Lecture 25 - Relativistic Energy Conservation

05/03/16

1. Energy-Momentum 4-Vector
2. Collision Examples (conserves E-P)
3. Hamiltonian Formulation of Relativistic Motion
   - Charged Particle in a "Wiggler" Field

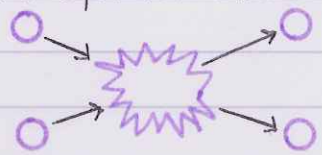
1. Energy-Momentum 4-Vector

Non-relativistically:

$\vec{p} = m\vec{u}$ $(\vec{u} = \vec{v})$

$E = \frac{1}{2}mu^2$

for a system of bodies that interact elastically



energy & momentum conserved

Relativistically:

$\left.\begin{array}{l} \vec{p} = m(u)\vec{u} \\ E = \mathcal{E}(u) \end{array}\right\}$ demand conservation in all frames

evolve under elastic collisions according to the Lorentz factor

$\vec{p} = \gamma m\vec{u}$ $\quad \gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$

$E = \gamma mc^2$

NOW: Say that we know momentum...

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right)$$

where we know that Maxwell's equations are correct and behave according to Poynting's Theorem

$$\frac{d}{dt}U + \int_S dA\, \hat{n}\cdot\vec{S} = \underbrace{-\int_V d^3x\, \vec{J}\cdot\vec{E}}_{\text{the rate at which energy is being given to the fields}}$$

$$\frac{d}{dt}\mathcal{E} = \vec{v}\cdot\frac{d\vec{p}}{dt} = q\underbrace{\vec{v}\cdot\vec{E}}$$

$\vec{J} = \sum_i q_i \vec{v}_i\, \delta(\vec{x} - \vec{x}_i(t))$

$\longrightarrow$

$\vec{p} = \gamma m \vec{v}$

$\hookrightarrow \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \rightarrow \gamma = \sqrt{1 + \frac{p^2}{m^2c^2}}$

$$\frac{d}{dt}\gamma = \frac{\vec{p}}{m^2c^2\gamma}\frac{d}{dt}\vec{p}$$

$$= \frac{1}{\sqrt{1+p^2/m^2c^2}} \cancel{\tfrac{1}{2}}\cancel{2}\frac{\vec{p}}{m^2c^2} \cdot \frac{d\vec{p}}{dt}$$

$$= \frac{1}{mc^2}\vec{v} \cdot \frac{d\vec{p}}{dt}$$

$\Rightarrow \underbrace{\mathcal{E} = \gamma mc^2}$

we have formed a 4-vector here

How do we know this is a 4-vector?

start with: $(ct, \vec{x})$, our standard 4-vector

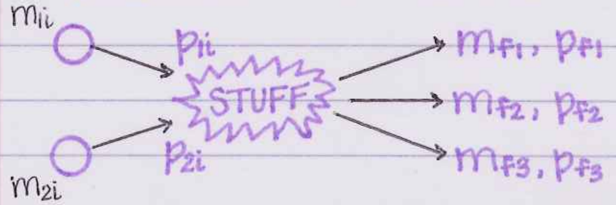
$d\tau = \frac{d\gamma}{d(\vec{v})}$, $\vec{v} = \frac{d\vec{x}}{dt}$

proper time; invariant

$$\frac{(cdt, d\vec{x})}{d\tau} = \left(c\frac{dt}{d\tau}, \frac{dt}{d\tau}\frac{d\vec{x}}{dt}\right) = (c\gamma, \gamma\vec{v})$$

$$= \underbrace{\left(\frac{E}{mc}, \frac{p}{m}\right)}_{\text{our 4-vector, confirmed}} \checkmark$$

ex. Conservation in a Collision



for our general four-vector $(A_0, \vec{A})$

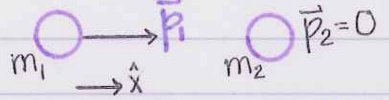
$$\sum_{j=1}^{N}(m_{ij}\gamma_{ij}c, p_{ij}) = \sum_{k=1}^{N}(m_{fk}\gamma_{fk}c, p_{fk})$$

a statement of conservation of energy & momentum

$A_0^2 - \vec{A}\cdot\vec{A} =$ invariant

$\rightarrow \sum_i \frac{E_i}{c} - |\Sigma \vec{p}|^2 =$ the same for all observers

The two particle case:



	Lab Frame	Center of Mass Frame
m_1	$\overset{p_x\ p_y\ p_z}{(\gamma_1 m_1 c;\ p_1, 0, 0)}$	$(\gamma_1' m_1 c;\ p_1', 0, 0)$
m_2	$(m_2 c;\ 0, 0, 0)$	$(\gamma_2' m_2 c;\ p_2', 0, 0)$
TOTAL	$(\gamma_1(m_1+m_2)c;\ p_1, 0, 0)$	$((\gamma_1' m_1 + \gamma_2' m_2)c;\ \underbrace{p_1'+p_2'}, 0, 0)$

Lorentz Transformation

We want to make this 0 (if there exists a COM frame where this is true)

transforms according to

$x_1' = \gamma(x_1 - \beta x_0)$

$(p_1' + p_2') = \gamma(v)(p_1 - \beta(\gamma_1 m_1 + m_2)c)$

in only the frame of m_1 (lab)

$\hookrightarrow \beta = \frac{p_1}{(\gamma_1 m_1 + m_2 c)} < \beta_1 < c$ ✓

$\hookrightarrow \beta_1 = p_1/\gamma_1 m_1$

So there does exist a COM frame where the momentum = 0

E' = energy available in the COM frame (for the final state)

$\hookrightarrow$ *may give rise to more particles

$$\frac{(E')^2}{c^2} = \overbrace{(\gamma_1 m_1 c + \gamma_2 m_2 c)^2}^{\text{initial energy of the two particles}} - \underbrace{(\vec{p}_1 + \vec{p}_2)^2}_{\text{initial momentum of the two particles}}$$

$$= \gamma_1^2 m_1^2 c^2 + \gamma_2^2 m_2^2 c^2 + 2\gamma_1\gamma_2 m_1 m_2 c^2 - (p_1^2 + p_2^2 + 2\vec{p}_1 \cdot \vec{p}_2)$$

$\gamma_1^2 = 1 + p_1^2/m_1^2 c^2, \quad \gamma_2^2 = 1 + p_2^2/m_2^2 c^2$

$$= m_1^2 c^2 + m_2^2 c^2 + 2(\gamma_1 \gamma_2 m_1 m_2 c^2 - \vec{p}_1 \cdot \vec{p}_2)$$

$\vec{p}_2 = 0,\ \gamma_2 = 1$ in the Lab frame

$$\frac{(E')^2}{c^2} = (m_1^2 + m_2^2)c^2 + 2\gamma_1 m_1 m_2 c^2$$

energy available in terms of initial masses

Define: $T_1 \equiv (\gamma_1 - 1)m_1c^2$; the kinetic energy of particle #1

extra energy it has by virtue of its motion

$$(E')^2 = (m_1^2 + m_2^2)c^4 + 2T_1m_2c^2 + m_1m_2c^4$$

⇒ How much energy does particle #1 need to have so that upon collision it makes new particles? (the accelerator problem)

$$(E')^2 = (m_1 + m_2)^2c^4 + 2T_1m_2c^2$$

$$E' \propto \sqrt{T_1}$$

if you increase the energy of one particle by 4, you only double the energy available post-collision (this is why it makes sense to accelerate both particles in a collider)

Threshold Calculation:

So how much energy is needed? The minimum energy needed to make particles is the rest energy of their masses.

$$E' = \sum_k m_k c^2$$

requirement on T_1

$$\frac{T_1}{m_1c^2} > \frac{(\sum_k m_k)^2 - (m_1 + m_2)^2}{2m_1m_2}$$

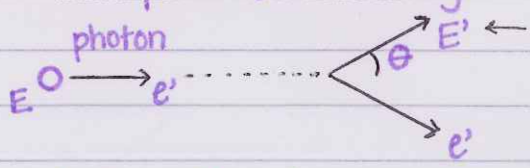
For four final particles (all of amu = 1)

$p + n$ (2 amu) $\rightarrow p + n + p + \bar{p}$ (4 amu)

$$\frac{T_1}{m_1c^2} > \frac{4^2 - 2^2}{2} = 6$$

actual experimental value = 5.63 GeV

It can also be helpful to apply this Lorentz invariance to Compton Scattering:



What is the energy of the scattered photon when it is at an angle θ with respect to the initial photon.

$\gamma + e^- \rightarrow \gamma' + e^{-'}$ $(p_1 + p_2) > (p_3 + p_4)$

$$\vec{p}_1 = \left(\frac{E}{c}, \frac{E}{c}\hat{z}\right) \qquad \vec{p}_3 = \left(\frac{E'}{c}, \frac{E'}{c}\hat{n}\right)$$

unit vector at an angle to $\hat{z}$

at rest $\Longrightarrow$

$$\vec{p}_2 = (m_e c, \vec{0}) \qquad \vec{p}_4 = \left(\frac{E'_{e^-}}{c}, \vec{p}\,'_{e^-}\right)$$

Such that we may relate the energies of the initial & resultant photon and electron as a 4-vector

$(\vec{p}_1 - \vec{p}_3) = (\vec{p}_4 - \vec{p}_2) = \vec{q}$, four-vector of the momentum transfer

→ use q to determine E'_{e^-}

$$\vec{q}\cdot\vec{q} = (\vec{p}_1 - \vec{p}_3)\cdot(\vec{p}_1 - \vec{p}_3)$$
$$= \underbrace{\vec{p}_1\cdot\vec{p}_1}_{0} + \underbrace{\vec{p}_3\cdot\vec{p}_3}_{0} - 2\underbrace{\vec{p}_1\cdot\vec{p}_3}_{\frac{EE'}{c^2}(1-\underbrace{\hat{n}\cdot\hat{z}}_{\cos(\theta)})}$$

$$= -2\frac{EE'}{c^2}(1-\cos(\theta))$$

$$\vec{q}\cdot\vec{q} = (\vec{p}_4 - \vec{p}_2)\cdot(\vec{p}_4 - \vec{p}_2)$$
$$= \underbrace{\vec{p}_4\cdot\vec{p}_4}_{m_{e^-}^2c^2} + \underbrace{\vec{p}_2\cdot\vec{p}_2}_{m_{e^-}^2c^2} - 2\underbrace{\vec{p}_2\cdot\vec{p}_4}_{}$$

$$(m_{e^-}c, \vec{0})\cdot\left(\frac{E'_{e^-}}{c}, \vec{p}\,'_{e^-}\right) = m_{e^-}E'_{e^-}$$

$$= 2m_{e^-}^2c^2 - 2m_{e^-}E'_{e^-}$$

$$-2\frac{EE'}{c^2}(1-\cos(\theta)) = 2m_{e^-}^2c^2 - 2m_{e^-}E'_{e^-}$$

$$\Rightarrow E'_{e^-} = m_{e^-}c^2 + \frac{EE'}{m_{e^-}c^2}(1-\cos(\theta))$$

→ Applying conservation of energy to find E'

$$E'_{e^-} = (\underbrace{E + m_{e^-}c^2}_{\text{initial energy of the system}} - E')$$

$$E + m_{e^-}c^2 = E' + E'_{e^-}$$

$$E - E' = \frac{EE'}{m_{e^-}c^2}(1-\cos(\theta))$$

$$E = E'\left(1 + \frac{E}{m_{e^-}c^2}(1-\cos(\theta))\right)$$

$$\Rightarrow E' = \frac{E}{(1 + \underbrace{E/m_{e^-}c^2}_{\text{photon energy}}(1-\cos(\theta)))}$$

* Note that if the initial photon energy is much less than the rest mass of the electron, there is hardly any change in the photon energy after scattering

Recall: for photons, $E = \frac{hc}{\lambda}$

so our equation for E' may be alternately expressed as:

$$\frac{1}{\lambda'} = \frac{1}{\lambda}\frac{1}{(1+\lambda_c/\lambda(1-\cos(\theta)))}, \quad \lambda_c = \frac{hc}{m_{e^-}c^2} \text{ the } \underline{\text{Compton Wavelength}}$$

$$E_{e^-}\lambda_c = m_{e^-}c^2$$

2. <u>Relativistic Formulation</u> (the Hamiltonian picture)

When you have an electromagnetic interaction, the Hamiltonian takes the form

$$\mathcal{H} = T + V$$

$\hookrightarrow T = p^2/2m$ ← mechanical momentum

$\hookrightarrow \vec{p} = \underbrace{\vec{P} - \frac{q}{c}\vec{A}}$ ← canonical momentum ($\vec{P}$)

this form is required to get back the Lorentz force

If $\mathcal{H} = \mathcal{H}(\vec{x}, \vec{P})$

then $\underbrace{\frac{d\vec{P}}{dt} = -\frac{\partial \mathcal{H}}{\partial \vec{x}}}, \quad \frac{d\vec{x}}{dt} = \frac{\partial \mathcal{H}}{\partial \vec{P}}$

non-relativistically, this is the same as:

$$\frac{d\vec{p}}{dt} = q\left(-\frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right) + \frac{\vec{v}}{c}\times(\nabla\times\vec{A}) + \frac{\vec{v}}{c}\cdot\nabla\vec{A}$$

because:

$$\frac{d\vec{P}}{dt} = \frac{d}{dt}\left(\vec{p} + \frac{q}{c}\vec{A}(\vec{x},t)\right) = \frac{d\vec{p}}{dt} + \frac{q}{c}\overbrace{\left(\frac{\partial \vec{A}}{\partial t} + \vec{v}\cdot\nabla\vec{A}\right)}^{\text{explicit and convective derivative}}$$

$$= -\frac{\partial \mathcal{H}}{\partial \vec{x}} = -\frac{\partial}{\partial \vec{x}}\frac{(\vec{P} - \frac{q}{c}\vec{A})^2}{2m}$$

$$= -\frac{1}{m}\underbrace{\left(\vec{P} - \frac{q}{c}\vec{A}\right)}_{\vec{p}}\cdot\left(-\frac{q}{c}\frac{\partial \vec{A}}{\partial \vec{x}}\right)$$

$$= +\frac{\vec{p}}{m}\cdot\frac{q}{c}\frac{\partial}{\partial x}\vec{A} = \frac{d\vec{p}}{dt} + \underbrace{\frac{q}{c}\left(\frac{\partial\vec{A}}{\partial t} + \vec{v}\cdot\nabla\vec{A}\right)}$$

move this to the other side and see if you get a Lorentz force law (the rate of change of $\vec{p}$)

$$\frac{d\vec{p}}{dt} = q\left\{\frac{-1}{c}\frac{\partial\vec{A}}{\partial t} + \frac{1}{c}\left[\underbrace{\left(\vec{v}\cdot\frac{\partial}{\partial x}\vec{A}\right) - \vec{v}\cdot\nabla\vec{A}}_{\vec{v}\times(\nabla\times\vec{A})}\right]\right\}$$

$$\left(\vec{v}\cdot\frac{\partial}{\partial x}\vec{A} - \vec{v}\cdot\nabla\vec{A}\right)_i$$

$$= \sum_j\left(v_j\frac{\partial}{\partial x_i}A_j - v_j\frac{\partial}{\partial x_j}A_i\right)$$

*be careful about your component directions!

$$= q\left\{\underbrace{\frac{-1}{c}\frac{\partial\vec{A}}{\partial t}}_{\vec{E}} + \frac{1}{c}\vec{v}\times\underbrace{(\nabla\times\vec{A})}_{\vec{B}}\right\}$$

$$= q\left\{\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right\}$$

$\Rightarrow$ Hamilton's equations are the same as Newton's Law for the Lorentz force when the Hamiltonian has been expressed in terms of the canonical momentum.

So relativistically:

$$\mathcal{H} = T + V$$

$$= p^2/2m + q\phi \longrightarrow \gamma mc^2 + q\phi$$

scalar potential

$\uparrow$ γ must be expressed in terms of momentum

$$\gamma = \sqrt{1 + \frac{p^2}{m^2c^2}} = \sqrt{1 + \left(\vec{P} - \frac{q}{c}\vec{A}\right)^2 / m^2c^2}$$

we must have an expression in terms of the canonical momentum so that we know Newton's laws still hold true here

for $\mathcal{H} = \mathcal{H}(\vec{P}, \vec{x}, t)$

$$\frac{d\mathcal{H}}{dt} = \frac{\partial\mathcal{H}}{\partial t} + \frac{d\vec{x}}{dt}\cdot\frac{\partial\mathcal{H}}{\partial\vec{x}} + \frac{d\vec{P}}{dt}\cdot\frac{\partial\mathcal{H}}{\partial\vec{P}}$$

$$= \frac{\partial\mathcal{H}}{\partial t} + \underbrace{\frac{\partial\mathcal{H}}{\partial\vec{P}}\cdot\frac{\partial\mathcal{H}}{\partial\vec{x}} + \left(-\frac{\partial\mathcal{H}}{\partial\vec{x}}\right)\cdot\frac{\partial\mathcal{H}}{\partial\vec{P}}}_{0}$$

$$\frac{d\mathcal{H}}{dt} = \frac{\partial\mathcal{H}}{\partial t}$$

$\Rightarrow$ If the Hamiltonian does not depend explicitly on time, $\partial\mathcal{H}/\partial t = 0$ and $\mathcal{H}$ may be said to be a constant of motion

$\mathcal{H}$ as a Constant of Motion

Check that Hamilton's equations $\Rightarrow \frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right)$ for $\vec{v} = \frac{d\vec{x}}{dt}$

$$\frac{\partial\mathcal{H}}{\partial\vec{P}} = \frac{\vec{p}}{m\gamma} = \frac{d\vec{x}}{dt} = \vec{v}$$

look at the x-component of $\vec{P}$

$$\frac{\partial\vec{P}}{dt} = -\frac{\partial\mathcal{H}}{\partial\vec{x}} \to \frac{dP_x}{dt} = -\underbrace{\frac{\partial\mathcal{H}}{\partial x}}$$

* P_x is a constant of motion if $\partial\mathcal{H}/\partial x = 0$

$$\frac{d}{dt}\left[p_x + \frac{q}{c}A_x(\vec{x},t)\right] = -q\frac{\partial\phi}{\partial x} - \frac{\frac{1}{2}\frac{\partial}{\partial x}\left[(\vec{P}-\frac{q}{c}\vec{A})\cdot(\vec{P}-\frac{q}{c}\vec{A})\right]}{m\underbrace{\sqrt{1+(\vec{P}-q/c\vec{A})^2/m^2c^2}}_{\gamma}}$$

$A_x(\vec{x},t) = A_x(\vec{x}(t),t)$

$\to \frac{d}{dt}A_x = \frac{\partial A_x}{\partial t} + \vec{v}\cdot\nabla A_x$

$$\frac{dP_x}{dt} + \frac{q}{c}\left(\frac{\partial A_x}{\partial t} + \cancel{v_x\frac{\partial A_x}{\partial x}} + v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z}\right) = q\left[\frac{-\partial\phi}{\partial x} + \frac{1}{c}\underbrace{\left(\cancel{v_x\frac{\partial A_x}{\partial x}} + v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x}\right)}_{\vec{v}\cdot\partial\vec{A}/\partial x}\right]$$

$$\frac{dp_x}{dt} = q\left[\underbrace{-\left(\frac{\partial\phi}{\partial x} + \frac{1}{c}\frac{\partial A_x}{\partial t}\right)}_{E_x} + \frac{v_y}{c}\underbrace{\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)}_{B_z} - \frac{v_z}{c}\underbrace{\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)}_{B_y}\right]$$

$$\frac{dp_x}{dt} = q\left[E_x + \frac{1}{c}\underbrace{(v_yB_z - v_zB_y)}_{x\text{-component of }\vec{v}\times\vec{B}}\right] \checkmark$$

ex. Applying Hamiltonian formulation to an $\vec{E}\times\vec{B}$ configuration

$\left.\begin{array}{l}\vec{E} = E_0\hat{y}\\ \vec{B} = B_0\hat{z}\end{array}\right\}$ $\begin{array}{l}\phi = -E_0y\\ \vec{A} = -B_0y\hat{x}\end{array}$ — one of many possible choices for $\vec{A}$

$$\mathcal{H} = \sqrt{c^2(\vec{p} + q/cB_0y\hat{x})^2 + m^2c^4} - qE_0y$$

$\mathcal{H}$ depends on y but not on x, z, or t

$\Rightarrow \frac{\partial\mathcal{H}}{\partial x} = 0 \to P_x = p_x - \frac{q}{c}B_0y = \text{constant}$

$\Rightarrow \frac{\partial\mathcal{H}}{\partial z} = 0 \to P_z = p_z = \text{constant}$

$\Rightarrow \frac{\partial\mathcal{H}}{\partial t} = 0 \to \mathcal{H} = \text{constant}$

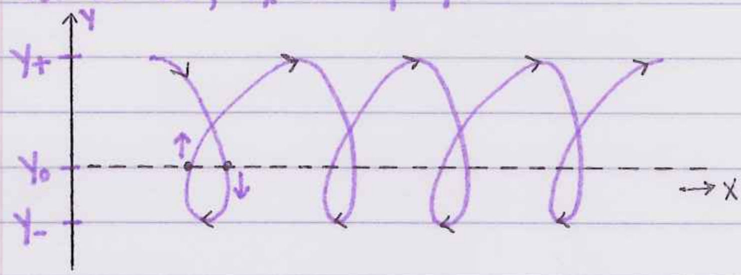
$\longrightarrow$

$p_x = \frac{q}{c}B_0 y + \underbrace{const.} \equiv \frac{q}{c}B_0(y-y_0)$

↳ our constant may be defined as $-\frac{q}{c}B_0 y_0$

following this: $\gamma v_x = \frac{qB_0}{mc}(y-y_0)$

for $B_0 > E_0$, $v_x = 0$ @ $y = y_0$



→ Employ the fact that $\mathcal{H}$ = constant to find y_+ and y_-:

$\mathcal{H} = \sqrt{c^2(p_x^2 + p_y^2) + m^2c^4} - qE_0 y$ = constant

└ plug in the expression defined above

$= \sqrt{c^2\{[q/cB_0(y-y_0)]^2 + p_y^2\} + m^2c^4} - qE_0 y$

$\mathcal{H} + qE_0 y = \sqrt{c^2\{[q/cB_0(y-y_0)]^2 + p_y^2\} + m^2c^4}$

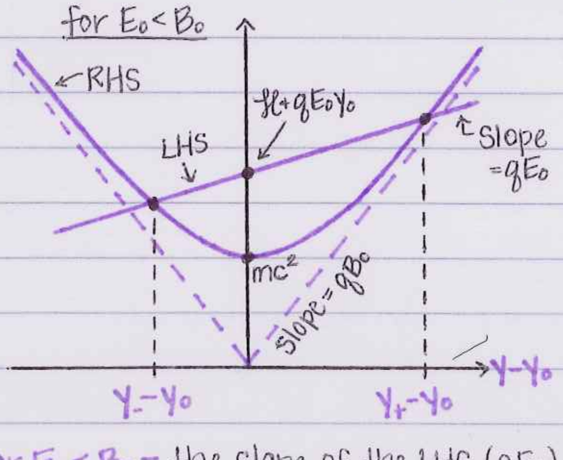
@ $y = y_0$: $\mathcal{H} + qE_0 y_0 = \sqrt{p_y^2c^2 + m^2c^4} > mc^2$

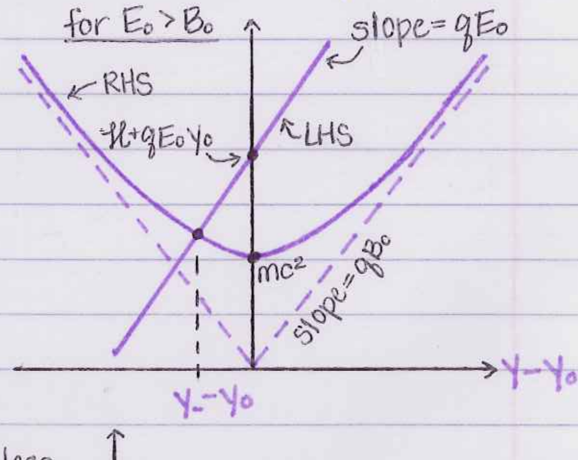
↑ we can use this to find p_y @ $y = y_0$

@ $y = y_\pm$: $p_y = 0$

↳ $\underbrace{\mathcal{H} + qE_0 y}_{\text{LHS}} = \underbrace{\sqrt{[qB_0(y-y_0)]^2 + m^2c^4}}_{\text{RHS}}$

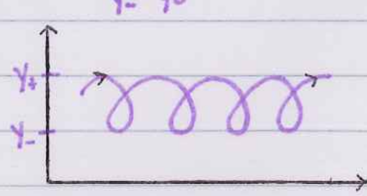
Plot the LHS & RHS versus $(y - y_0)$ to find constraints on $y_\pm$



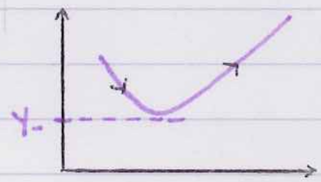


for $E_0 < B_0$ – the slope of the LHS (qE_0) is less than the slope of the RHS asymptotes (qB_0)

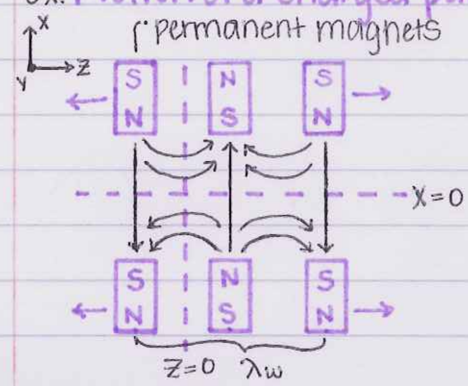
⇒ two roots for y: y_+ & y_-



for $E_0 > B_0$ – the slope of the LHS (qE_0) is greater than the slope of the RHS asymptotes (qB_0)
⇒ only one root for y: y_-



ex. Motion of a charged particle in a periodic magnetic field



*assume infinite in y
(no fringing field)

$(B_x(x,z);\ 0\ ;\ B_z(x,z))$

⇒ We want to find the $A_y(x,z)$ that generates this $\vec{B}$

↑ not A_x or A_z because they would require a component in B_y, which we know we do not have

Parity:

- B_x is an even function in x because it points away on either side of x=0
- B_z is an odd function in z because it changes direction along z=0

@ x=0, $\vec{A}$ & $\vec{B}$ must satisfy:

$\nabla \cdot \vec{B} = 0$

no changing $\vec{E}$ (no current in the region between the magnets)

$\nabla \times \vec{B} = 0 \rightarrow \nabla \times (\nabla \times \vec{A}) = 0$

$$\underbrace{\cancel{\nabla(\nabla \cdot \vec{A})}}_{0,\ \vec{A} = A_y \hat{y}} \underbrace{- \nabla^2 \vec{A} = 0}_{\text{must satisfy this!}}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) A_y(x,z) = 0$$

↑ A_y will be periodic in z with λ_w

choose:

cosine chosen for max value @ z=0

$A_y = A_0(x)\cos(k_w z)$

separation of variables

$k_w = \frac{2\pi}{\lambda_w}$

where

$B_x = -\frac{\partial}{\partial z} A_y \ ; \ B_z = \frac{\partial}{\partial x} A_y$

$$\underbrace{\left[\frac{\partial^2}{\partial x^2} A_0 - k_w^2 A_0\right]}_{\text{must} = 0 \text{ for this to hold for all values of } z} \cos(k_w z) = 0$$

Because of up-down symmetry, we can choose a hyperbolic solution for A_0

$$\frac{\partial^2}{\partial x^2}A_0 - k_w^2 A_0 = 0$$

there are actually higher-order-terms, but we ignore them in the near-axis approximation

$$\rightarrow A_0 = \underbrace{\cosh(k_w x)}_{\text{choose this solution}} \text{ or } A_0 = \sinh(k_w x)$$

$B_x = -\frac{\partial}{\partial z} \rightarrow A_0(x)$ must have the same parity in x as B_x, so $A_0(x)$ must also be even (choose cosh over sinh)

$$\Rightarrow A_y = \frac{B_w}{k_w}\cosh(k_w x)\cos(k_w z)$$

B_w = B-wiggle

such that

$$B_x = B_w \cosh(k_w x)\sin(k_w z)$$

$$B_z = B_w \sinh(k_w x)\cos(k_w z)$$

→ From these expressions we can see that a charge shot into this configuration will wiggle in and out of the board.

- accelerating charges radiate, and these wiggles cause <u>Synchrotron radiation</u>

Consider the motion in the xz-plane:

*because the fields are not changing in time, $\mathcal{H} = \mathcal{H}(\vec{x}, \vec{P})$

no explicit time-dependence

Hamilton's equations

$$\frac{dP_x}{dt} = \frac{-\partial\mathcal{H}}{\partial x} \; ; \; \cancel{\frac{dP_y}{dt} = \frac{-\partial\mathcal{H}}{\partial y}} \; ; \; \frac{dP_z}{dt} = \frac{-\partial\mathcal{H}}{\partial z}$$

because our fields do not depend on y, P_y = constant

$$\frac{dx}{dt} = \frac{\partial\mathcal{H}}{\partial P_x} \; ; \; \cancel{\frac{dy}{dt} = \frac{\partial\mathcal{H}}{\partial P_y}} \; ; \; \frac{dz}{dt} = \frac{\partial\mathcal{H}}{\partial P_z}$$

also ignorable

→ yields a Hamiltonian of the form

$$\mathcal{H} = mc^2\sqrt{1 + \frac{p_x^2}{m^2c^2} + \frac{p_z^2}{m^2c^2} + \frac{(P_y - q/c\,A_y)^2}{m^2c^2}}$$

where if we initially assume $p_x = 0$ ...

$$\frac{dz}{dt} = \frac{\partial\mathcal{H}}{\partial p_z} = \frac{p_z}{\gamma m} = v_z(z)$$

$$P_z(z) = mc\sqrt{\gamma_0^2 - \left[1 + \frac{(P_y - q/c A_y)^2}{m^2c^2}\right]}$$

from this, we can track trajectory over all time

$$\int dt = \int \frac{dz}{v_z(z)} \quad , \quad \int dt = \int \frac{dz}{\partial\mathcal{H}/\partial P_z}$$

BUT! We don't care about the evolution in time! It is therefore more intuitive to concern ourselves only with the evolution in z.

⇒ Canonical Transformation!

→ to get rid of dt

$$\frac{dP_x}{dt}\Big/\frac{dz}{dt} = \frac{dP_x}{dz} = \frac{-\partial\mathcal{H}}{\partial x}\Big/\frac{\partial\mathcal{H}}{\partial P_z} = \frac{\partial P_z}{\partial x}\Big|_{\mathcal{H}}$$

$$\frac{dx}{dt}\Big/\frac{dz}{dt} = \frac{dx}{dz} = \frac{\partial\mathcal{H}}{\partial P_x}\Big/\frac{\partial\mathcal{H}}{\partial P_z} = \frac{-\partial P_z}{\partial P_x}\Big|_{\mathcal{H}}$$

We've reduced 4 equations to 2 equations by canonical transformation

P_z becomes our new "Hamiltonian" to describe the motion!

$$P_z(p_x, x, z) = mc\sqrt{\gamma_0^2 - \left[\frac{(P_y - q/c A_y)^2 + p_x^2}{m^2c^2}\right]}$$

we've effectively reduced the amount of variables in our problem by 1