2}\left(\cos(\varphi)\hat{r} - \sin(\varphi)\hat{\varphi}\right) + \dfrac{2a^2b^2\varepsilon_0(\varepsilon-\varepsilon_0)}{b^2(\varepsilon-\varepsilon_0)^2 - a^2(\varepsilon-\varepsilon_0)^2}\left(\dfrac{-1}{r^2}\cos(\varphi)\hat{r} - \dfrac{1}{r^2}\sin(\varphi)\hat{\varphi}\right) & , a<r<b \\[2ex] \dfrac{+b^2(b^2-a^2)(\varepsilon_0^2-\varepsilon^2)}{b^2(\varepsilon+\varepsilon_0)^2 - a^2(\varepsilon-\varepsilon_0)^2}\left(\dfrac{-1}{r^2}\cos(\varphi)\hat{r} - \dfrac{1}{r^2}\sin(\varphi)\hat{\varphi}\right) + \left(\cos(\varphi)\hat{r} - \sin(\varphi)\hat{\varphi}\right) & , r>b \end{cases}$$

(b) For the typical case of $b \simeq 2a$:

$$\vec{E}(r,\varphi) = E_0 \begin{cases} \dfrac{16\varepsilon\varepsilon_0 \cancel{a^2}}{\cancel{a^2}[4(\varepsilon+\varepsilon_0)^2 - (\varepsilon-\varepsilon_0)^2]}\left(\cos(\varphi)\hat{r} - \sin(\varphi)\hat{\varphi}\right) & , r<a \\[2ex] \dfrac{8\varepsilon_0 \cancel{a^2}(\varepsilon+\varepsilon_0)}{\cancel{a^2}[4(\varepsilon+\varepsilon_0)^2 - (\varepsilon-\varepsilon_0)^2]}\left(\cos(\varphi)\hat{r} - \sin(\varphi)\hat{\varphi}\right) - \dfrac{8\cancel{a^2}a^2\varepsilon_0(\varepsilon-\varepsilon_0)}{\cancel{a^2}[4(\varepsilon+\varepsilon_0)^2 - (\varepsilon-\varepsilon_0)^2]}\dfrac{1}{r^2}\left(\cos(\varphi)\hat{r} + \sin(\varphi)\hat{\varphi}\right) & , a<r<b \\[2ex] \dfrac{-4\cancel{a^2}(4a^2-a^2)(\varepsilon_0^2-\varepsilon^2)}{\cancel{a^2}[4(\varepsilon+\varepsilon_0)^2 - (\varepsilon-\varepsilon_0)^2]}\dfrac{1}{r^2}\left(\cos(\varphi)\hat{r} + \sin(\varphi)\hat{\varphi}\right) + \left(\cos(\varphi)\hat{r} - \sin(\varphi)\hat{\varphi}\right) & , r>b \end{cases}$$

$$= E_0 \begin{cases} \dfrac{16\varepsilon\varepsilon_0}{4(\varepsilon+\varepsilon_0)^2 - (\varepsilon-\varepsilon_0)^2}\left(\cos(\varphi)\hat{r} - \sin(\varphi)\hat{\varphi}\right) & , r<a \\[2ex] \dfrac{8\varepsilon_0(\varepsilon+\varepsilon_0)}{4(\varepsilon+\varepsilon_0)^2 - (\varepsilon-\varepsilon_0)^2}\left(\cos(\varphi)\hat{r} - \sin(\varphi)\hat{\varphi}\right) - \dfrac{8a^2\varepsilon(\varepsilon-\varepsilon_0)}{4(\varepsilon+\varepsilon_0)^2 - (\varepsilon-\varepsilon_0)^2}\dfrac{1}{r^2}\left(\cos(\varphi)\hat{r} + \sin(\varphi)\hat{\varphi}\right) & , a<r<b \\[2ex] \dfrac{-12a^2(\varepsilon_0^2-\varepsilon^2)}{4(\varepsilon+\varepsilon_0)^2 - (\varepsilon-\varepsilon_0)^2}\dfrac{1}{r^2}\left(\cos(\varphi)\hat{r} + \sin(\varphi)\hat{\varphi}\right) + \left(\cos(\varphi)\hat{r} - \sin(\varphi)\hat{\varphi}\right) & , r>b \end{cases}$$

- In the region $r<a$

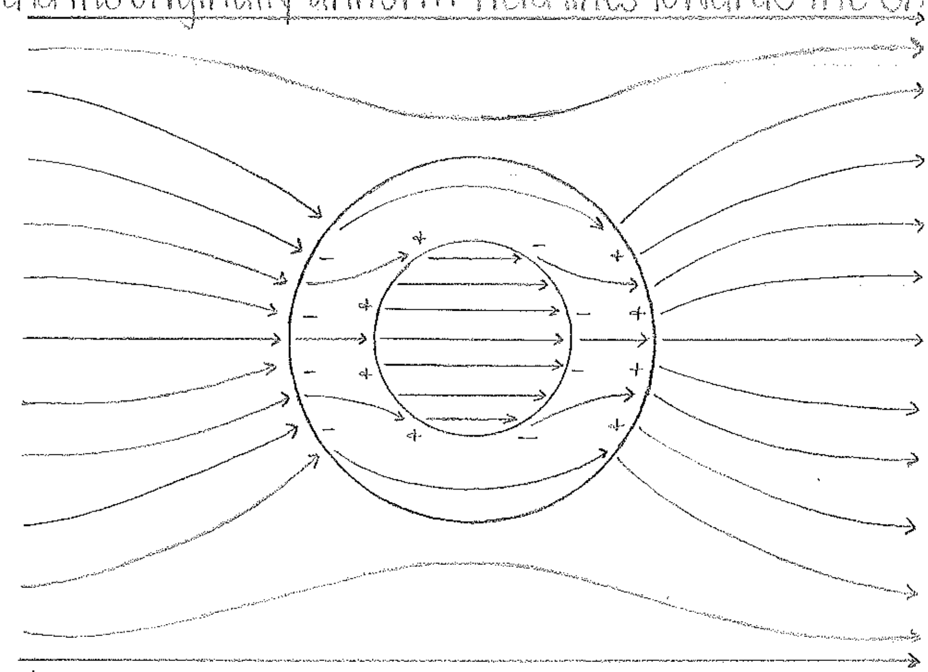
  The field is uniform in the interior of the cylindrical shell

- In the region $a<r<b$

  Charges would build up on the perimeter within the material, reducing the number density of field lines here compared to the area around the cylinder and the hollow within the cylinder

- In the region $r>b$

The charges built up on the perimeter of the cylindrical shell will bend the originally uniform field lines towards the shell.



└ for larger r, the field lines are once again uniform, as discussed in part (a)

(c) For a solid dielectric cylinder in a uniform field: $a \to 0$

$$\vec{E}(r,\varphi) = E_0 \begin{cases} \dfrac{2\varepsilon_0 \cancel{b^2}(\varepsilon+\varepsilon_0)}{\cancel{b^2}(\varepsilon+\varepsilon_0)^{\cancel{2}}}(\cos(\varphi)\hat{r} - \sin(\varphi)\hat{\varphi}) + 0 & , r<b \\ \dfrac{-\cancel{b^2}(b^2)(\varepsilon_0^2-\varepsilon^2)}{\cancel{b^2}(\varepsilon+\varepsilon_0)^2}\dfrac{1}{r^2}(\cos(\varphi)\hat{r} + \sin(\varphi)\hat{\varphi}) + (\cos(\varphi)\hat{r} - \sin(\varphi)\hat{\varphi}) & , r>b \end{cases}$$

$$= E_0 \begin{cases} \dfrac{2\varepsilon_0}{(\varepsilon+\varepsilon_0)}(\cos(\varphi)\hat{r} - \sin(\varphi)\hat{\varphi}) & , r<b \\ \dfrac{-(\varepsilon_0^2-\varepsilon^2)}{(\varepsilon+\varepsilon_0)^2}\left(\dfrac{b}{r}\right)^2(\cos(\varphi)\hat{r} + \sin(\varphi)\hat{\varphi}) + (\cos(\varphi)\hat{r} - \sin(\varphi)\hat{\varphi}) & , r>b \end{cases}$$

→

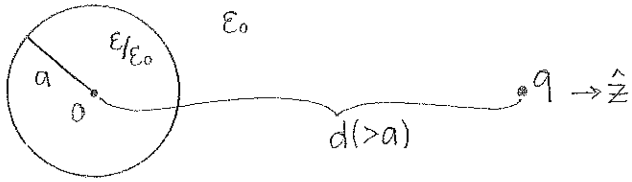
For a cylindrical cavity in a uniform dielectric: $b \to \infty$

- for all terms, the denominator $b^2(\varepsilon+\varepsilon_0)^2 - a^2(\varepsilon-\varepsilon_0)^2 \approx b^2(\varepsilon+\varepsilon_0)^2$ in this case as the a^2-contribution is negligible.

$$\vec{E}(r,\varphi) = E_0 \begin{cases} \dfrac{4\varepsilon\varepsilon_0 \cancel{b^2}}{\cancel{b^2}(\varepsilon+\varepsilon_0)^2}(\cos(\varphi)\hat{r} - \sin(\varphi)\hat{\varphi}) & , r<a \\ \dfrac{2\varepsilon_0 \cancel{b^2}\cancel{(\varepsilon+\varepsilon_0)}}{\cancel{b^2}(\varepsilon+\varepsilon_0)^{\cancel{2}}}(\cos(\varphi)\hat{r} - \sin(\varphi)\hat{\varphi}) - \dfrac{2a^2 \cancel{b^2}\varepsilon_0(\varepsilon-\varepsilon_0)}{\cancel{b^2}(\varepsilon+\varepsilon_0)^2}\dfrac{1}{r^2}(\cos(\varphi)\hat{r} + \sin(\varphi)\hat{\varphi}) & , r>a \end{cases}$$

$$= E_0 \begin{cases} \dfrac{4\varepsilon\varepsilon_0}{(\varepsilon+\varepsilon_0)^2}(\cos(\varphi)\hat{r} - \sin(\varphi)\hat{\varphi}) & , r<a \\ \dfrac{2\varepsilon_0}{(\varepsilon+\varepsilon_0)}(\cos(\varphi)\hat{r} - \sin(\varphi)\hat{\varphi}) - \dfrac{2\varepsilon_0(\varepsilon-\varepsilon_0)}{(\varepsilon+\varepsilon_0)^2}\left(\dfrac{a}{r}\right)^2(\cos(\varphi)\hat{r} + \sin(\varphi)\hat{\varphi}) & , r>a \end{cases}$$

3. Jackson Problem 4.9



(a) No free charges within the sphere: $\nabla\cdot\vec{D}=0$

↳ uniform permittivity within the sphere: $\nabla\cdot\vec{E}=\nabla\cdot(\vec{D}\frac{1}{\varepsilon})=0$ also

⇒ Polarization charge exists only on the surface of the sphere

$\Phi(r,\theta)=\sum_\ell A_\ell r^\ell P_\ell\cos(\theta)$ for $r<a$ (à la Jackson eq. 3.33)

In the area external to the sphere, the potential has contributions from both the polarization charge on the surface of the sphere and the point charge.

- contribution from the polarization charge

$\Phi(r,\theta)=\sum_\ell B_\ell r^{-(\ell+1)}P_\ell\cos(\theta)$ for $r>a$ (à la Jackson eq. 3.33)

- contribution from the point charge

$$\Phi(r,\theta)=\frac{q}{4\pi\varepsilon_0}\begin{cases}\sum_\ell \frac{r^\ell}{d^{\ell+1}}P_\ell\cos(\theta) & , a<r<d\\ \sum_\ell \frac{d^\ell}{r^{\ell+1}}P_\ell\cos(\theta) & , r>d\end{cases}$$

↑ from Jackson pg. 104, where $P_\ell\cos(\alpha)=1$ (azimuthally symmetric)

Combining these gives us three regions of interest:

$$\Phi(r,\theta)=\begin{cases}\sum_\ell A_\ell r^\ell P_\ell\cos(\theta) & , r<a\\ \sum_\ell\left(B_\ell r^{-(\ell+1)}+\frac{q}{4\pi\varepsilon_0}\frac{r^\ell}{d^{\ell+1}}\right)P_\ell\cos(\theta) & , a<r<d\\ \sum_\ell\left(B_\ell+\frac{q}{4\pi\varepsilon_0}d^\ell\right)r^{-(\ell+1)}P_\ell\cos(\theta) & , r>d\end{cases}$$

→ Use the boundary conditions @ $r=a$ to solve for the unknown constants, A_ℓ & B_ℓ

- The normal boundary condition @ $r=a$

$$\frac{\varepsilon}{\varepsilon_0}\frac{\partial\Phi}{\partial r}\bigg|_{a-0}=\frac{\partial\Phi}{\partial r}\bigg|_{a+0}$$

$$\frac{\varepsilon}{\varepsilon_0}\cancel{\ell}A_\ell\cancel{a^{\ell-1}}\cancel{P_\ell\cos(\theta)}=\left[\frac{-(\ell+1)}{\ell}B_\ell a^{\overset{(2\ell+1)}{\cancel{-(\ell+2)}}}+\frac{\cancel{\ell}q}{4\pi\varepsilon_0}\frac{\cancel{a^{\ell-1}}}{d^{\ell+1}}\right]\cancel{P_\ell\cos(\theta)}$$

$$A_\ell = \frac{\varepsilon_0}{\varepsilon}\left[\frac{-(\ell+1)}{\ell} B_\ell a^{-(2\ell+1)} + \frac{q}{4\pi\varepsilon_0}\frac{1}{d^{\ell+1}}\right]$$

• The tangential boundary condition @ r=a

$$\left.\frac{\partial\Phi}{\partial\theta}\right|_{a-0} = \left.\frac{\partial\Phi}{\partial\theta}\right|_{a+0}$$

$$A_\ell a^\ell = B_\ell a^{-(\ell+1)} + \frac{q}{4\pi\varepsilon_0}\frac{a^\ell}{d^{\ell+1}}$$

$$B_\ell a^{-(\ell+1)} = A_\ell a^\ell - \frac{q}{4\pi\varepsilon_0}\frac{a^\ell}{d^{\ell+1}}$$

$$B_\ell = a^{2\ell+1}\left(A_\ell - \frac{q}{4\pi\varepsilon_0}\frac{1}{d^{\ell+1}}\right)$$

$$A_\ell = \frac{\varepsilon_0}{\varepsilon}\left[\frac{-(\ell+1)}{\ell}\cancel{a^{2\ell+1}}\left(A_\ell - \frac{q}{4\pi\varepsilon_0}\frac{1}{d^{\ell+1}}\right)\cancel{a^{-(2\ell+1)}} + \frac{q}{4\pi\varepsilon_0}\frac{1}{d^{\ell+1}}\right]$$

$$= \frac{\varepsilon_0}{\varepsilon}\left[\frac{-(\ell+1)}{\ell}A_\ell + \frac{(\ell+1)}{\ell d^{\ell+1}}\frac{q}{4\pi\varepsilon_0} + \frac{q}{4\pi\varepsilon_0}\frac{1}{d^{\ell+1}}\right]$$

$$A_\ell\left[1 + \frac{\varepsilon_0(\ell+1)}{\varepsilon\ell}\right] = \frac{q}{4\pi\varepsilon}\left[\frac{\ell+1}{\ell d^{\ell+1}} + \frac{\ell}{\ell d^{\ell+1}}\right] \quad * \frac{\varepsilon}{\varepsilon_0}$$

$$A_\ell\left[\frac{\varepsilon}{\varepsilon_0} + \frac{\ell+1}{\ell}\right] = \frac{q}{4\pi\varepsilon_0}\left(\frac{2\ell+1}{\ell d^{\ell+1}}\right)$$

$$\longrightarrow A_\ell = \frac{\frac{q}{4\pi\varepsilon_0}\left(\frac{2\ell+1}{\ell d^{\ell+1}}\right)}{\left(\frac{\varepsilon}{\varepsilon_0} + \frac{\ell+1}{\ell}\right)}$$

$$B_\ell = a^{2\ell+1}\left[\frac{\frac{q}{4\pi\varepsilon_0}\left(\frac{2\ell+1}{\ell d^{\ell+1}}\right)}{\left(\frac{\varepsilon}{\varepsilon_0} + \frac{\ell+1}{\ell}\right)}\right] - \frac{q}{4\pi\varepsilon_0}\frac{a^{2\ell+1}}{d^{\ell+1}} \quad * \frac{\left(\frac{\varepsilon}{\varepsilon_0} + \frac{\ell+1}{\ell}\right)}{\left(\frac{\varepsilon}{\varepsilon_0} + \frac{\ell+1}{\ell}\right)}$$

$$= \left[\frac{\frac{q}{4\pi\varepsilon_0}\frac{a^{2\ell+1}}{d^{\ell+1}}}{\left(\frac{\varepsilon}{\varepsilon_0} + \frac{\ell+1}{\ell}\right)}\right]\left(\frac{2\ell+1}{\ell} - \frac{\varepsilon}{\varepsilon_0} - \frac{\ell+1}{\ell}\right)$$

$$\longrightarrow B_\ell = \left[\frac{\frac{q}{4\pi\varepsilon_0}\frac{a^{2\ell+1}}{d^{\ell+1}}}{\left(\frac{\varepsilon}{\varepsilon_0} + \frac{\ell+1}{\ell}\right)}\right]\left(1 - \frac{\varepsilon}{\varepsilon_0}\right)$$

$\longrightarrow$

So in all three regions of interest, we know the potential to be:

$$\Phi(r,\theta)=\begin{cases}\sum_\ell \frac{q}{4\pi\varepsilon_0}\left(\frac{2\ell+1}{\ell d^{\ell+1}}\right)\left(\frac{\varepsilon}{\varepsilon_0}+\frac{\ell+1}{\ell}\right)^{-1} r^\ell P_\ell\cos(\theta) & , r<a \\ \sum_\ell\left[\left(1-\frac{\varepsilon}{\varepsilon_0}\right)\frac{q}{4\pi\varepsilon_0}\frac{a^{2\ell+1}}{d^{\ell+1}}\left(\frac{\varepsilon}{\varepsilon_0}+\frac{\ell+1}{\ell}\right)^{-1} r^{-(\ell+1)}+\frac{q}{4\pi\varepsilon_0}\frac{r^\ell}{d^{\ell+1}}\right]P_\ell\cos(\theta) & , a<r<d \\ \sum_\ell\left[\left(1-\frac{\varepsilon}{\varepsilon_0}\right)\frac{q}{4\pi\varepsilon_0}\frac{a^{2\ell+1}}{d^{\ell+1}}\left(\frac{\varepsilon}{\varepsilon_0}+\frac{\ell+1}{\ell}\right)^{-1}+\frac{q}{4\pi\varepsilon_0}d^\ell\right]r^{-(\ell+1)}P_\ell\cos(\theta) & , r>d\end{cases}$$

(b) Near the center of the sphere, $\Phi(r<a,\theta)\to r\ll d\to \frac{r}{d}\ll 1$

$$\Phi(r<a,\theta)=\sum_\ell A_\ell r^\ell P_\ell\cos(\theta)$$

$$\hookrightarrow A_\ell=\frac{q}{4\pi\varepsilon_0}\left(\frac{2\ell+1}{\ell d^{\ell+1}}\right)\left(\frac{\varepsilon}{\varepsilon_0}+\frac{\ell+1}{\ell}\right)^{-1}$$

$$=A_1 r P_1\cos(\theta)+A_2 r^2 P_2\cos(\theta)+A_3 r^3 P_3\cos(\theta)+\ldots$$

$$A_1=\frac{q}{4\pi\varepsilon_0}\left(\frac{3}{d^2}\right)\left(\frac{\varepsilon}{\varepsilon_0}+2\right)^{-1}*\frac{\varepsilon_0}{\varepsilon_0}=\frac{q}{4\pi\varepsilon_0}\frac{3\varepsilon_0}{d^2(\varepsilon+2\varepsilon_0)},\quad A_1 r\propto\frac{r}{d^2}\quad\to P_1\cos(\theta)=\cos(\theta)$$

$$\left.\begin{aligned}A_2&=\frac{q}{4\pi\varepsilon_0}\left(\frac{5}{2d^3}\right)\left(\frac{\varepsilon}{\varepsilon_0}+\frac{3}{2}\right)^{-1}*\frac{2\varepsilon_0}{2\varepsilon_0}=\frac{q}{4\pi\varepsilon_0}\frac{5\varepsilon_0}{d^3(2\varepsilon+3\varepsilon_0)},\quad A_2r^2\propto\frac{r^2}{d^3}\\ A_3&=\frac{q}{4\pi\varepsilon_0}\left(\frac{7}{3d^4}\right)\left(\frac{\varepsilon}{\varepsilon_0}+\frac{4}{3}\right)^{-1}*\frac{3\varepsilon_0}{3\varepsilon_0}=\frac{q}{4\pi\varepsilon_0}\frac{7\varepsilon_0}{d^4(3\varepsilon+4\varepsilon_0)},\quad A_3r^3\propto\frac{r^3}{d^4}\end{aligned}\right\}\text{ because } \frac{r}{d}\ll 1\text{, these higher-order terms are negligible}$$

$$\Phi(r<a,\theta)\approx\frac{q}{4\pi\varepsilon_0}\frac{3\varepsilon_0}{d^2(\varepsilon+2\varepsilon_0)}\underbrace{r\cos(\theta)}_{\to\, r\cos(\theta)=z \text{ in Cartesian}}$$

$$\approx\frac{q}{4\pi\varepsilon_0}\frac{3\varepsilon_0}{d^2(\varepsilon+2\varepsilon_0)}z$$

$$\vec{E}=-\nabla\Phi$$

$$\vec{E}(\vec{x})=-\frac{q}{4\pi\varepsilon_0}\frac{3\varepsilon_0}{d^2(\varepsilon+2\varepsilon_0)}\hat{z}$$

(c) In the limit $\frac{\varepsilon}{\varepsilon_0}\to\infty$

$$A_\ell=\frac{q}{4\pi\varepsilon_0}\left(\frac{2\ell+1}{\ell d^{\ell+1}}\right)\left(\overbrace{\frac{\varepsilon}{\varepsilon_0}+\frac{\ell+1}{\ell}}^{\infty}\right)^{-1}\to 0$$

$$B_\ell=\overset{0}{\cancel{A_\ell}}a^{2\ell+1}-\frac{q}{4\pi\varepsilon_0}\frac{a^{2\ell+1}}{d^{\ell+1}}\longrightarrow-\frac{q}{4\pi\varepsilon_0}\frac{a^{2\ell+1}}{d^{\ell+1}}$$

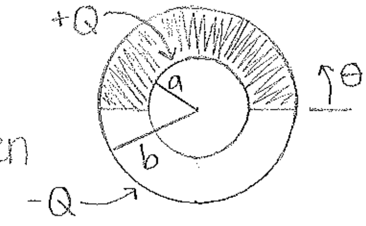
$$\Phi(r,\theta) = \sum_{\ell}\left(-\frac{q}{4\pi\varepsilon_0}\frac{a^{2\ell+1}}{d^{\ell+1}}r^{-(\ell+1)} + \frac{q}{4\pi\varepsilon_0}\frac{r^{\ell}}{d^{\ell+1}}\right)P_\ell\cos(\theta)$$

$$= \left(\frac{-q}{4\pi\varepsilon_0}\frac{a}{d}\right)\sum_{\ell}\underbrace{\left(\frac{a^2}{d}\right)}_{z}^{\ell}\frac{1}{r^{\ell+1}}P_\ell\cos(\theta)$$

This is the potential for a point charge, qa/d, located at $z = a^2/d$, analogous to the image charge calculation for a point charge near a <u>conducting sphere</u> as done in Jackson chapter 2, section 2.

4. Jackson Problem 4.10

Two concentric conducting spheres of inner and outer radii a and b, respectively, carry charges $\pm Q$. The empty space between the spheres is half-filled by a hemispherical shell of dielectric constant $\varepsilon/\varepsilon_0$



(a). $$\Phi(r,\theta)=\begin{cases}\frac{1}{4\pi\varepsilon}\sum_\ell\left(A_\ell r^\ell+B_\ell r^{-(\ell+1)}\right)P_\ell\cos(\theta) & , \quad 0<\theta<\pi \\ \frac{1}{4\pi\varepsilon_0}\sum_\ell\left(C_\ell r^\ell+D_\ell r^{-(\ell+1)}\right)P_\ell\cos(\theta) & , \quad \pi<\theta<2\pi\end{cases}$$

Where the electric potential on the surfaces of each sphere is constant

$$\Phi_\varepsilon(a,\theta)=\Phi_{\varepsilon_0}(a,\theta)=V_a$$
$$\Phi_\varepsilon(b,\theta)=\Phi_{\varepsilon_0}(b,\theta)=V_b$$
V_a & V_b are functions of Q

→ Use boundary conditions @ $\theta=\pi$ to solve for the unknown constants

- The normal boundary condition @ $\theta=\pi$

$$\left.\frac{\partial\Phi}{\partial r}\right|_{\pi-0}=\left.\frac{\partial\Phi}{\partial r}\right|_{\pi+0}$$

$$\sum_\ell\left(\ell A_\ell r^{\ell-1}-(\ell+1)B_\ell r^{-(\ell+2)}\right)P_\ell(0)=\sum_\ell\left(\ell C_\ell r^{\ell-1}-(\ell+1)D_\ell r^{-(\ell+2)}\right)P_\ell(0)$$
$$\sum_\ell\left(\ell(A_\ell-C_\ell)r^{\ell-1}P_\ell(0)-(\ell+1)(B_\ell-D_\ell)r^{-(\ell+2)}P_\ell(0)\right)=0$$

- The tangential boundary condition @ $\theta=\pi$

$$\frac{\varepsilon_0}{\varepsilon}\left.\frac{\partial\Phi}{\partial\theta}\right|_{\pi+0}=\left.\frac{\partial\Phi}{\partial\theta}\right|_{\pi-0}$$

$$\frac{\varepsilon_0}{\varepsilon}\sum_\ell\left(A_\ell r^\ell+B_\ell r^{-(\ell+1)}\right)P_\ell'(0)=\sum_\ell\left(C_\ell r^\ell+D_\ell r^{-(\ell+1)}\right)P_\ell'(0)$$
$$\sum_\ell\left[\left(\frac{\varepsilon_0}{\varepsilon}A_\ell-C_\ell\right)r^\ell P_\ell'(0)+\left(\frac{\varepsilon_0}{\varepsilon}B_\ell-D_\ell\right)r^{-(\ell+1)}P_\ell'(0)\right]=0$$

In order for these equations to hold for all $a<r<b$, the coefficients of r must vanish identically for both equations.

- for even values of ℓ, $P_\ell'(0)$ vanishes & the tangential condition is taken care of
- for odd values of ℓ (and $\ell=0$), $P_\ell'(0)$ does <u>not</u> vanish and we must address the tangential condition

$$\sum_\ell\Big[\underbrace{\left(\frac{\varepsilon_0}{\varepsilon}A_\ell-C_\ell\right)}r^\ell P_\ell'(0)+\underbrace{\left(\frac{\varepsilon_0}{\varepsilon}B_\ell-D_\ell\right)}r^{-(\ell+1)}P_\ell'(0)=0$$

these must $=0$ for this statement to hold true

$$\frac{\varepsilon_0}{\varepsilon}A_\ell=C_\ell\,,\quad\frac{\varepsilon_0}{\varepsilon}B_\ell=D_\ell$$

✓

By orthogonality of the Legendre Polynomials and the consistency requirements above, we can see that the only P_ℓ that can be in the dielectric-filled region AND the free space region is P_0. Thus, the only acceptable ℓ-value that satisfies the normal and tangential boundary conditions on $\Phi(r,\theta)$ and the orthogonality of Legendre Polynomials is $\ell=0$.

$$\Phi(r,\theta)=\begin{cases}\frac{1}{4\pi\varepsilon}\left(A_0\cancelto{1}{r^0}+\frac{B_0}{r^{(0+1)}}\right)\cancelto{1}{P_0\cos(\theta)}, & 0<\theta<\pi\\[2ex] \frac{1}{4\pi\varepsilon_0}\left(C_0\cancelto{1}{r^0}+\frac{D_0}{r^{(0+1)}}\right)\cancelto{1}{P_0\cos(\theta)}, & \pi<\theta<2\pi\end{cases}$$

$$=\begin{cases}\frac{1}{4\pi\varepsilon}\left(A_0+\frac{B_0}{r}\right), & 0<\theta<\pi\\[2ex] \frac{1}{4\pi\varepsilon_0}\left(C_0+\frac{D_0}{r}\right), & \pi<\theta<2\pi\end{cases}$$

$$\vec{E}(r,\theta)=-\nabla\Phi(r,\theta)=\begin{cases}\frac{-1}{4\pi\varepsilon}\frac{B_0}{r^2}\hat{r}, & 0<\theta<\pi\\[2ex] \frac{-1}{4\pi\varepsilon_0}\frac{D_0}{r^2}\hat{r}, & \pi<\theta<2\pi\end{cases}$$

From this, we can see that we only actually need to solve for the constants B_0 and D_0.

→ use the potential difference between the inner and outer conducting spheres

- for B_0 (valid in the dielectric hemisphere)

$$V_a-V_b=\Phi_\varepsilon(a,\theta)-\Phi_\varepsilon(b,\theta)$$

$$=\left(\frac{1}{4\pi\varepsilon}\frac{B_0}{a}\right)-\left(\frac{1}{4\pi\varepsilon}\frac{B_0}{b}\right)$$

$$=\frac{B_0}{4\pi\varepsilon}\left(\frac{1}{a}-\frac{1}{b}\right) \quad * ab$$

$$ab(V_a-V_b)=\frac{B_0}{4\pi\varepsilon}(b-a)$$

$$\rightarrow B_0=\frac{4\pi\varepsilon ab(V_a-V_b)}{(b-a)}$$

→

• then, as found earlier, $D_\ell = \frac{\varepsilon_0}{\varepsilon} B_\ell$

$$D_0 = \frac{\varepsilon_0}{\cancel{\varepsilon}}\left[\frac{4\pi\cancel{\varepsilon}ab(V_a - V_b)}{(b-a)}\right]$$

$$\rightarrow D_0 = \frac{4\pi\varepsilon_0 ab(V_a - V_b)}{(b-a)}$$

$$\vec{E}(r,\theta) = \begin{cases} \frac{-1}{4\pi\varepsilon}\left[\frac{4\pi\varepsilon ab(V_a - V_b)}{(b-a)}\right]\frac{1}{r^2}\hat{r}, & 0<\theta<\pi \\ \frac{-1}{4\pi\varepsilon_0}\left[\frac{4\pi\varepsilon_0 ab(V_a - V_b)}{(b-a)}\right]\frac{1}{r^2}\hat{r}, & \pi<\theta<2\pi \end{cases}$$

$$= \frac{-ab(V_a - V_b)}{r^2(b-a)}\hat{r}$$ $\Rightarrow$ The electric field is the same for all values of θ, even with the dielectric hemisphere!

$\rightarrow$ Apply Gauss' Law to determine $(V_a - V_b)$ as a function of Q:

$$\int \varepsilon\vec{E}\cdot\hat{n}\,dA = Q_{encl}$$

Gaussian shape encloses inner sphere

$$\underbrace{\int \varepsilon\vec{E}\cdot\hat{n}\,dA}_{0<\theta<\pi} + \underbrace{\int \varepsilon_0\vec{E}\cdot\hat{n}\,dA}_{\pi<\theta<2\pi} = Q$$

$$\int \frac{-\varepsilon ab(V_a - V_b)}{r^2(b-a)}\,dA - \int \frac{\varepsilon_0 ab(V_a - V_b)}{r^2(b-a)}\,dA = Q$$

$\int dA$ for a hemisphere $= \frac{1}{2}(4\pi r^2) = 2\pi r^2$

$$\frac{-\varepsilon ab(V_a - V_b)}{\cancel{r^2}(b-a)}(2\pi\cancel{r^2}) - \frac{\varepsilon_0 ab(V_a - V_b)}{\cancel{r^2}(b-a)}(2\pi\cancel{r^2}) = Q$$

$$\frac{-ab(V_a - V_b)}{(b-a)}2\pi(\varepsilon + \varepsilon_0) = Q$$

$$\rightarrow \frac{ab(V_a - V_b)}{(b-a)} = \frac{-Q}{2\pi(\varepsilon + \varepsilon_0)}$$

$$\vec{E}(r,\theta) = \frac{Q}{2\pi(\varepsilon + \varepsilon_0)r^2}\hat{r}$$

(b) The reason the electric field in part (a) was found to be the same for all values of θ is because the charges can redistribute on the conductors such that the surface charge density for $0<\theta<\pi$ is greater than the surface charge density for $\pi<\theta<2\pi$. Following this, the equation for surface charge density will differ depending on what region of interest you are in.

→ In general:

$$\sigma(a,\theta) = -\varepsilon \frac{\partial \Phi}{\partial r}\Big|_{r=a} \quad \text{(Jackson eq. 2.15)}$$

$$\sigma(a,\theta) = \begin{cases} \varepsilon E(a,\theta), & 0<\theta<\pi \\ \varepsilon_0 E(a,\theta), & \pi<\theta<2\pi \end{cases}$$

$$= \begin{cases} \dfrac{\varepsilon Q}{2\pi(\varepsilon+\varepsilon_0)a^2}, & 0<\theta<\pi \\ \dfrac{\varepsilon_0 Q}{2\pi(\varepsilon+\varepsilon_0)a^2}, & \pi<\theta<2\pi \end{cases}$$

$$(\varepsilon+\varepsilon_0) = \varepsilon\left(1+\frac{\varepsilon_0}{\varepsilon}\right)$$
$$= \varepsilon_0\left(\frac{\varepsilon}{\varepsilon_0}+1\right)$$

$$\sigma(a,\theta) = \frac{Q}{2\pi a^2}\begin{cases} \left(1+\frac{\varepsilon_0}{\varepsilon}\right)^{-1}, & 0<\theta<\pi \\ \left(\frac{\varepsilon}{\varepsilon_0}+1\right)^{-1}, & \pi<\theta<2\pi \end{cases}$$

(c) In general:

$$\sigma_{pol} = -(\vec{P}_2-\vec{P}_1)\cdot\hat{n}_{21} \qquad \text{Jackson eq. 4.46}$$
$$\hookrightarrow \vec{P} = \varepsilon_0\chi_e\vec{E} \qquad \text{eq. 4.36}$$
$$\hookrightarrow \varepsilon = \varepsilon_0(1+\chi_e) \qquad \text{eq. 4.38}$$

Where σ_{pol} is the surface charge induced in the dielectric due to the external electric field.

$$\sigma_{pol}\,@\,r=a = -(\vec{P}_2 - \cancel{\vec{P}_1})\cdot\hat{r} \qquad \vec{P}_1 = 0\text{, inside the } r=a \text{ conducting sphere}$$
$$= -\vec{P}_2\cdot\hat{r}$$
$$= -(\varepsilon-\varepsilon_0)\vec{E}\big|_{r=a}\hat{r}$$
$$= -(\varepsilon-\varepsilon_0)\frac{Q}{2\pi(\varepsilon+\varepsilon_0)a^2}\underbrace{\hat{r}\cdot\hat{r}}_{=1}$$

$$\sigma_{pol} = -\frac{(\varepsilon-\varepsilon_0)}{(\varepsilon+\varepsilon_0)}\frac{Q}{2\pi a^2}$$

✓

5. Jackson Problem 4.11

Clausius-Mossotti:

For any given substance, $(\varepsilon/\varepsilon_0 - 1)/(\varepsilon/\varepsilon_0 + 2) \propto$ density of the substance

molecular polarizability

$$\gamma_{mol} = \frac{3}{N}\left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right) \qquad \text{Jackson eq. 4.70}$$

→ What's expected:

for low density, $\chi \propto N$
for high density, $\chi \uparrow N$

$$\left.\begin{array}{l}\text{for low density, } \chi \propto N \\ \text{for high density, } \chi \uparrow N\end{array}\right\} \chi = \frac{\gamma N}{1 - \frac{1}{3}\gamma N}$$

- For Air

$$\rho \propto \frac{(\varepsilon/\varepsilon_0 - 1)}{(\varepsilon/\varepsilon_0 + 2)} \longrightarrow \rho\frac{(\varepsilon/\varepsilon_0 + 2)}{(\varepsilon/\varepsilon_0 - 1)} = \text{constant}$$

Relative density of air as a function of pressure

$$\rho = \frac{P}{R_d T} \qquad R_d = 287 \ \mathrm{N\cdot m/kg/K}, \quad 1\ \mathrm{atm} = 1.01325 \times 10^5\ \mathrm{N/m^2}$$

$T = 292\ \mathrm{K}$

20 atm $= 2.02665 \times 10^6\ \mathrm{N/m^2}$ 80 atm $= 8.1060 \times 10^6\ \mathrm{N/m^2}$

40 atm $= 4.0530 \times 10^6\ \mathrm{N/m^2}$ 100 atm $= 1.01325 \times 10^7\ \mathrm{N/m^2}$

60 atm $= 6.0795 \times 10^6\ \mathrm{N/m^2}$

@ 20 atm:

$$\rho = \frac{(2.02665 \times 10^6)}{83804} = 24.181\ \mathrm{kg/m^3} \qquad \underbrace{\frac{(\varepsilon/\varepsilon_0 - 1)}{(\varepsilon/\varepsilon_0 + 2)}}_{\equiv \text{C-M}} = \frac{(1.0108 - 1)}{(1.0108 + 2)} = 3.587 \times 10^{-3}$$

@ 40 atm:

$$\rho = \frac{(4.053 \times 10^6)}{83804} = 48.363\ \mathrm{kg/m^3} \qquad \frac{(\varepsilon/\varepsilon_0 - 1)}{(\varepsilon/\varepsilon_0 + 2)} = \frac{(1.0218 - 1)}{(1.0218 + 2)} = 7.214 \times 10^{-3}$$

@ 60 atm:

$$\rho = \frac{(6.0795 \times 10^6)}{83804} = 72.544\ \mathrm{kg/m^3} \qquad \frac{(\varepsilon/\varepsilon_0 - 1)}{(\varepsilon/\varepsilon_0 + 2)} = \frac{(1.0333 - 1)}{(1.0333 + 2)} = 10.978 \times 10^{3}$$

✓

→

@ 80 atm:

$$\rho = \frac{(8.106\times10^{6})}{83804} = 96.726 \ ^{kg}/_{m^3}$$

$$\frac{(\varepsilon/\varepsilon_0 - 1)}{(\varepsilon/\varepsilon_0 + 2)} = \frac{(1.0439-1)}{(1.0439+2)} = 14.422\times10^{-3}$$

@100 atm:

$$\rho = \frac{(1.01325\times10^{7})}{83804} = 120.907 \ ^{kg}/_{m^3}$$

$$\frac{(\varepsilon/\varepsilon_0 - 1)}{(\varepsilon/\varepsilon_0 + 2)} = \frac{(1.0548-1)}{(1.0548+2)} = 17.939\times10^{-3}$$

Air at 292 K

Pressure (atm)	Density ($^{kg}/_{m^3}$)	$\varepsilon/\varepsilon_0$	C-M	ρ/C-M	F Var
20	24.181	1.0108	3.587×10^{-3}	6741.243	0.00617
40	48.363	1.0218	7.214×10^{-3}	6703.800	0.00058
60	72.544	1.0333	10.978×10^{-3}	6608.064	0.01371
80	96.726	1.0439	14.422×10^{-3}	6706.682	0.0010
100	120.907	1.0548	17.939×10^{-3}	6739.910	0.00597

avg. 6699.940

The Clausius-Mossotti relation does not hold exactly here, but the largest deviation is only 1.37%, so it is very close.

- For Pentane

→ Similarly to Clausius-Mossotti, it is expected that for $\rho \propto (\varepsilon/\varepsilon_0 - 1)$, $\rho/(\varepsilon/\varepsilon_0 - 1) = \text{constant}$.

@ 1 atm:

$$\frac{(\varepsilon/\varepsilon_0 - 1)}{(\varepsilon/\varepsilon_0 + 2)} = \frac{(1.82-1)}{(1.82+2)} = 0.2147$$

@ 10^3 atm:

$$\frac{(\varepsilon/\varepsilon_0 - 1)}{(\varepsilon/\varepsilon_0 + 2)} = \frac{(1.96-1)}{(1.96+2)} = 0.2424$$

@ 4×10^3 atm:

$$\frac{(\varepsilon/\varepsilon_0 - 1)}{(\varepsilon/\varepsilon_0 + 2)} = \frac{(2.12-1)}{(2.12+2)} = 0.2718$$

@ 8×10^3 atm:

$$\frac{(\varepsilon/\varepsilon_0 - 1)}{(\varepsilon/\varepsilon_0 + 2)} = \frac{(2.24-1)}{(2.24+2)} = 0.2925$$

→

@ 12×10^3 atm

$$\frac{(\varepsilon/\varepsilon_0-1)}{(\varepsilon/\varepsilon_0+2)} = \frac{(2.33-1)}{(2.33+2)}$$

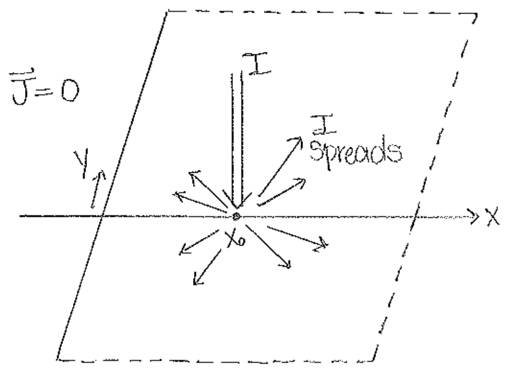
$$= 0.3072$$

Pentane (C_5H_{12}) at 303 K

Pressure (atm)	Density (g/cm^3)	$\varepsilon/\varepsilon_0$	C-M	ρ/C-M	FVar	$\rho/(\varepsilon/\varepsilon_0-1)$	FVar
1	0.613	1.82	0.2147	2.856	0.0209	0.748	0.0476
10^3	0.701	1.96	0.2424	2.892	0.0086	0.730	0.0224
4×10^3	0.796	2.12	0.2718	2.928	0.0038	0.711	0.0042
8×10^3	0.865	2.24	0.2925	2.958	0.0141	0.698	0.0229
12×10^3	0.907	2.33	0.3702	2.953	0.0123	0.682	0.0448
				avg. 2.917		avg. 0.714	

While the largest deviation for the Clausius-Mossotti relation is only 2.09%, this is a little more than half again larger than the largest fractional variance in Air. This is consistent with the expectation that the Clausius-Mossotti relation holds more closely for lower density substances. Also as expected, there are much larger fractional variances for the cruder relation, $(\varepsilon/\varepsilon_0-1) \propto$ density. The largest deviation here is 4.76%, more than twice the largest variance found by applying Clausius-Mossotti to Pentane.

6. Antonsen Problem



$\nabla\cdot\vec{J}=0 \rightarrow$ the current density at any point $\vec{x}$ is independent of time. Charges are in motion, however, there are a large number of charges executing the same amount of charge is passing through any fixed surface.

Surface current density flowing in the sheet

$\nabla \vec{J}_s = I\delta(\vec{x}-\vec{x}_0)$

- If the sheet were infinite, not half-infinite...

$I = \int_s \vec{J}(\vec{r})\cdot\hat{n}\,dA$

$J_s = \frac{I}{2\pi r}$

$r=\sqrt{(x-x_0)^2+y^2}$

However, in this case, we must use superposition to add another current source (analogous to an image charge) to satisfy the half-infinite sheet boundary condition.

$J_s = \frac{I_0}{2\pi r_0} + \frac{I_s}{2\pi r_s}$, $I_0 I_s < 0$

$r_0 = \sqrt{(x-x_0)^2+y^2}$, $r_s = \sqrt{(x-x_s)+y^2}$

- Boundary Condition: $J_s(x=0, y)=0$

$$J_s = \frac{1}{2\pi}\left[\frac{I_0}{\sqrt{(x-x_0)^2+y^2}} - \frac{I_s}{\sqrt{(x-x_s)^2+y^2}}\right]$$

$$0 = \frac{1}{2\pi}\left[\frac{I_0}{\sqrt{x_0^2+y^2}} - \frac{I_s}{\sqrt{x_s^2+y^2}}\right]$$

$I_0\sqrt{x_s^2+y^2} = I_s\sqrt{x_0^2+y^2}$

$I_0^2(x_s^2+y^2) = I_s^2(x_0^2+y^2)$

$I_0^2 x_s^2 - I_s^2 x_0^2 = y^2(I_s^2 - I_0^2)$

this equation must hold for all values of y as the boundary condition is only specified in x, so I_s must $= I_0 = I$

It follows then, as is predictable by inspection, that the location of the superimposed current source must be $(-x_0, 0)$.

$\longrightarrow$

→ Our equation for $\vec{J}_s$ for the infinite half-sheet then becomes

$$\vec{J}_s(\vec{x},\vec{y}) = \frac{I}{2\pi}\left[\frac{1}{\sqrt{(\vec{x}-\vec{x}_0)^2+y^2}} - \frac{1}{\sqrt{(\vec{x}+\vec{x}_0)^2+y^2}}\right] \quad \text{for } x \geq 0$$

We can then apply Ohm's law to find the electric field:

$$\vec{J} = \sigma\vec{E}$$

$$\vec{E}(\vec{x},\vec{y}) = \frac{I}{2\pi\sigma}\left[\frac{1}{\sqrt{(\vec{x}-\vec{x}_0)^2+\vec{y}^2}} - \frac{1}{\sqrt{(\vec{x}+\vec{x}_0)^2+\vec{y}^2}}\right] \quad \text{for } x \geq 0$$

And finally, we can determine the potential:

$$\vec{E} = -\nabla\varphi$$

$$\varphi = -\left(\int E\,dx + \int E\,dy\right)$$

$$= +\Big(\Big[+\ln\left(\sqrt{(\vec{x}_0-\vec{x})^2+y^2}+\vec{x}_0-\vec{x}\right) + \ln\left(\sqrt{(\vec{x}_0+\vec{x})^2+y^2}+\vec{x}_0+\vec{x}\right)\Big]$$
$$-\Big[\ln\left(\sqrt{x_0^2-2x_0x+x^2+y^2}+y\right) - \ln\left(\sqrt{x_0^2-2x_0x+x^2+y^2}+y\right)\Big]\Big)\frac{I}{2\pi\sigma}$$

$$= \frac{I}{2\pi\sigma}\left\{\ln\left[\left(\sqrt{(x_0-x)^2+y^2}+x_0-x\right)\left(\sqrt{(x_0+x)^2+y^2}+x_0+x\right)\right]\right.$$
$$\left. - \ln\left[\cancelto{0}{\frac{\left(\sqrt{x_0^2-2x_0x+x^2+y^2}+y\right)}{\left(\sqrt{x_0^2-2x_0x+x^2+y^2}+y\right)}}\right]\right\}$$

$$\varphi = \frac{I}{2\pi\sigma}\ln\left[\left(\sqrt{(x_0-x)^2+y^2}+x_0-x\right)\left(\sqrt{(x_0+x)^2+y^2}+x_0+x\right)\right]$$