Lecture 5 – Solutions to Fields & Enhancements

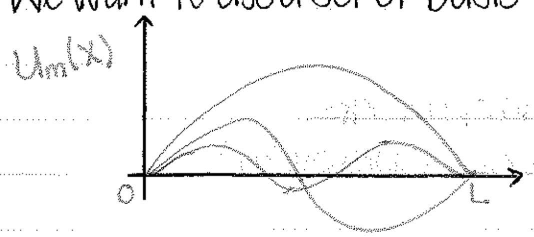
02/11/16

1. Non-Absolutely Converging System
2. Field Enhancements
3. Line Charge
   - over a planar conductor
   - over a groove/blade
4. Spherical Harmonics

1. Non-Absolutely Converging System

$\frac{\partial^2 \varphi}{dx^2} = f(x)$ Boundary Conditions: $\varphi(x=0) = 0$ – one plate held at ground; $\varphi(x=L) = V$ – one plate held at V

We want to use a set of basis functions



Even though these basis functions do not satisfy the Boundary Conditions, we can still use them to describe the potential, the solution just won't converge absolutely.

How can we use $U_m(x)$?

We can redefine $\tilde{\varphi}$

$$\frac{d^2\tilde{\varphi}}{dx^2} = f(x) \qquad \begin{array}{l}\tilde{\varphi}(0) = 0 \\ \tilde{\varphi}(L) = 0\end{array}$$

new boundary conditions satisfy the basis functions

to remove the part of the function that prevents the series from converging

But what if we don't redefine φ?

→ Use superposition

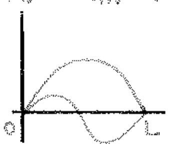
$$\varphi(x) = \sum_m a_m U_m(x), \quad \underbrace{\int_0^L dx\, U_m(x) U_{m'}(x) = \delta_{mm'}}_{\text{the basis functions are orthonormal}}$$

$$\frac{d^2\varphi}{dx^2} = f(x)$$

$$\underbrace{\int_0^L dx\, U_{m'}(x) \frac{d^2}{dx^2}\varphi}_{\text{integration by parts}} = \int_0^L dx\, U_m(x) f(x) = f(m')$$

*want to stipulate that you choose a basis function that has eigenvalues k_m

$$\frac{d^2}{dx^2}U_m(x) = -k_m^2 U_m(x)$$



$k_m = \frac{m\pi}{L}$ (only sines satisfy boundary conditions)

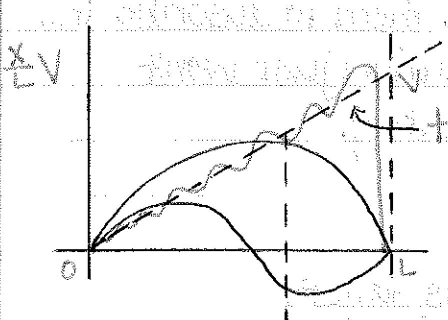
$$\int_0^L dx\, U_{m'}(x)\frac{d^2}{dx^2}\varphi = \Big[\underbrace{U_{m'}(x)\frac{d\varphi}{dx}}_{=0} - \underbrace{\varphi(x)\frac{dU_m}{dx}}_{=-V\frac{dU_{m'}}{dx}\big|_L}\Big]_0^L + \underbrace{\int_0^L dx\,\varphi(x)\frac{d^2}{dx^2}U_{m'}(x)}_{=-k_{m'}^2 \underline{a_{m'}}} = f_{m'}$$

coefficients of $\varphi(x)$'s

assuming $f(x)$ is a smooth function so it has a well-defined fourier coefficient

$$a_{m'} = \frac{1}{k_{m'}^2}\left[-f_{m'} + V\frac{d}{dx}U_{m'}\Big|_L\right]$$

this is the term that prevents absolute convergence (goes as $1/n$)

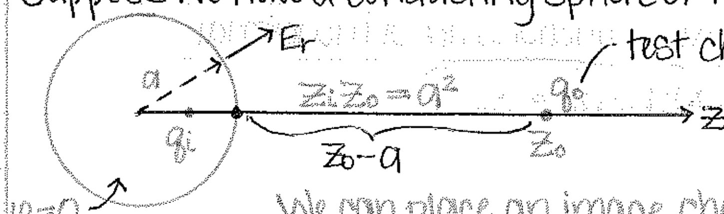


the ringing caused by the superposition to approximate a convergent function is known as Gibb's Phenomenon

x – larger x = closer to convergent (# of terms)

2. <u>Field Enhancements</u>

Suppose we have a conducting sphere of radius $= a$



test charge @ $z = z_0$

We can place an image charge q_i at some point $z = z_i$ within the sphere to ensure $\varphi = 0$ inside

define: $q_i = -q_0\frac{a}{z_0}$

$$E_r(\theta) = \frac{q_0}{4\pi\varepsilon_0}\left(1 - \frac{z_0^2}{a^2}\right)\frac{a}{[a^2 + z_0^2 - 2z_0 a\cos(\theta)]^{3/2}}$$

$\theta = 0$ @ black dot above

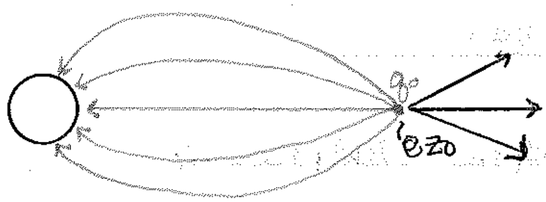
How does E_r without the sphere there compare? $(\theta=0)$

$$E_r = \frac{-q_0}{4\pi\epsilon_0}\frac{a}{|z_0-a|^2}$$

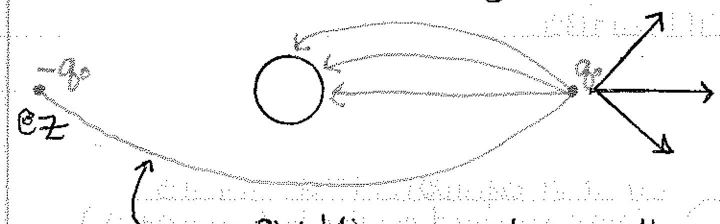
with sphere $(\theta=0)$

$$E_r = \frac{1}{4\pi\epsilon_0}\left(1-\frac{z_0^2}{a^2}\right)\frac{a}{|z_0-a|^3}$$

$$\frac{E_{r\,w/}}{E_{r\,w/o}} = \left(1+\frac{z_0}{a}\right)$$



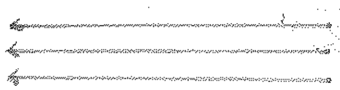
now, add a second charge:



some field lines now bypass the sphere to go to $-q_0$

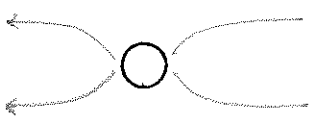
Adding the second charge decreases the field on the sphere!

→ If we were to take out the sphere...



resulting uniform field

...reinsert small sphere into the uniform field



It turns out that the field is always 3X BIGGER at the location of the center of the sphere, no matter the size of the sphere in the field.

For 2 charges:

With sphere

$$E_r = \frac{q_0}{4\pi\epsilon_0}\left(1-\frac{z_0^2}{a^2}\right)\left[\underbrace{\frac{a}{|z_0-a|^3}}_{\text{contribution from right charge}} - \overbrace{\frac{a}{|z+a|^3}}^{\text{contribution from left charge}}\right]$$

Without sphere

$$E_r = \frac{-q}{4\pi\varepsilon_0}\left(\frac{2}{z_0^2}\right)$$

- take the limit $z_0 \gg a$

$$\frac{1}{|z_0 \pm a|^3} = \frac{1}{z_0^3\left|1 \pm \frac{a}{z_0}\right|^3}$$

$$\simeq \frac{1}{z_0^3}\left(1 \mp 3\frac{a}{z_0}\right)$$

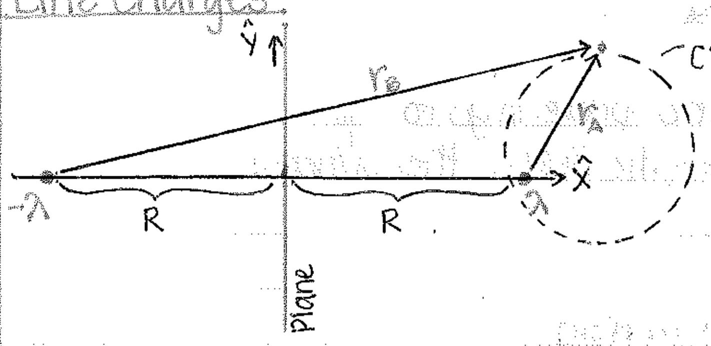
this causes the field with the sphere becomes ⤵

$$E_r \simeq \frac{-q_0}{4\pi\varepsilon_0}\frac{6}{z_0^2}$$

3x larger than if no sphere was present; independent of a!

*a sphere actually gives the smallest enhancement because of its smooth surface

3. Line Charges



$$r_A = \sqrt{y^2 + (x-R)^2}$$
$$r_B = \sqrt{y^2 + (x+R)^2}$$

Want to find ⤵

$$\varphi(x,y) = \frac{\lambda}{2\pi\varepsilon_0}\ln\left|\frac{r_A}{r_B}\right|$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln\left|\frac{r_A^2}{r_B^2}\right|$$

We can find the equipotential points by solving ⤵

$$\frac{r_A^2}{r_B^2} = \exp\left\{\frac{4\pi\varepsilon_0}{\lambda}\varphi\right\}$$

holding φ fixed

→ yields the equation for a circle

Want to expand this solution...

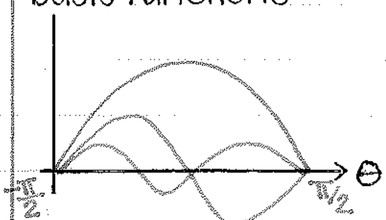
Now solve for potential using basis functions in θ

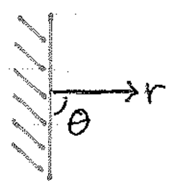
$$\nabla^2\varphi = -\frac{\lambda}{\varepsilon_0}\underbrace{\frac{\delta(\theta)\,\delta(r-R)}{R}}_{\delta\text{- function in 2D cylindrical coordinates}}$$

where in cylindrical coordinates we express φ as

$$\varphi = \sum_{n=1}^{\infty}\varphi_n(r)\sin\left(n\left(\theta+\frac{\pi}{2}\right)\right)$$

basis functions:





θ measured such that $\pi/2$ and $-\pi/2$ are located on the wall.

- multiply original equation, then integrate

$$\int_{-\pi/2}^{\pi/2}\sin\left(n\left(\theta+\pi/2\right)\right)d\theta\left\{\nabla^2\varphi = -\frac{\lambda}{\varepsilon_0}\frac{\delta(\theta)\,\delta(r-R)}{R}\right\}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\varphi_n - \frac{n}{r^2}\varphi_n(r) = -\frac{\delta(r-R)}{R}\underbrace{\left(\frac{2\lambda}{\pi\varepsilon_0}\sin\left(\frac{n\pi}{2}\right)\right)}_{\equiv\mu_N}$$

look for solutions @ $r=R$ that satisfy B.C.'s

approximate $\varphi_n \sim r^{\alpha}$, $\alpha = \pm n$

$$\hookrightarrow \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}r^{\alpha} = \alpha^2 r^{\alpha-2} = n^2 r^{\alpha-2}$$

- for $r<R$: $\varphi_n(r) = a_n r^n$
- for $r>R$: $\varphi_n(r) = b_n r^{-n}$

What can we say about these coefficients?

- By continuity of φ_n:

$$a_n R^n = b_n R^{-n}$$

- By evaluating jump conditions:

$$\int_{R-0}^{R+0}dr\left\{\frac{\partial}{\partial r}r\frac{\partial\varphi_n}{\partial r} - \frac{n^2}{r}\varphi_n = -\frac{r}{R}\delta(r-R)\,\mu_N\right\}$$

→ satisfying $\delta(r-R)$ ↴

$$R \frac{\partial \varphi_n}{\partial r}\Big|_{R-0}^{R+0} = -\mu_N$$

for both $r>R$ and $r<R$ conditions

$$-nb_n R^{-n} - (na_n R^n) = -\mu_n$$

↑ substitute for b_n in favor of a_n

$$-2na_n R^n = -\mu_n$$

$$a_n = \frac{\mu_n}{2n} R^{-n}$$

Then, for $r<R$:

$$\varphi(r,\theta) = \sum_n \frac{\mu_n}{2n}\left(\frac{r}{R}\right)^n \sin\left[n(\theta + \pi/2)\right]$$

and for $r>R$:

$$\varphi(r,\theta) = \sum_n \frac{\mu_n}{2n}\left(\frac{R}{r}\right)^n \sin\left[n(\theta + \pi/2)\right]$$

Now, evaluate the potential close to the origin:
(only odd-n have non-zero μ_n)
→ in the limit $r \ll R$

$$\varphi(r,\theta) \simeq \frac{2\lambda}{\pi\varepsilon_0}\frac{1}{2}\left(\frac{r}{R}\right)\sin\left(\theta + \frac{\pi}{2}\right)$$

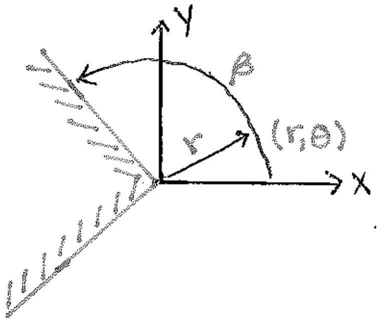
$$E_r\Big|_{\theta=0} = \frac{-\partial\varphi}{\partial r} = \underbrace{\frac{\lambda}{\pi\varepsilon_0}\frac{1}{R}}$$

Previously, using MOI, we saw that each line charge contributed $\frac{\lambda}{2\pi\varepsilon_0}\frac{1}{R}$, so this is consistent ✓

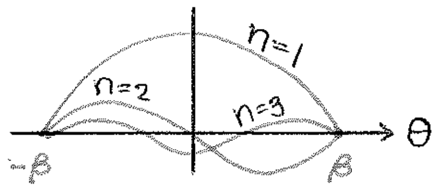
$$E(R)\Big|_{\theta=0} = \frac{\lambda}{2\pi\varepsilon_0 R}$$

* doing such an expansion allows us to evaluate in the case of, e.g., a groove or a blade

→



Boundary Conditions: Want basis functions that go to zero when $\theta = \pm\beta$



$$\varphi(r,\theta) = \sum_n \varphi_n \sin\left[\frac{\pi n}{2\beta}(\theta+\beta)\right]$$

a stretching of θ to satisfy the B.C.'s of the desired basis function set

We now must satisfy:

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\varphi_n - \frac{(n\pi/2\beta)^2}{r^2}\varphi_n(r) = -\frac{\delta(r-R)}{R}\left(\frac{2\lambda}{\pi\varepsilon_0}\sin\left(\frac{n\pi}{2}\right)\right)$$

such that $\beta = \pm\pi/2$ restores the original equation

$$\rightarrow \varphi(r,\theta) \simeq \frac{2\lambda}{\pi\varepsilon_0}\frac{1}{2}\left(\frac{r}{R}\right)^{(\pi/2\beta)}\sin\left(\frac{\pi}{2\beta}(\theta+\beta)\right) \quad \text{for } r \ll R$$

$$E_r = \frac{-\partial\varphi}{\partial r} = -C\frac{\pi}{2\beta}\frac{r^{(\pi/2\beta - 1)}}{R^{\pi/2\beta}}\sin\left(\frac{\pi}{2\beta}(\theta+\beta)\right)$$

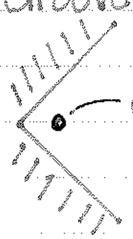
C — some constant

* fractional power has important consequences!

$E_r \rightarrow \infty$ as $r \rightarrow 0$ if $1 > \pi/2\beta$

↳ if $\beta > \pi/2$

Groove ($\beta < \pi/2$)



electric field very small inside this groove

Blade ($\beta > \pi/2$)

but $E \rightarrow \infty$ on the [perfect] tip of this blade (in reality, we round metal corners to prevent such field breakdowns)

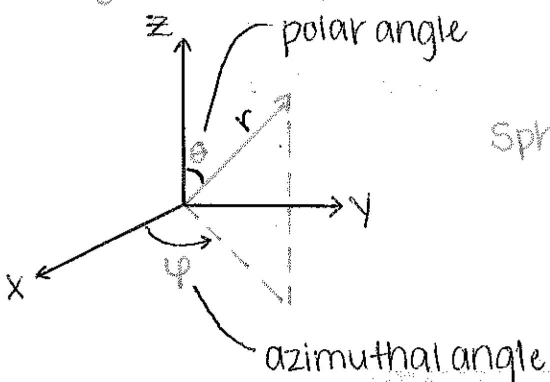
What about a conical tip?

→ the same breakdown will happen, just in spherical coordinates.

→

4. Spherical Harmonics

Solving Laplacian problems in spherical geometries.



Spherical geometries yield coordinates of (r, θ, φ)

Laplace's Equation in spherical coordinates:

$$\nabla^2 \phi = \underbrace{\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi)}_{=\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}} + \frac{1}{r^2}\underbrace{\left[\frac{1}{\sin(\theta)}\frac{\partial}{\partial \theta}\sin(\theta)\frac{\partial}{\partial \theta}\phi + \frac{1}{\sin^2(\theta)}\frac{\partial^2 \phi}{\partial \varphi^2}\right]}_{\text{angular contribution}} = 0$$

We want to find the eigenfunction Ψ and eigenvalues λ that satisfy this angular contribution:

$$\frac{1}{\sin(\theta)}\frac{\partial}{\partial \theta}\sin(\theta)\frac{\partial \Psi}{\partial \theta} + \frac{1}{\sin^2(\theta)}\frac{\partial^2 \Psi}{\partial \varphi^2} = -\lambda \Psi$$ spherical harmonic?

- demand that Ψ be periodic in φ

$$\Psi(\theta, \varphi) = e^{im\varphi}\Psi_m(\theta)$$

↳ removes φ-dependence

$$\frac{1}{\sin(\theta)}\frac{\partial}{\partial \theta}\sin(\theta)\frac{\partial}{\partial \theta}\Psi_m(\theta) - \frac{m^2}{\sin^2(\theta)}\Psi_m(\theta) = -\lambda \Psi_m(\theta)$$

If there are no restrictions on the movement of Ψ, then the Boundary conditions will be at $\theta = \pm\pi$

$\Rightarrow \lambda = \ell(\ell+1), \; \ell \geq m$

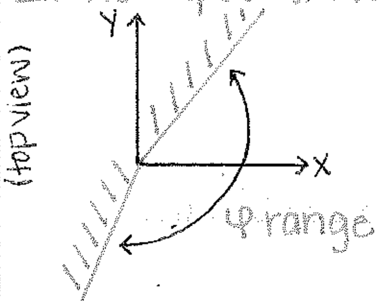
↑ integer

There's a degeneracy here as there are many λ's for one value of m

* this expression for λ results from writing a series solution for Ψ and then demanding that it terminates after a certain number of terms

→

If we impose a boundary such that φ is restricted...



m would no longer be an integer in this case.

ψ_m acquires an index ℓ

$$\psi_m^\ell = Y_{\ell m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, e^{im\varphi} P_\ell^m(\cos(\theta))$$

Legendre Polynomials