# Physics 606: Homework #2

Due: Thursday Feb. 18, 2016

Jackson: Problems 1.15
2.1,
2.3,
2.8,
2.9,

2.A Two dimensional solutions of Laplace's equations in Cartesian coordinates are easy to come by. Let $z = x + iy$ be a complex number, and $f(z)$ <u>any</u> complex analytic function of $z$. Examples of analytic functions are: $z^n$, $\sin z$, $e^z$ ... The complex function $f$ will have a real part $f_R(x,y)$ and an imaginary part $f_I(x,y)$ each of which depend on $x$ and $y$. Both $f_R$ and $f_I$ can be regarded as the real functions of $x$ and $y$ as well as the real and imaginary parts of the complex function $f = f_R + i f_I$.

(a) Show that $f_R$ and $f_I$ are both solutions of Laplace's Equation. Along the way you must first show the following (Cauchy-Riemann) equations,

$$\frac{\partial f_R}{\partial x} = \frac{\partial f_I}{\partial y}, \qquad \frac{\partial f_I}{\partial x} = -\frac{\partial f_R}{\partial y}.$$

(b) Take $f(z)$ to be arcsin($z$). Make contour plots of the potential corresponding to the real part of $f$. What problem is this the potential for?

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# Homework #2

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1. Jackson Problem 1.15

Thompson's theorem: If a number of surfaces are fixed in position and a given total charge is placed on each surface, then the electrostatic energy in the region bounded by the surfaces is an absolute minimum when the charges are placed so that every surface is an equipotential, as happens when they are conductors.

→ The theorem must hold for any number of surfaces and conductors
→ Every surface for the minimum energy state should be an equipotential $\Phi_i = C$

Electrostatic energy: (potential energy)

$$W = q_i \Phi(\vec{x})$$
$$= \frac{\varepsilon_0}{2} \int_V |\vec{E}|^2 d^3x$$ — Jackson eq. (1.54)

Let the field with equipotential surfaces be denoted $\vec{E}$ and any other field that is not bounded by equipotential surfaces be denoted $\vec{E}'$

Applicable relations:

$\nabla \cdot \vec{E} = \rho/\varepsilon_0$, $\vec{\nabla} \times \vec{E} = 0$, $\vec{E} = -\nabla\varphi$, $\oint \vec{E} \cdot \hat{n}\, dA = q_{encl}/\varepsilon_0$ $\quad (\vec{E}_2 - \vec{E}_1) \cdot \hat{n} = \frac{\sigma}{\varepsilon_0}$

$\nabla \cdot \vec{E}' = \rho/\varepsilon_0$, $\vec{E}' = -\nabla\varphi$, $\oint \vec{E}' \cdot \hat{n}\, dA = q_{encl}/\varepsilon_0$

We expect this to always be larger if the theorem holds $(W' - W > 0)$

$$W' - W = \frac{\varepsilon_0}{2} \int_V E'^2 d^3x - \frac{\varepsilon_0}{2} \int_V E^2 d^3x$$
$$= \frac{\varepsilon_0}{2} \int_V \underbrace{(E'^2 - E^2)} d^3x$$

$$E'^2 - E^2 = \underbrace{(\vec{E}' - \vec{E})^2} - 2\vec{E} \cdot (\vec{E}' - \vec{E})$$

This term is guaranteed to be >0, so we just need to look at the second term

$$2\vec{E} \cdot (\vec{E}' - \vec{E})$$
$\vec{E} = -\nabla\varphi$

$$= -2\nabla\varphi \cdot (\vec{E}' - \vec{E})$$
$$= -2\nabla(\varphi(\vec{E}' - \vec{E})) - \varphi \overbrace{\nabla \cdot \cancel{(\vec{E}' - \vec{E})}}^{\nabla \cdot \vec{E}' = \rho/\varepsilon_0} \to 0$$
$\nabla \cdot \vec{E} = \rho/\varepsilon_0$

So the part of the integral we care about becomes

$$= \varepsilon_0 \int_V \nabla(\varphi(\vec{E}' - \vec{E}))\, d^3x$$

$$\underbrace{\int_S \vec{E}\cdot\hat{n}\,dA}_{} = q/\varepsilon_0$$

$$= \varepsilon_0 \int_S \varphi(\vec{E}' - \vec{E})\cdot\hat{n}\,dA$$

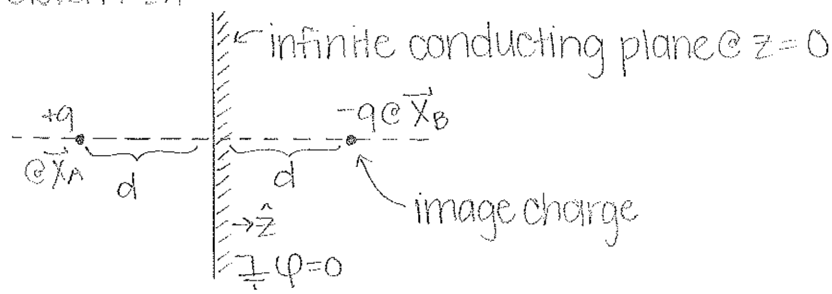
$$\uparrow \int_S \vec{E}'\cdot\hat{n}\,dA = q/\varepsilon_0$$

So this term also goes to zero

Thus, since the first term of the integrand will always be positive and the second term integrates to zero, it is guaranteed that $W' - W > 0$

$\Rightarrow$ And therefore $W' > W$ and Thomson's Theorem holds

2. Jackson Problem 2.1




(a) Want $\sigma(x,y)$

electric field due to an image charge

$$\vec{E}(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\frac{(\vec{x}-\vec{x}_A)}{|\vec{x}-\vec{x}_A|^3} - \underbrace{\frac{q}{4\pi\varepsilon_0}\frac{(\vec{x}-\vec{x}_B)}{|\vec{x}-\vec{x}_B|^3}}$$

image component representing the $\vec{E}$-contribution due to the conducting plane

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{(\vec{x}-\vec{x}_A)}{|\vec{x}-\vec{x}_A|^3} - \frac{(\vec{x}-\vec{x}_B)}{|\vec{x}-\vec{x}_B|^3}\right]$$

$\vec{x}_A = -d\hat{e}_z$, $\vec{x}_B = d\hat{e}_z$

$\hookrightarrow |\vec{x}-\vec{x}_A| = \sqrt{x^2+y^2+(z+d)^2}$; $|\vec{x}-\vec{x}_B| = \sqrt{x^2+y^2+(z-d)^2}$

$$\vec{E}(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{(\vec{x}-\vec{x}_A)}{(x^2+y^2+(z+d)^2)^{3/2}} - \frac{(\vec{x}-\vec{x}_B)}{(x^2+y^2+(z-d)^2)^{3/2}}\right]$$

normal field at the surface of the conductor

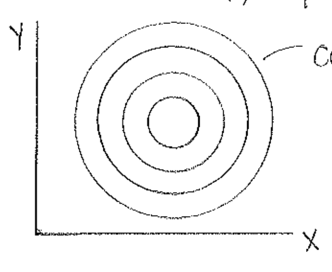
$$E_n = \hat{n}\cdot\vec{E}(\vec{x}) = -E_z\Big|_{z=0} = \frac{-q}{4\pi\varepsilon_0}\left[\frac{-(-d)}{(x^2+y^2+d^2)^{3/2}} - \frac{(-d)}{(x^2+y^2+d^2)^{3/2}}\right]$$

$$E_n = \frac{-\cancel{2}q}{_2\cancel{4}\pi\varepsilon_0}\frac{d}{(x^2+y^2+d^2)^{3/2}}$$

$$\sigma(x,y) = \varepsilon_0 E_n$$

$$= \frac{-\cancel{\varepsilon_0}q}{2\pi\cancel{\varepsilon_0}}\frac{d}{(x^2+y^2+d^2)^{3/2}}$$

$$\sigma(x,y) = \frac{-q}{2\pi}\frac{d}{(x^2+y^2+d^2)^{3/2}}$$




contour plot in the xy-plane

Surface charge is a circle extending into the z-plane (so, a cylinder)

(b) $\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{z^2} \hat{z}$

$= \frac{1}{4\pi\varepsilon_0} \frac{q(-q)}{(2d)^2} \hat{z}$

$\vec{F} = \frac{-1}{16\pi\varepsilon_0} \frac{q^2}{d^2} \hat{z}$

the force between the plane and the charge

(c) the force per unit area on the surface of a conductor

$\vec{f} = \frac{\vec{F}}{dA} = \sigma \langle \vec{E} \rangle$ (as given by Dr. Sprangle's lecture)

$\hookrightarrow \langle \vec{E} \rangle = \frac{1}{2} \frac{\sigma}{\varepsilon_0} \hat{n}$

$\frac{d\vec{F}}{dA} = \frac{\sigma^2}{2\varepsilon_0} \hat{n}$

choose cylindrical coordinates as suggested by the plot in (a)

$\vec{F} = \int \frac{\sigma^2}{2\varepsilon_0} dA \, \hat{n}$

$= \frac{1}{2\varepsilon_0} \int_0^\infty \int_0^{2\pi} \frac{q^2 d^2}{4\pi^2} \frac{r}{(\underbrace{x^2+y^2}_{r^2}+d^2)^3} dr d\varphi \, \hat{z}$

$= \frac{q^2 d^2}{4\pi\varepsilon_0} \int_0^\infty \frac{r dr}{(r^2+d^2)^3} \hat{z}$

$= \frac{q^2 d^2}{4\pi\varepsilon_0} \left(\frac{-1}{4}\right) \left[\frac{1}{(r^2+d^2)^2}\right]_0^\infty \hat{z}$

$= \frac{-q^2 d^2}{16\pi\varepsilon_0} \left[0 - \frac{1}{d^4}\right] \hat{z}$

$\vec{F} = \frac{q^2}{16\pi\varepsilon_0 d^2} \hat{z}$

Equal value but opposite magnitude as the force on the charge, as expected

→

(d) $W = -\int_d^\infty \vec{F}\cdot d\vec{\ell}$ (Jackson eq. (1.18))

$$= -\int_d^\infty \frac{-1}{16\pi\varepsilon_0}\frac{q^2}{\ell^2}\,d\vec{\ell}$$

$$= \frac{q^2}{16\pi\varepsilon_0}\int_d^\infty \frac{1}{\ell^2}\,d\vec{\ell}$$

$$= \frac{q^2}{16\pi\varepsilon_0}\left[\frac{-1}{\ell}\right]_d^\infty$$

$$= \frac{q^2}{16\pi\varepsilon_0}\left(0 - \frac{-1}{d}\right)$$

$$W = \frac{q^2}{16\pi\varepsilon_0 d}$$ ✓

the work required to move the charge q from d to $\infty$

(e) $W = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{z}$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q(-q)}{(2d)}$$ ✓

$$W = \frac{-q^2}{8\pi\varepsilon_0 d}$$

The magnitude of the potential energy between the charge and its image is twice the work required to move the charge from $d \to \infty$. This discrepancy is because placing an image charge "induces" (and includes) field lines through the conductor, when in reality there are no fields within the conductor, just an induced surface charge density.

(f) $W = \frac{q^2}{16\pi\varepsilon_0 d}$ ✓

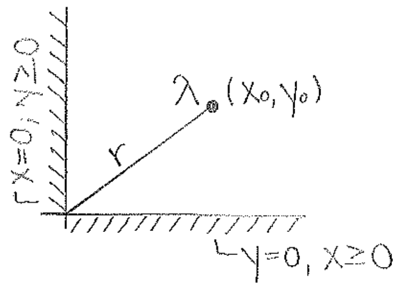
- for an electron originally 1 Å from the surface

$q = 1.6\times10^{-19}\,J = 1\,eV$, $Å = 10^{-10}\,m$, $\varepsilon_0 = 8.85\times10^{-12}\,F/m$

$$W = \frac{(1.6\times10^{-19}\,J)^2}{16\pi(8.85\times10^{-12}\,F/m)(10^{-10}\,m)}$$

$= 3.60\,J^2/F = 3.60\,eV$ required ✓

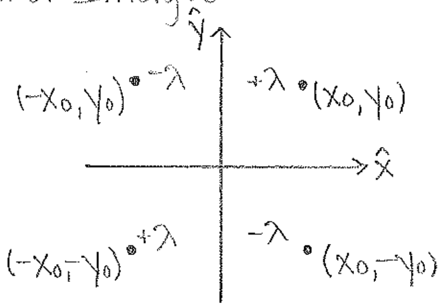
3. Jackson Problem 2.3




(a) Potential for an isolated line charge @ $(x_0, y_0)$:

$$\Phi(x,y) = \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{R^2}{r^2}\right) \quad \text{for } r^2 = (x-x_0)^2 + (y-y_0)^2$$

As shown in Lecture 4 (02/09), this problem can be solved using Method of Images




Where the line charges in the $2^{nd}$, $3^{rd}$, and $4^{th}$ quadrants are the image charges.

The contributions to the potential for each of these line charges can then simply be summed.

$$\Phi(x,y) = \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{R^2}{(x-x_0)^2+(y-y_0)^2}\right) - \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{R^2}{(x+x_0)^2+(y-y_0)^2}\right)$$

$$+ \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{R^2}{(x+x_0)^2+(y+y_0)^2}\right) - \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{R^2}{(x-x_0)^2+(y+y_0)^2}\right)$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left[\ln\left(\frac{R^2}{(x-x_0)^2+(y-y_0)^2}\right) - \ln\left(\frac{R^2}{(x+x_0)^2+(y-y_0)^2}\right)\right.$$

$$\left. + \ln\left(\frac{R^2}{(x+x_0)^2+(y+y_0)^2}\right) - \ln\left(\frac{R^2}{(x-x_0)^2+(y+y_0)^2}\right)\right]$$

$\ln(ab) = \ln(a) + \ln(b)$; $\ln(c/d) = \ln(c) - \ln(d)$

$$= \frac{\lambda}{4\pi\varepsilon_0} \ln\left[\frac{\cancel{R^2}}{((x-x_0)^2+(y-y_0)^2)} \frac{((x+x_0)^2+(y-y_0)^2)}{\cancel{R^2}}\right]$$

$$+ \ln\left[\frac{\cancel{R^2}}{((x+x_0)^2+(y+y_0)^2)} \frac{((x-x_0)^2+(y+y_0)^2)}{\cancel{R^2}}\right]$$

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$$\Phi(x,y)=\frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{((x+x_0)^2+(y-y_0)^2)((x-x_0)^2+(y+y_0)^2)}{((x-x_0)^2+(y-y_0)^2)((x+x_0)^2+(y+y_0)^2)}\right]$$

→ Verify the potential goes to 0 on the boundary

$$\Phi(0,y)=\frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{(x_0^2+(y-y_0)^2)(x_0^2+(y+y_0)^2)}{(x_0^2+(y-y_0)^2)(x_0^2+(y+y_0)^2)}\right]$$

$$=\frac{\lambda}{4\pi\varepsilon_0}\cancel{\ln(1)}^{\to 0}$$

$\Phi(0,y)=0$ ✓

$$\Phi(x,0)=\frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{((x+x_0)^2+y_0^2)((x-x_0)^2+y_0^2)}{((x-x_0)^2+y_0^2)((x+x_0)^2+y_0^2)}\right]$$

$$=\frac{\lambda}{4\pi\varepsilon_0}\cancel{\ln(1)}^{\to 0}$$

$\Phi(x,0)=0$ ✓

→ Verify the tangential electric field goes to 0 on the boundary

$$E_x=\left.\frac{-\partial\Phi}{\partial x}\right|_{y=0}$$

$$E_x=\frac{-\lambda}{4\pi\varepsilon_0}\frac{\partial}{\partial x}\left[\ln((x+x_0)^2+y_0^2)+\ln((x-x_0)^2+y_0^2)-\ln((x+x_0)+y_0^2)-\ln((x-x_0)^2+y_0^2)\right]$$

$$=\frac{-\lambda}{4\pi\varepsilon_0}\left[\cancel{\frac{2(x+x_0)}{(x+x_0)^2+y_0^2}}+\cancel{\frac{2(x-x_0)}{(x-x_0)^2+y_0^2}}-\cancel{\frac{2(x+x_0)}{(x+x_0)^2+y_0^2}}-\cancel{\frac{2(x-x_0)}{(x-x_0)^2+y_0^2}}\right]$$

$E_x=0$ ✓

$$E_y=\left.\frac{-\partial\Phi}{\partial y}\right|_{x=0}$$

$$=\frac{-\lambda}{4\pi\varepsilon_0}\frac{\partial}{\partial y}\left[\ln(x_0^2+(y+y_0)^2)+\ln(x_0^2+(y-y_0)^2)-\ln(x_0^2+(y+y_0)^2)-\ln(x_0^2+(y-y_0)^2)\right]$$

$$=\frac{-\lambda}{4\pi\varepsilon_0}\left[\cancel{\frac{2(y+y_0)}{x_0^2+(y+y_0)^2}}+\cancel{\frac{2(y-y_0)^2}{x_0^2+(y-y_0)^2}}-\cancel{\frac{2(y+y_0)^2}{x_0^2+(y+y_0)^2}}-\cancel{\frac{2(y-y_0)^2}{x_0^2+(y-y_0)^2}}\right]$$

$E_y=0$ ✓

(b) For a surface with a charge density $\sigma$ and electric fields that may be characterized on both sides, we can use Jackson eq. (1.22)

$$(\vec{E}_2-\vec{E}_1)\cdot\hat{n}=\sigma/\varepsilon_0$$

where $\vec{E}_1$, the electric field beneath the plane $= 0$

$\vec{E}_2 \cdot \hat{n} = \sigma/\varepsilon_0$

normal direction for this plane is $\hat{y}$

$\sigma = \varepsilon_0 E_y \big|_{y=0}$

$$\sigma = \frac{\lambda}{4\pi}\left[\frac{2y_0}{(x+x_0)^2+y_0^2} - \frac{2y_0}{(x-x_0)^2+y_0^2} + \frac{2y_0}{(x+x_0)^2+y_0^2} - \frac{2y_0}{(x-x_0)^2+y_0^2}\right]$$

$$\sigma = \frac{-\lambda y_0}{\pi}\left[\frac{1}{(x-x_0)^2+y_0^2} - \frac{1}{(x+x_0)^2+y_0^2}\right]$$

- for $(x_0=2, y_0=1)$

$$\frac{\sigma}{\lambda} = \frac{-1}{\pi}\left[\frac{1}{(x-2)^2+1} - \frac{1}{(x+2)^2+1}\right]$$

- for $(x_0=1, y_0=1)$

$$\frac{\sigma}{\lambda} = \frac{-1}{\pi}\left[\frac{1}{(x-1)^2+1} - \frac{1}{(x+1)^2+1}\right]$$

- for $(x_0=1, y_0=2)$

$$\frac{\sigma}{\lambda} = \frac{-2}{\pi}\left[\frac{1}{(x-1)^2+4} - \frac{1}{(x+1)^2+4}\right]$$

*Plots immediately follow this problem

(c) $Q_x = \int_0^\infty \sigma\, dx$

$$= \frac{-\lambda y_0}{\pi}\left[\int_0^\infty \frac{1}{(x-x_0)^2+y_0^2}dx - \int_0^\infty \frac{1}{(x+x_0)^2+y_0^2}dx\right]$$

shift lower bounds to $-x_0$ and $x_0$, respectively, to create a more recognizable integral

$$= \frac{-\lambda y_0}{\pi}\left[\int_{-x_0}^\infty \frac{1}{x^2+y_0^2}dx - \int_{x_0}^\infty \frac{1}{x^2+y_0^2}dx\right]$$

$$= \frac{-\lambda y_0}{\pi}\left[\frac{1}{y_0}\arctan\left(\frac{x}{y_0}\right)\Big|_{-x_0}^{\infty} - \frac{1}{y_0}\arctan\left(\frac{x}{y_0}\right)\Big|_{x_0}^{\infty}\right]$$

$$= \frac{-\lambda y_0}{\pi}\left[\cancel{\frac{\pi}{2y_0}} - \underbrace{\frac{1}{y_0}\arctan\left(\frac{-x_0}{y_0}\right)}_{=-\arctan\left(\frac{x_0}{y_0}\right)} - \cancel{\frac{\pi}{2y_0}} + \arctan\left(\frac{x_0}{y_0}\right)\right]$$

$$Q_x = \frac{-2\lambda}{\pi} \arctan\left(\frac{x_0}{y_0}\right)$$

Because there is total symmetry between the x- and y-axes, it may just be stated that the total charge on the plane $x=0, y\geq 0$ is

$$Q_y = \frac{-2\lambda}{\pi} \arctan\left(\frac{y_0}{x_0}\right)$$

(d) For $\rho \gg \rho_0$; $\rho = \sqrt{(x^2+y^2)}$ and $\rho_0 = \sqrt{(x_0^2+y_0^2)}$

$$\Phi(x,y) = (-\lambda/4\pi\varepsilon_0)[\ln((x+x_0)^2+(y+y_0)^2) - \ln((x-x_0)^2+(y+y_0)^2) + \ln((x-x_0)^2+(y-y_0)^2) - \ln((x+x_0)^2-y-y_0)^2)]$$

-Transition this to cylindrical coordinates

$$\Phi(\rho,\theta) = (-\lambda/4\pi\varepsilon_0)[\ln(\rho^2+\rho_0^2-2\rho\rho_0\cos(\theta-\theta_0)) - \ln((\rho^2+\rho_0^2+2\rho\rho_0\cos(\theta+\theta_0)) - \ln((\rho^2+\rho_0^2-2\rho\rho_0\cos(\theta+\theta_0)) + \ln((\rho^2+\rho_0^2+2\rho\rho_0\cos(\theta-\theta_0))]$$

-In order to determine where the far-from-origin approximation applies, divide by $\rho^2$ so a statement can be made about $\frac{\rho_0}{\rho}$

$$\Phi = (-\lambda/4\pi\varepsilon_0)[\ln(\rho^2) + \ln(1+(\tfrac{\rho_0}{\rho})^2 - 2\tfrac{\rho_0}{\rho}\cos(\theta-\theta_0)) - \ln(\rho^2) - \ln(1+(\tfrac{\rho_0}{\rho})^2 + 2\tfrac{\rho_0}{\rho}\cos(\theta+\theta_0)) - \ln(\rho^2) - \ln(1+(\tfrac{\rho_0}{\rho})^2 - 2\tfrac{\rho_0}{\rho}\cos(\theta+\theta_0)) + \ln(\rho^2) + \ln(1+(\tfrac{\rho_0}{\rho})^2 + 2\tfrac{\rho_0}{\rho}\cos(\theta-\theta_0))]$$

-Because $\rho_0/\rho$ is small, we can perform a Taylor expansion

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \ldots$$

and because $\rho \gg \rho_0$, only keep up to the terms of order $x^2$

$$\Phi = (-\lambda/4\pi\varepsilon_0)[\{((\tfrac{\rho_0}{\rho})^2 - 2\tfrac{\rho_0}{\rho}\cos(\theta-\theta_0)) - \tfrac{1}{2}((\tfrac{\rho_0}{\rho})^2 - 2\tfrac{\rho_0}{\rho}\cos(\theta-\theta_0))^2 + \ldots\}$$
$$- \{((\tfrac{\rho_0}{\rho})^2 + 2\tfrac{\rho_0}{\rho}\cos(\theta+\theta_0)) + \tfrac{1}{2}((\tfrac{\rho_0}{\rho})^2 + 2\tfrac{\rho_0}{\rho}\cos(\theta+\theta_0))^2 + \ldots\}$$
$$- \{((\tfrac{\rho_0}{\rho})^2 - 2\tfrac{\rho_0}{\rho}\cos(\theta+\theta_0)) + \tfrac{1}{2}((\tfrac{\rho_0}{\rho})^2 - 2\tfrac{\rho_0}{\rho}\cos(\theta+\theta_0))^2 + \ldots\}$$
$$+ \{((\tfrac{\rho_0}{\rho})^2 + 2\tfrac{\rho_0}{\rho}\cos(\theta-\theta_0)) - \tfrac{1}{2}((\tfrac{\rho_0}{\rho})^2 + 2\tfrac{\rho_0}{\rho}\cos(\theta-\theta_0))^2 + \ldots\}]$$

$$= (-\lambda/\cancel{4}\pi\varepsilon_0)(\cancel{4}(\tfrac{\rho_0}{\rho})^2 \underbrace{(\cos^2(\theta+\theta_0) - \cos^2(\theta-\theta_0))}_{\text{"trig cheer" expansion}} + \text{higher order terms})$$

$$= (-\lambda/\pi\varepsilon_0)(\tfrac{\rho_0}{\rho})^2(-4\cos(\theta)\cos(\theta_0)\sin(\theta)\sin(\theta_0))$$

$$\rho_0^2 = x_0^2 + y_0^2, \quad \rho^2 = x^2 + y^2$$

$$\Phi = \frac{4\lambda}{\pi\varepsilon_0}\frac{(x_0 y_0)(xy)}{\rho^4} \checkmark$$

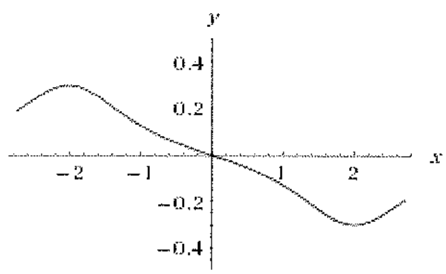
This is the quadrupole potential resulting from the four line charges (3 of them image charges)

Case 1: for $(x_0=2, y_0=1)$

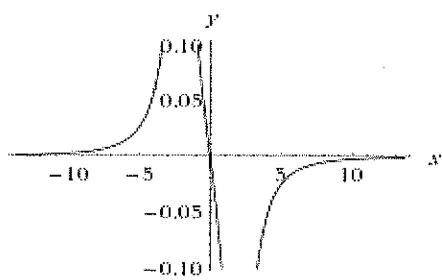
Input:

$$y = -\frac{\frac{1}{(x-2)^2+1} - \frac{1}{(x+2)^2+1}}{\pi}$$

Plots:




(x from −2.7 to 2.7)



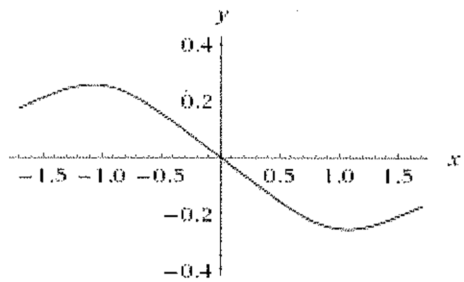

(x from −13.7 to 13.7)

Case 2: for $(x_0 = 1, y_0 = 1)$

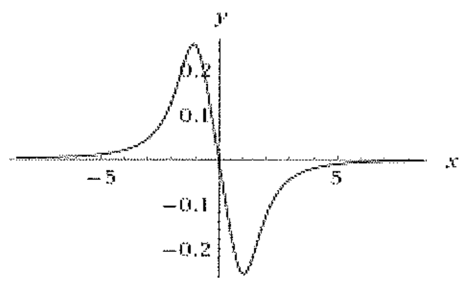
Input:

$$y = -\frac{\frac{1}{(x-1)^2+1} - \frac{1}{(x+1)^2+1}}{\pi}$$

Plots:




(x from −1.7 to 1.7)



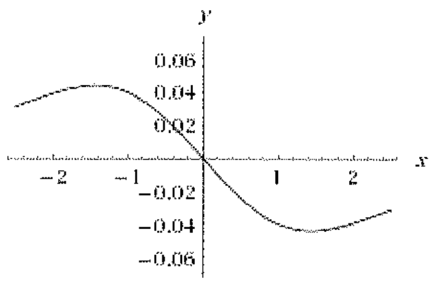

(x from −8.5 to 8.5)

Case 3: for $(x_0=1, y_0=2)$

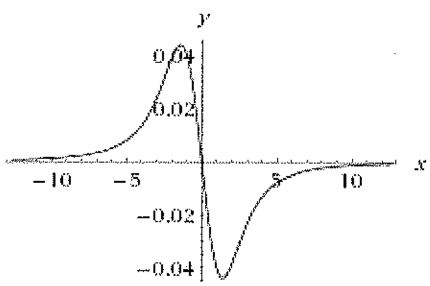
Input:

$$y = -\frac{\frac{1}{(x-1)^2+4} - \frac{1}{(x+1)^2+4}}{\pi}$$

Plots:



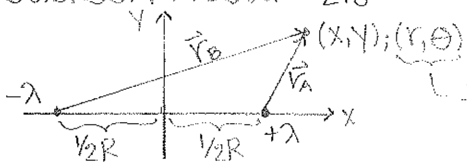

(x from −2.5 to 2.5)




(x from −12.6 to 12.6)

4. Jackson Problem 2.8

(a)




└ the equipotential coordinates

$$\varphi(x,y) = \frac{\lambda}{2\pi\varepsilon_0}\ln\left|\frac{r_A}{r_B}\right|$$ ← as derived in Lecture 4 on 02/09

Then, as discussed in Lecture 5 on 02/11...

-- We can find the equipotential constraints on $\vec{r}_A$ and $\vec{r}_B$ by solving

$$\frac{r_A^2}{r_B^2} = \exp\left\{\frac{4\pi\varepsilon_0}{\lambda}\varphi\right\}$$

which for a fixed value of $\varphi = V$ is equivalently

$$\frac{r_A}{r_B} = \underbrace{\exp\left\{\frac{2\pi\varepsilon_0 V}{\lambda}\right\}}_{\text{a constant value} \equiv \alpha}$$

Using the law of cosines:

$$\begin{cases} r_A^2 = (R+x)^2 + r^2 - 2(R+x)r\cos(\theta) \\ r_B^2 = x^2 + r^2 - 2xr\cos(\theta) \end{cases}$$

$$\longrightarrow \alpha^2 = \frac{(R+x)^2 + r^2 - 2(R+x)r\cos(\theta)}{x^2 + r^2 - 2xr\cos(\theta)}$$

In order to hold for all $\theta$, the ratios of the terms with and without $\cos(\theta)$ must be equal

$$\alpha^2 = \frac{(R+x)^2 + r^2}{x^2 + r^2} \qquad \alpha^2 = \frac{\cancel{2}(R+x)\cancel{r}}{\cancel{2}x\cancel{r}}$$

$$\alpha^2 x^2 + \alpha^2 r^2 = (R+x)^2 + r^2$$

$$x = \frac{R}{\alpha^2 - 1}$$

$$\alpha^2 x^2 - (R+x)^2 = r^2(\alpha^2 - 1)$$

$$r^2 = \frac{\alpha^2 x^2 - (R+x)^2}{\alpha^2 - 1}$$

$$= \left[\alpha^2\left(\frac{R}{\alpha^2-1}\right)^2 - \left(R + \frac{R}{\alpha^2-1}\right)^2\right]\frac{1}{\alpha^2-1}$$

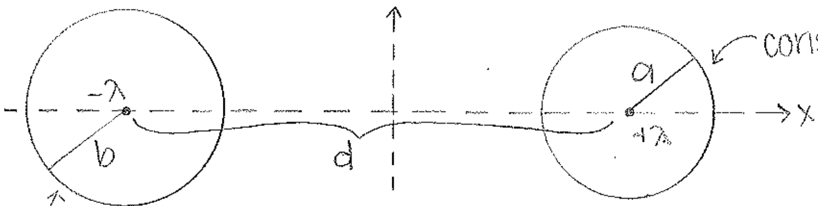
$$= \left[\frac{\alpha^2 R^2}{(\alpha^2-1)^2} - \left(\frac{R\alpha^2 - R}{\alpha^2-1} + \frac{R}{\alpha^2-1}\right)^2\right]\frac{1}{\alpha^2-1}$$

→

$$r = \left( \left[ \frac{\alpha^2 R^2}{(\alpha^2-1)^2} - \frac{R^2\alpha^4}{(\alpha^2-1)^2} \right] \frac{1}{\alpha^2-1} \right)^{1/2}$$

$$r = \left| \frac{\alpha R}{\alpha^2 - 1} \right|$$

(b)




constant $\alpha_+ = \exp\left\{ \frac{2\pi\varepsilon_0 V_+}{\lambda} \right\}$

constant $\alpha_- = \exp\left\{ \frac{2\pi\varepsilon_0 V_-}{\lambda} \right\}$

Using $r = \frac{\alpha R}{\alpha^2 - 1}$ from part (a):

$$a = \frac{\alpha_+ R}{\alpha_+^2 - 1} \;;\; b = -\frac{\alpha_- R}{\alpha_-^2 - 1} \;;\; d = \frac{\alpha_+ R}{\alpha_+^2 - 1} - \frac{\alpha_- R}{\alpha_-^2 - 1}$$

$$2ab = -2R^2 \frac{\alpha_+ \alpha_-}{(\alpha_+^2 - 1)(\alpha_-^2 - 1)}$$ ← solving by construction

$$d^2 - a^2 - b^2 = -R^2 \frac{\alpha_+^2 + \alpha_-^2}{(\alpha_+^2 - 1)(\alpha_-^2 - 1)}$$

$$\frac{d^2 - a^2 - b^2}{2ab} = \frac{\cancel{R^2}(\alpha_+^2 + \alpha_-^2)}{\cancel{(\alpha_+^2 - 1)(\alpha_-^2 - 1)}} \frac{\cancel{(\alpha_+^2 - 1)(\alpha_-^2 - 1)}}{2\cancel{R^2}\alpha_+\alpha_-}$$

$$= \frac{1}{2}\frac{\alpha_+}{\alpha_-} + \frac{1}{2}\frac{\alpha_-}{\alpha_+}$$

which, by above definition of $\alpha_+$ and $\alpha_-$ is equal to

$$= \cosh\left( \frac{2\pi\varepsilon_0 (V_+ - V_-)}{\lambda} \right)$$

$$\operatorname{arccosh}\left( \frac{d^2 - a^2 - b^2}{2ab} \right) = (V_+ - V_-) \frac{2\pi\varepsilon_0}{\lambda}$$ ← $\lambda = Q/L$

$$= \underbrace{(V_+ - V_-)\frac{2\pi\varepsilon_0 L}{Q}}_{C = qV} = \frac{2\pi\varepsilon_0 L}{C}$$

→

$$\frac{C}{L} = \frac{2\pi\varepsilon_0}{\operatorname{arccosh}(d^2-a^2-b^2/2ab)}$$

(c) "Appropriate" limit: $d^2 \gg a^2+b^2$

$\hookrightarrow \frac{d^2-a^2-b^2}{2ab} \gg 1$ in this limit

for $x \gg 1$, $\operatorname{arccosh}(x) \approx \ln(2x)$

$$\operatorname{arccosh}\left(\frac{d^2-a^2-b^2}{2ab}\right) \approx \ln\left(\frac{d^2-a^2-b^2}{ab}\right)$$

$$= \ln\left[\frac{d^2}{ab}\left(1-\frac{a^2+b^2}{d^2}\right)\right]$$

$$\approx 2\ln\left(\frac{d}{\sqrt{ab}}\right) + \cancelto{0}{\ln(1)} - \frac{a^2+b^2}{d^2}$$

$$\frac{C}{L} \approx \frac{2\pi\varepsilon_0}{2\ln(d/\sqrt{ab}) - \frac{a^2+b^2}{d^2}}$$

For $a \,\&\, b \to 0$, we expect $V \to \infty$ from our result in part (b), which it does as $C = qV$ and this expression blows up.

$\to$ this also holds for the lowest order correction in $a^2+b^2/d^2$, i.e., $a/d$ & $b/d$

(d) Because the expressions found in part (b) are very general by design, they are applicable here as well:

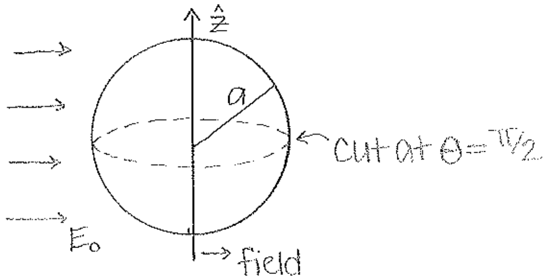
$$\frac{C}{L} = \frac{2\pi\varepsilon_0}{\operatorname{arcosh}(d^2-a^2-b^2/2ab)}$$

In the concentric cylinder case: $d=0$

$$\operatorname{arccosh}\left(\frac{a^2+b^2}{2ab}\right) \approx \ln\left(\frac{a}{b}\right)$$

$$\Rightarrow \frac{C}{L} = \frac{2\pi\varepsilon_0}{\ln(a/b)}$$

5. Jackson Problem 2.9




(a) If the shell is uncharged, we can solve this problem by placing point charges (at strategic locations to maintain symmetry) as described in Jackson section 2.5 to obtain Jackson eq. (2.12)

$$\phi = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{r^2+R^2+2rR\cos(\theta)}} - \frac{1}{\sqrt{r^2+R^2+2rR\cos(\theta)}}\right.$$

$$\left. - \frac{a/R}{\sqrt{r^2+(a^2/R)^2+2a^2 r/R\cos(\theta)}} + \frac{a/R}{\sqrt{r^2+(a^2/R)^2-2a^2 r/R\cos(\theta)}}\right]$$

with $E_0 = \frac{2Q}{4\pi\varepsilon_0 R^2}$.

and in the $R \to \infty$ limit...

$$\phi = -E_0\left(r - \frac{a^3}{r^2}\right)\cos(\theta)$$

Jackson eq. (2.14)

To find the force on a hemisphere,

$$\vec{F} = \int \sigma(\vec{x})\vec{E}(\vec{x})dA$$

we must first find the electric field at the surface of the sphere and the surface charge density of the sphere.

$$\vec{E} = -\nabla\phi$$

$$\vec{E} = -E_0\nabla\left[r\cos(\theta) - \frac{a^3}{r^2}\cos(\theta)\right]$$

$$= E_0\left(\cos(\theta)\hat{r} - r\sin(\theta)\hat{\theta} + 2\frac{a^3}{r^3}\cos(\theta)\hat{r} + \frac{a^3}{r^2}\sin(\theta)\hat{\theta}\right)$$

the electric field induced at the surface of the sphere

$$\vec{E}\Big|_{r=a} = E_0\left[\cos(\theta)\hat{r} - \cancel{a\sin(\theta)\hat{\theta}} + 2\cancelto{1}{\frac{a^3}{a^3}}\cos(\theta)\hat{r} + \cancel{\frac{a^3}{a^2}\sin(\theta)\hat{\theta}}\right]$$

$$\vec{E}\Big|_{r=a} = 3E_0\cos(\theta)\hat{r}$$

$$\sigma = \varepsilon_0\hat{r}\cdot\vec{E}\Big|_{r=a}$$

$\sigma = 3\varepsilon_0 E_0 \cos(\theta)$

$\vec{F} = \int \sigma(\vec{x})\vec{E}(\vec{x})dA$

but in order not to include the self-force of the hemisphere on itself, we need to use the form of the Electric field, $E = \sigma/2\varepsilon_0$

$\vec{F} = \int \frac{\sigma^2}{2\varepsilon_0} dA$

$= \int \frac{9\varepsilon_0^{\cancel{2}} E_0^2 \cos^2(\theta)}{2\cancel{\varepsilon_0}} \hat{r} dA$

$= \frac{9\varepsilon_0 E_0^2}{2} \int \cos^2(\theta)\hat{r} dA$

with respect to the bottom hemisphere: $\pi/2 \leq \theta \leq \pi$

$= \frac{9\varepsilon_0 E_0^2}{2} a^2 \hat{z} \int_0^{2\pi}\int_{\pi/2}^{\pi} \cos^2(\theta)(\cos(\theta)\sin(\theta)d\theta d\varphi)$

$= \frac{9\varepsilon_0 E_0^2 a^2}{\cancel{2}} \cancel{2}\pi\hat{z} \int_{\pi/2}^{\pi} \cos^3(\theta)\sin(\theta)d\theta$

$= 9\varepsilon_0 E_0^2 a^2 \pi\hat{z}\left[-\frac{1}{4}\cos^4(\theta)\right]_{\pi/2}^{\pi}$

$= 9\varepsilon_0 E_0^2 a^2 \pi\hat{z}\left[-\frac{1}{4} - 0\right]$

$\vec{F} = -\frac{9\varepsilon_0 E_0^2 a^2 \pi}{4}\hat{z}$

But this is the force pushing on the lower hemisphere, so the force needed to keep it in place is equal, but of opposite magnitude

$\vec{F}_{lower} = \frac{9}{4}\varepsilon_0 E_0^2 a^2 \pi \hat{z}$

And the upper hemisphere, by symmetry, is held in place by a force of opposite magnitude than the lower hemisphere (so, the original result)

$\vec{F}_{upper} = -\frac{9}{4}\varepsilon_0 E_0^2 a^2 \pi\hat{z}$ ✓

(b) If a charge is placed on a conducting shell, it will spread out evenly over the shell → you can add this charge/surface area to the induced surface charge density found in part (a)

$\sigma = 3\varepsilon_0 E_0 \cos(\theta) + \underbrace{\frac{Q}{4\pi a^2}}_{\text{even distribution of applied charge}}$

And then, similarly to in part (a), in order to avoid self-charge effects, the force is given by

$$\vec{F} = \frac{1}{2\varepsilon_0}\int \sigma^2 \hat{r}\, dA$$

again starting with the lower hemisphere

$$= \frac{a^2}{2\varepsilon_0}\hat{z}\int_0^{2\pi}\int_{\pi/2}^{\pi}\left(3\varepsilon_0 E_0\cos(\theta) + \frac{Q}{4\pi a^2}\right)\left(3\varepsilon_0 E_0 \cos(\theta) + \frac{Q}{4\pi a^2}\right)\cos(\theta)\sin(\theta)\, d\theta\, d\varphi$$

$$= \frac{a^2\pi}{\varepsilon_0}\hat{z}\int_{\pi/2}^{\pi}\left(9\varepsilon_0^2 E_0^2\cos^2(\theta) + \underbrace{6\varepsilon_0 E_0\cos(\theta)\frac{Q}{4\pi a^2}} + \frac{Q^2}{16\pi^2 a^4}\right)\cos(\theta)\sin(\theta)\, d\theta$$

This term represents the force on the applied charge due to the external field $E_0$. Because $E_0$ "sees" Q as a point charge, it does not affect the two hemispheres separately so we can neglect it (does not contribute to separation)

$$= \frac{a^2\pi}{\varepsilon_0}\hat{z}\int_{\pi/2}^{\pi} 9\varepsilon_0^2 E_0^2\cos^3(\theta)\sin(\theta)\, d\theta + \frac{\cancel{a^2}\pi}{\varepsilon_0}\hat{z}\int_{\pi/2}^{\pi}\frac{Q^2}{16\pi^2 a^{\cancel{4}\,2}}\cos(\theta)\sin(\theta)\, d\theta$$

$$= 9a^2\varepsilon_0 E_0^2\pi\hat{z}\int_{\pi/2}^{\pi}\cos^3(\theta)\sin(\theta)\, d\theta + \frac{Q^2}{16\pi a^2\varepsilon_0}\hat{z}\int_{\pi/2}^{\pi}\cos(\theta)\sin(\theta)\, d\theta$$

$$= 9a^2\varepsilon_0 E_0^2\pi\hat{z}\underbrace{\left[\frac{-1}{4}\cos^4(\theta)\right]_{\pi/2}^{\pi}}_{-1/4} + \frac{Q^2}{16\pi a^2\varepsilon_0}\hat{z}\underbrace{\left[\frac{-1}{2}\cos^2(\theta)\right]_{\pi/2}^{\pi}}_{-1/2}$$

$$\vec{F} = \frac{-9}{4}a^2\varepsilon_0 E_0^2\pi\hat{z} - \frac{Q^2}{32\pi a^2\varepsilon_0}\hat{z}$$

But again, as with part (a), this is the force on the lower hemisphere, so the force required to keep the lower hemisphere in place is equal but of opposite magnitude:

$$\vec{F}_{lower} = \left[\frac{9}{4}a^2\varepsilon_0 E_0^2\pi + \frac{Q^2}{32\pi a^2\varepsilon_0}\right]\hat{z}$$

and the force to keep the upper hemisphere in place, by symmetry, must be equal but opposite magnitude to this (so again, the original solution)

$$\vec{F}_{upper} = \left[\frac{-9}{4}a^2\varepsilon_0 E_0^2\pi - \frac{Q^2}{32\pi a^2\varepsilon_0}\right]\hat{z}$$

6. Antonsen Problem 2.A

$z = x + iy$ ; $f(z) = f_R + i f_I$

(a) Laplace's equation: $\nabla^2 f = 0$

First, prove the Cauchy-Riemann condition for existence:

$\frac{df}{dz}$ exists @ z IFF the value is independent of the path $\longrightarrow \frac{df}{dx} = \frac{df}{dy}$

- In the x-direction

$$\frac{df}{dz} \equiv \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \frac{\partial f}{\partial x} = \frac{\partial f_R}{\partial x} + i \frac{\partial f_I}{\partial x}$$

- In the y-direction

$$\frac{df}{dz} \equiv \lim_{\Delta y \to 0} \frac{\Delta f}{i \Delta y} = \frac{1}{i} \frac{\partial f}{\partial y} = -i \frac{\partial f_R}{\partial y} + \frac{\partial f_I}{\partial y}$$

$$\to \frac{\partial f_R}{\partial x} + i \frac{\partial f_I}{\partial x} = -i \frac{\partial f_R}{\partial y} + \frac{\partial f_I}{\partial y}$$

$\to$ combine like terms $\left\{ \frac{\partial f_R}{\partial x} = \frac{\partial f_I}{\partial y} \right.$ ; $\frac{\partial f_I}{\partial x} = \frac{-\partial f_R}{\partial y}$

$$-\frac{\partial f_I}{\partial x} = \frac{\partial f_R}{\partial y}$$

Then, if Cauchy-Riemann is satisfied:

$$\frac{\partial}{\partial x} \frac{\partial f_R}{\partial x} = \underbrace{\frac{\partial}{\partial x} \frac{\partial f_I}{\partial y}} \qquad \underbrace{\frac{\partial}{\partial y} \left( \frac{-\partial f_I}{\partial x} \right)} = \frac{\partial}{\partial y} \frac{\partial f_R}{\partial y}$$

these two

cancel out because you can flip the order of differentiation for nice analytic functions

$$\frac{\partial^2 f_R}{\partial x^2} + \frac{\partial^2 f_R}{\partial y^2} = 0$$

and for $f_I$

$$\underbrace{\frac{\partial}{\partial y} \frac{\partial f_R}{\partial x}} = \frac{\partial}{\partial y} \frac{\partial f_I}{\partial y} \qquad \frac{\partial}{\partial x} \frac{\partial f_I}{\partial x} = \underbrace{\frac{\partial}{\partial x} \left( \frac{-\partial f_R}{\partial y} \right)}$$

these cancel, as above

$$\frac{\partial^2 f_I}{\partial y^2} + \frac{\partial^2 f_I}{\partial x} = 0$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\nabla^2 f_R = \frac{\partial^2 f_R}{\partial x^2} + \frac{\partial^2 f_I}{\partial y^2} = 0$$

$$\nabla^2 f_I = \frac{\partial^2 f_I}{\partial x^2} + \frac{\partial^2 f_I}{\partial y^2} = 0$$

$\Rightarrow$ Both $f_R$ and $f_I$ can both be seen here to satisfy Laplace's equation

(b) $f(z) = \arcsin(z)$

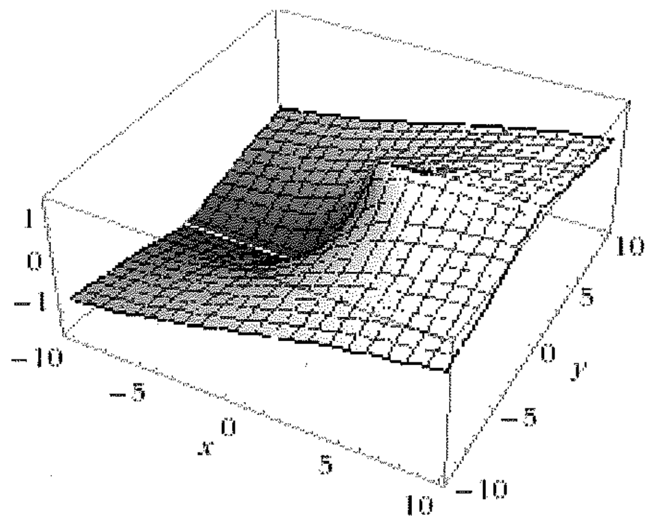
Contour plot of the real part of $f$ can be found on the following page.

$\rightarrow$ This is the potential for turbulent flow, such as air passing through part of a plane wing.

Input:

$$\mathrm{Re}\left(\sin^{-1}(x + i\,y)\right)$$

3D plot:




Contour plot: