Physics 606: Homework #1

Due: Tuesday Feb 9, 2016

Jackson: Problems 1.1, 1.5, 1.6, 1.12, and 1.13

Also,

Prob. 1A) Consider electrostatics in one dimension; i.e., $\rho = \rho(x)$, and $\Phi = \Phi(x)$. (i) What is the Dirichilet Green's function for the domain $0 \le x \le b$? (ii) What is the one-dimensional analog to Eq. (1.44)

Prob 1B) The potential surrounding a point charge, q, located at position $\mathbf{x}_0$ in a plasma is given by,

$$\Phi_0(\mathbf{x}) = \frac{q \exp(-|\mathbf{x}-\mathbf{x}_0|/\lambda_d)}{4\pi\varepsilon_0 |\mathbf{x}-\mathbf{x}_0|},$$

where λ_d is known as the Debye length. a) Find the charge density induced in the plasma. b) Show that it is proportional to the local potential.

Read "The Green on Green Functions" in the HNDO directory of the course web site.

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Homework 1

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1. Jackson Problem 1.1

Gauss' theorem:

$\int_S dA\, \hat{n}\cdot\vec{E} = \int_V d^3x\, \nabla\cdot\vec{E} = q/\varepsilon_0$

Eq. (1.21):

$\oint \vec{E}\cdot d\vec{l} = 0$

(a) – Fact: The electric field everywhere inside a conductor in static equilibrium is zero.

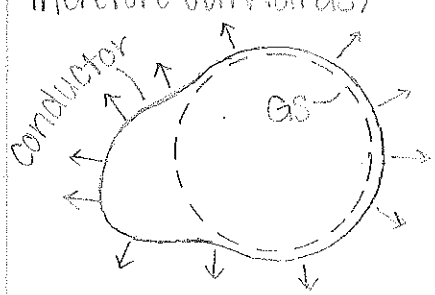
$\oint_S dA\, \hat{n}\cdot\vec{E} = \frac{q_{encl}}{\varepsilon_0} = 0$ ✓

Therefore q_{encl}, the charge enclosed within the Gaussian surface, must = 0 so long as the Gaussian surface is entirely inside the conductor.
→ Because we can make the GS any size we like, we may choose to make it take up the entirety of the interior of the conductor. If at this point we place excess charge on the conductor, we know it must reside on the surface as in order to continue to satisfy the aforementioned fact; the charge enclosed by our interior-encompassing GS must still = 0

(b) – Fact: The electric field at the surface of a conductor is normal to the surface.
Because the interior of this conductor is hollow by definition, that means it (and any GS within the hollow) contain no charge, allowing us to make the same electric field statement as in (a)

$\oint_S dA\, \hat{n}\cdot\vec{E} = \frac{q_{encl}}{\varepsilon_0} = 0$ ✓

and from the above fact, we know that the electric field vectors due to any charge on the surface will point in a direction normal to the surface (and therefore outwards)



We have already shown that the portion of the GS along the conductor must = 0, and since this is a closed surface integral, we also know that the non-touching portion must also have $\vec{E} = 0$, confirming

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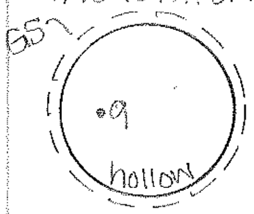
that <u>there can be no inward portion of an electric field even if there are external charges present.</u>

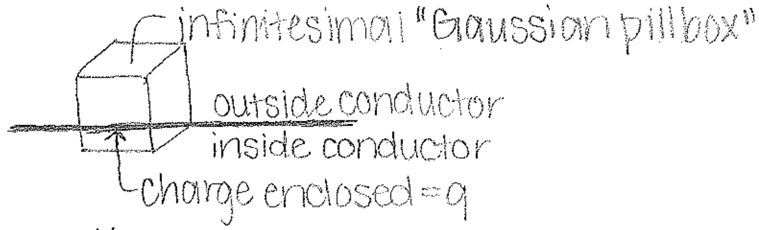
→ If we now place a charge q inside the hollow:

If we draw a GS just around the conducting shell, it must depend on the total charge enclosed.



The charge enclosed in this case must take into account not only any charge on the conductor's surface, but <u>also the charge placed in the hollow of the conductor</u> as it <u>is contained within the GS.</u>

(c) – Fact: The tangential component of a conductor's [surface] electric field equals zero.



Gauss' Law:

$$\underbrace{\oint_S \vec{E}\cdot\hat{n}\,dA}= \frac{q_{encl}}{\varepsilon_0}$$

= 0 for the sides of the "pillbox" because of the above fact concerning tangential contribution of $\vec{E}$

= 0 for the bottom because $q_{encl} = 0$ since it is inside the conductor, as seen in part (a)

$\int_{top} E\,dA = q_{encl}/\varepsilon_0 = q/\varepsilon_0$

↳ because the "pillbox" is infinitesimal this integral becomes ↓

$EA = q/\varepsilon_0$

$E = \frac{q}{\varepsilon_0 A}$ — Surface charge density = charge/area = σ

$\Rightarrow E = \frac{\sigma}{\varepsilon_0}$ ✓

2. Jackson Problem 1.5

The time-averaged potential of a neutral hydrogen atom is given by

$$\Phi = \frac{q}{4\pi\varepsilon_0}\frac{e^{-\alpha r}}{r}\left(1+\frac{\alpha r}{2}\right)$$

where q is the magnitude of the electric charge, and $\alpha^{-1} = a_0/2$, a_0 being the Bohr radius.

→ Use Poisson's equation to find the charge density

$$\nabla^2\Phi = -\rho/\varepsilon_0$$

└ want the Laplacian in spherical coordinates (best coordinate system for an atom)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)+\frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2(\theta)}\frac{\partial^2\Phi}{\partial\varphi^2} = \frac{-\rho}{\varepsilon_0}$$

But the potential is spherically symmetric

$$\frac{\partial\Phi}{\partial\theta} = \frac{\partial\Phi}{\partial\varphi} = \frac{\partial^2\Phi}{\partial\varphi^2} = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) = \frac{-\rho}{\varepsilon_0}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left[\frac{q}{4\pi\cancel{\varepsilon_0}}\left(\frac{e^{-\alpha r}}{r}+\frac{\alpha e^{-\alpha r}}{2}\right)\right]\right) = \frac{-\rho}{\cancel{\varepsilon_0}}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left[\frac{1}{r}\left(-\alpha e^{-\alpha r}\right)+e^{-\alpha r}\left(\frac{\partial}{\partial r}\frac{1}{r}\right)+\frac{-\alpha^2 e^{-\alpha r}}{2}\right]\right) = \frac{-4\pi\rho}{q}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(-r\alpha e^{-\alpha r}+r^2e^{-\alpha r}\left(\frac{\partial}{\partial r}\frac{1}{r}\right)+\frac{-r^2\alpha^2e^{-\alpha r}}{2}\right) = \frac{-4\pi\rho}{q}$$

$$\frac{1}{r^2}\left(\cancel{r\alpha^2e^{-\alpha r}}-\alpha e^{-\alpha r}+r^2e^{-\alpha r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\frac{1}{r}\right)+\left(\frac{\partial}{\partial r}\frac{1}{r}\right)\frac{\partial}{\partial r}\left(r^2e^{-\alpha r}\right)+\frac{r^2\alpha^3e^{-\alpha r}}{2}\cancel{-r\alpha^2e^{-\alpha r}}\right) = \frac{-4\pi\rho}{q}$$

$$\cancel{\frac{1}{r^2}}\left(\frac{-\alpha e^{-\alpha r}}{r^2}+\cancel{r^2}e^{-\alpha r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\frac{1}{r}\right)+\left(\frac{\partial}{\partial r}\frac{1}{r}\right)\left(-\alpha\cancel{r^2}e^{-\alpha r}+\frac{2re^{-\alpha r}}{r}\right)+\frac{\cancel{r^2}\alpha^3e^{-\alpha r}}{2}\right) = \frac{-4\pi\rho}{q}$$

$$\frac{-\alpha e^{-\alpha r}}{r^2}+e^{-\alpha r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\frac{1}{r}\right)+e^{-\alpha r}\left(\frac{\partial}{\partial r}\frac{1}{r}\right)\left(\frac{2}{r}-\alpha\right)+\frac{\alpha^3e^{-\alpha r}}{2} = \frac{-4\pi\rho}{q}$$ ✓

$$\rho = \frac{q}{4\pi}e^{-\alpha r}\left(\frac{\alpha}{r^2}-\frac{\alpha^3}{2}\right)-\frac{q}{4\pi}e^{-\alpha r}\left[\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\frac{1}{r}\right)+\left(\frac{2}{r}-\alpha\right)\left(\frac{\partial}{\partial r}\frac{1}{r}\right)\right]$$

→ because $1/r$ blows up at 0, we must break this up into two conditions

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– for $r>0$ (i.e., we can evaluate $1/r$):

$$\rho = \frac{q}{4\pi} e^{-\alpha r}\left(\cancel{\frac{\alpha}{r^2}} - \frac{\alpha^3}{2}\right) - \frac{q}{4\pi} e^{-\alpha r}\left[\frac{\partial}{\partial r}\left(\frac{-1}{r^2}\right) + \left(\frac{2}{r} - \cancel{\alpha}\right)\left(\frac{-1}{r^2}\right)\right]$$

$$= \frac{q}{4\pi} e^{-\alpha r}\left[\frac{-\alpha^3}{2} - \cancel{\frac{2}{r^3}} + \cancel{\frac{2}{r^3}}\right]$$

$$\rho = \frac{-\alpha^3 q}{8\pi} e^{-\alpha r} \text{ for } r>0$$

– for $r \simeq 0$ ($e^{-\alpha r} \sim 1$):

$$\rho = \frac{q}{4\pi}\left(\frac{\alpha}{r^2} - \frac{\alpha^3}{2}\right) - \frac{q}{4\pi}\underbrace{\left[\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\frac{1}{r}\right) + \left(\frac{2}{r} - \alpha\right)\frac{\partial}{\partial r}\frac{1}{r}\right]}_{\nabla^2(1/r) = -4\pi\delta(r)}$$

$$= \frac{q}{4\pi}\left(\frac{\alpha}{r^2} - \frac{\alpha^3}{2}\right) + \frac{q}{\cancel{4\pi}}\left[+\cancel{4\pi}\delta(r)\right]$$

$$\rho = \frac{q}{4\pi}\left(\frac{\alpha}{r^2} - \frac{\alpha^3}{2}\right) + q\delta(r) \text{ for } r \simeq 0$$

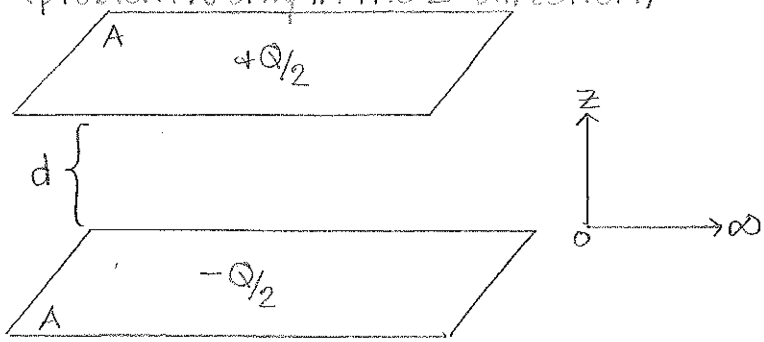
– then, for all r:

$$\rho = \frac{-\alpha^3 q}{8\pi} e^{-\alpha r} + q\delta(r)$$ ✓

3. Jackson Problem 1.6

(a) Two large, flat, conducting sheets of area A, separated by a small distance d

* Large sheets separated by a small distance ⇒ estimate the sheets as ∞ (problem is only in the z-direction)



– To calculate capacitance, $C = Q/V$, we must first determine the electric field between the plates.

$\vec{E} = -\nabla\varphi$

↳ for a uniform electric field (which we know this is because of symmetry) in a single direction (because the plates are essentially infinite in x&y), this equation becomes ↴

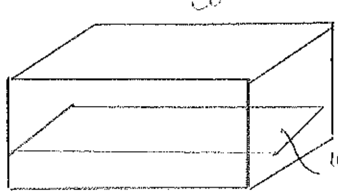
$$E = \frac{\varphi}{z}$$

Where $\varphi = V$, the potential difference between the plates.

$E = \frac{V}{d} \rightarrow C = \frac{Q}{Ed}$

– Use Gauss' law to determine the electric field and because the problem is symmetric, we only need to calculate it for one plate (here, the top plate)

$\oint \vec{E}\cdot\hat{n}\,dA = \frac{q_{encl}}{\varepsilon_0}$



$\uparrow\hat{n} \uparrow\vec{E} \Rightarrow \vec{E}\cdot\hat{n} = E$

"Gaussian Pillbox" same size as plate

$$\underbrace{\vec{E}\cdot\hat{n}}_{E_{1/2}}\int dA = \frac{(Q/2)}{\varepsilon_0}$$

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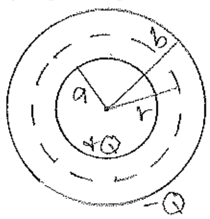
$$E_{1/2} = \frac{Q}{2A\varepsilon_0}$$ for one plate

total electric for both plates:

$$E = 2(E_{1/2}) = \frac{Q}{A\varepsilon_0}$$

$$\Rightarrow C = \frac{Q}{Ed} = \frac{\cancel{Q}}{d}\left(\frac{A\varepsilon_0}{\cancel{Q}}\right) = \frac{A\varepsilon_0}{d}$$

(b) Two concentric conducting spheres with radii a and b (b>a)



- Draw a spherical Gaussian surface between the conducting spheres ($a<r<b$)

$$\int_S dA\, \vec{E}\cdot\hat{n} = \frac{q_{encl}}{\varepsilon_0}$$

→ the electric field is symmetric between the spheres

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r}$$

$$\vec{E} = -\nabla\varphi$$

$$-\nabla\varphi = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r}$$

$$(-\nabla\varphi)\cdot\hat{r} = (Q/4\pi\varepsilon_0 r^2)\underbrace{\hat{r}\cdot\hat{r}}_{1}$$

$$\frac{d\varphi}{dr} = \frac{-Q}{4\pi\varepsilon_0 r^2}$$

Separation of variables method

$$\int_b^a d\varphi = \frac{Q}{4\pi\varepsilon_0}\int_b^a \frac{-1}{r^2}dr$$

$$\varphi = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r}\right]_b^a$$

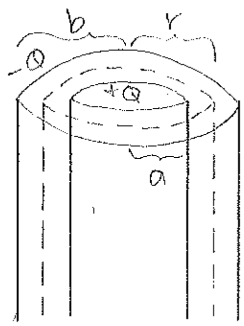
- constant potential: $\varphi = V$

$$V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$C = Q/V$

$C = \frac{4\pi\varepsilon_0}{\left(\frac{1}{a} - \frac{1}{b}\right)}$

(c) Two concentric conducting cylinders of length L, large compared to their radii a and b (b>a)



- As with (b), draw the Gaussian surface between the two conducting cylinders ($a<r<b$)

$\oint_S dA\, \vec{E}\cdot\hat{n} = \frac{q_{encl}}{\varepsilon_0}$

↳ the electric field is symmetric between the cylinders

$\vec{E} = \frac{Q}{2\pi\varepsilon_0 L}\frac{1}{r}\hat{r}$

$\vec{E} = -\nabla\varphi$

$-\nabla\varphi = \frac{Q}{2\pi\varepsilon_0 L}\frac{1}{r}\hat{r}$

$(-\nabla\varphi)\cdot\hat{r} = (Q/2\pi\varepsilon_0 L r)\underbrace{\hat{r}\cdot\hat{r}}_{1}$

$\frac{d\varphi}{dr} = \frac{-Q}{2\pi\varepsilon_0 L r}$

└ separation of variables method

$\int_b^a d\varphi = \frac{Q}{2\pi\varepsilon_0 L}\int_b^a -\frac{1}{r}dr$

$\varphi = \frac{Q}{2\pi\varepsilon_0 L}\left[-\ln(r)\right]_b^a$

- constant potential: $\varphi = V$

$V = \frac{Q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right)$

$C = Q/V$

$C = \frac{2\pi\varepsilon_0 L}{\ln(b/a)}$

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(a) The inner diameter of the outer conductor in an air-filled coaxial cable whose center conductor is a cylindrical wire of diameter 1 mm.

- Using the equation for capacitance of concentric cylinders found in part (c)...

$$C = \frac{2\pi\varepsilon_o L}{\ln(b/a)}$$

$$C\ln(b/a) = 2\pi\varepsilon_o L$$

$$\frac{b}{a} = e^{2\pi\varepsilon_o L/C}$$

$$b = ae^{2\pi\varepsilon_o L/C}$$

* $\varepsilon_o = 8.85\times10^{-12}$ F/m $= \varepsilon_o L$ for our purposes

→ for $C = 3\times10^{-11}$ F/m:

$$b = (1mm)\exp\left[\frac{2\pi(0.885\times10^{-11} F/m)}{(3\times10^{-11} F/m)}\right]$$

$$= (1mm)e^{2\pi(0.295)}$$

$$b = 6.38 \text{ mm}$$

→ for $C = 3\times10^{-12}$ F/m

$$b = (1mm)\exp\left[\frac{2\pi(8.85\times10^{-12} F/m)}{(3\times10^{-12} F/m)}\right]$$

$$= (1mm)e^{2\pi(2.95)}$$

$$b = 1.12\times10^{8} mm = 1.12\times10^{5} m$$

4. Jackson Problem 1.12

If φ is the potential due to a volume-charge density ρ within a volume V and a surface-charge density σ on the conducting surface S bounding the volume V, while φ' is the potential due to another charge distribution ρ' and σ', then:

$$\int_V \rho\varphi' d^3x + \int_S \sigma\varphi' dA = \int_V \rho'\varphi d^3x + \int_S \sigma'\varphi dA$$

↳ combine like-terms to resemble Green's theorem (from which we will start) ↓

$$\int_V d^3x\,[\rho\varphi' - \rho'\varphi] = \int_S dA\,[\sigma'\varphi - \sigma\varphi']$$

Green's Theorem:

$$\int_V (\varphi \underbrace{\nabla^2\varphi'} - \varphi'\nabla^2\varphi)\, d^3x = \oint_S \left(\varphi \frac{d\varphi'}{dn} - \varphi'\frac{d\varphi}{dn}\right) dA$$

↳ $\nabla^2\varphi' = -1/\varepsilon_0\,\rho'$; $\nabla^2\varphi = -1/\varepsilon_0\,\rho$; by Poisson's equation

$$\frac{-1}{\varepsilon_0}\int_V (\varphi\rho' - \varphi'\rho)\, d^3x = \oint_S \left(\varphi\frac{d\varphi'}{dn} - \varphi'\frac{d\varphi}{dn}\right) dA$$

→ From Jackson eq. (1.22)

$$(\vec{E}_2 - \vec{E}_1)\cdot\hat{n} = \sigma/\varepsilon_0$$

$$\underbrace{(-\nabla\varphi_2 + \nabla\varphi_1)\cdot\hat{n}} = \sigma/\varepsilon_0$$

$$\nabla\varphi\cdot\hat{n} = \frac{d\varphi}{dn}$$

↳ $\sigma = -\varepsilon_0\, d\varphi/dn$

$$\int_V (\varphi\rho' - \varphi'\rho)\, d^3x = \oint_S \Big(\underbrace{-\varepsilon_0\varphi\frac{d\varphi'}{dn}}_{=\varphi\sigma'} + \underbrace{\varepsilon_0\varphi'\frac{d\varphi}{dn}}_{=\varphi'\sigma}\Big) dA$$

$\int_V (\varphi\rho' - \varphi'\rho)\, d^3x = \oint_S (\varphi\sigma' - \varphi'\sigma)\, dA$, which matches the rearranged format presented above