Lecture 24 - Four-Vectors

04/28/16

1. Review of Relativistic Motions & Fields
2. Lorentz Transformation in 4-Variables
3. Four-Vectors

1. Review of Relativistic Motions & Fields

Maxwell's Equations

$$\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}, \quad \vec{D} = \varepsilon\vec{E}$$

$$\nabla\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}, \quad \vec{B} = \mu\vec{H}$$

we can substitute fields into the particle equations $\longrightarrow$

Newton's Laws

Particles q_i, $i = 1 \to N$

$$\frac{d\vec{p}_i}{dt} = q_i(\vec{E} + \vec{v}\times\vec{B})$$

$$\rho = \langle \sum_i q_i \delta(\vec{x}-\vec{x}_i) \rangle$$

$$\vec{J} = \langle \sum_i q_i \vec{v}_i \delta(\vec{x}-\vec{x}_i) \rangle$$

$$\vec{p}_i = m_i \vec{v}_i \gamma_i$$

$$\gamma_i = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$\nabla\cdot\vec{E} = \rho/\varepsilon$$

$$\nabla\cdot\vec{B} = 0$$

$\longleftarrow$ particles create charge densities and currents $\longleftarrow$ $\frac{d\vec{x}_i}{dt} = \vec{v}_i$

Under manifest covariance:

$$\partial_\alpha \partial^\alpha A^\beta = \frac{4\pi}{c} J^\beta$$

$$\partial^\alpha A^\alpha = 0$$

$$\frac{dp^\alpha}{d\tau} = \frac{q}{c} F^{\alpha\beta} v_\beta$$

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$$

*not necessarily a more useful form!

2. Lorentz Transformation in 4-Variables

*previously just one space and one time variable

Transformation of the three spatial coordinates in time:

linear transform

$$\begin{cases} t' = \gamma(t - \frac{v}{c^2}z) \\ z' = \gamma(z - vt) \\ x' = x \\ y' = y \end{cases} \Longrightarrow \begin{array}{l} t = \gamma(t' + \frac{v}{c^2}z') \\ z = \gamma(z' + vt') \\ x = x' \\ y = y' \end{array}$$

where $z(t) = vt$ and c = constant in any frame

must be linear because space & time are assumed homogeneous & isotropic

$$\begin{pmatrix} t' \\ \vec{x}' \end{pmatrix} = \left((\vec{v}) \right) \begin{pmatrix} t \\ \vec{x} \end{pmatrix}$$

What if we have two events?

$\vec{x}_1, t_1$ and $\vec{x}_2, t^2$

<u>Invariant Separation</u>; the same in <u>any</u> inertial frame
(the separation in space and time of two events)

$$S_{12}^2 = c^2(t_1 - t_2)^2 - |\vec{x}_1 - \vec{x}_2|^2$$

↓ now go to the primed reference frame ↓

$\vec{x}_1', t_1'$ and $\vec{x}_2', t_2'$

$$S_{1'2'}^2 = c^2(t_1' - t_2')^2 - |\vec{x}_1' - \vec{x}_2'|^2 = \underbrace{S_{12}^2}_{\text{both frames agree on this value}}$$

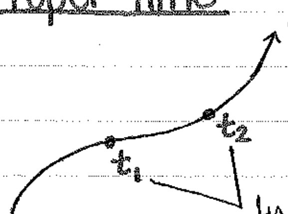
S_{12}^2 can allow us to determine if events are correlated
i.e., if one event caused the other

<u>$S_{12}^2 > 0$</u> this is a time-like separation; one event may have caused the other
- can find a frame where $\vec{x}_1 = \vec{x}_2$

<u>$S_{12}^2 < 0$</u> this is a space-like separation; neither event caused the other
- can find a frame where $t_1 = t_2$

*S_{12}^2 does NOT relate the regular frame to the primed frame, though

<u>Proper Time</u>



$\vec{u}(t)$ - velocity (assumed constant)

$t_2 = t_1 + dt$
$\vec{x}_2 = \vec{x}_1 + \vec{u}dt$

two events occuring at different times at different places

No matter what observance frame, all viewers would agree on S_{12}^2

$$S_{12}^2 = c^2dt^2 - |\vec{u}|^2 dt^2 = \frac{c^2dt^2}{\gamma_u^2} \quad \gamma_u = \frac{1}{\sqrt{1 - u^2/c^2}}$$

⇒ Interval of Proper Time

$$d\tau = \sqrt{\frac{S_{12}^2}{c^2}} = \frac{dt}{\gamma_u}$$

← you get the largest proper time interval for the smallest Lorentz factor

$\sqrt{S_{12}^2/c^2}$ – dependent only on frame independent (Lorentz invariant) quantities $\Rightarrow d\tau$ is frame independent

3. Four-Vectors

time-like component ↘ ↙ 3D space component

For a generic 4-vector $(A_0, \vec{A})$

where $A_0^2 - \vec{A}\cdot\vec{A} = \text{invariant}$

assume now a second vector $(B_0, \vec{B})$

generic, but different, four-vectors

Want to show

$$A_0B_0 - \vec{A}\cdot\vec{B} = A_0'B_0' - \vec{A}'\cdot\vec{B}'$$

a Lorentz invariant statement

First: Compactify the Lorentz Transformation

let $\beta \equiv v/c$

$$A_0' = \gamma(A_0 - \beta A_\parallel) \qquad B_0' = \gamma(B_0 - \beta B_\parallel)$$

$$A_\parallel' = \gamma(A_\parallel - \beta A_0) \qquad B_\parallel' = \gamma(B_\parallel - \beta B_0)$$

$$\vec{A}_\perp' = \vec{A}_\perp \qquad \vec{B}_\perp' = \vec{B}_\perp$$

- plugging in to obtain the above expression

$$A_0'B_0' - A_\parallel' B_\parallel' - \vec{A}_\perp'\cdot\vec{B}_\perp'$$

$$\gamma(A_0 - \beta A_\parallel)\gamma(B_0 - \beta B_\parallel) - \gamma(A_\parallel - \beta A_0)\gamma(B_\parallel - \beta B_0) - \vec{A}_\perp\cdot\vec{B}_\perp$$

$$= \gamma^2[A_0B_0 - \beta A_\parallel B_0 - \beta A_0 B_\parallel + \beta^2 A_\parallel B_\parallel] - [A_\parallel B_\parallel - \beta A_0 B_\parallel - \beta A_\parallel B_0 + \beta^2 A_0 B_0] - \vec{A}_\perp\cdot\vec{B}_\perp$$

$$= \underbrace{\gamma^2(1-\beta^2)}_{=1}\underbrace{[A_0B_0 - A_\parallel B_\parallel] - \vec{A}_\perp\cdot\vec{B}_\perp}_{\text{The same expression, just unprimed!}}$$

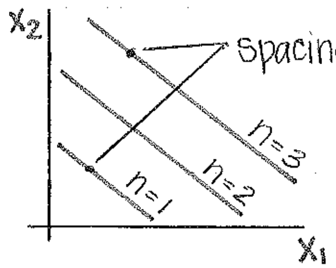
$\Rightarrow (B_0, \vec{B})$ must also be a 4-vector (proven, not just assumed so)

More invariant quantities:

→ Suppose we have a plane wave traveling through space

$\vec{E}(\vec{x}, t) = \mathrm{Re}\{\hat{E} e^{i\phi}\}$

↳ $\phi = \vec{k}\cdot\vec{x} - \omega t$



Spacing invariant

Let $\phi = 2\pi n$ be the wave crests
$\quad \llcorner n = \text{integer}$

$\rightarrow$ Under transform, we would not expect the number of crests between two points to change; ϕ must be invariant

$$\Rightarrow \phi' = \vec{k}'\cdot\vec{x}' - \omega' t' = \phi$$

If $(ct, \vec{x})$ is a four-vector, $(\omega/c, \vec{k})$ must be also!

alternate representation:

$$(\omega/c, \vec{k}) \longleftrightarrow \left(\frac{i}{c}\frac{\partial}{\partial t}, -i\nabla\right)$$
$$\hookrightarrow \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right)$$

$$ct = \gamma(ct' + \beta x')$$

$$\frac{\partial}{\partial t'} = \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} + \frac{\partial x}{\partial t'}\frac{\partial}{\partial x}$$

Applications to charge conservation:

– charge is conserved in all frames –

$$\left.\begin{aligned} &\frac{\partial \rho}{\partial t} + \nabla\cdot\vec{J} = 0 \\ &\frac{1}{c}\frac{\partial}{\partial t} c\rho + \nabla\cdot\vec{J} = 0 \end{aligned}\right\} \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right)\cdot \underbrace{(c\rho, \vec{J})}_{\text{must also be a 4-vector}} = 0$$

What about Maxwell's equations?

$\rightarrow$ switch to esu (where $\vec{E}$ & $\vec{B}$ have the same units here)

for scalar potential:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = -4\pi\rho \longrightarrow \nabla\cdot\vec{E} = 4\pi\rho, \quad \vec{F} = q\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right)$$

$$\nabla\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}, \quad \nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

for vector potential:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\frac{4\pi}{c}\vec{J}$$

$\Longrightarrow$ $\vec{A}$ & ϕ must transform like a 4-vector!

$$\begin{pmatrix}\phi \\ \vec{A}\end{pmatrix} = -\frac{4\pi}{c}\begin{pmatrix} c\rho \\ \vec{J}\end{pmatrix}$$

*the gauge condition for the vector potential expression is dependent on the Lorentz Gauge ⇒ our 4-vector statement is also dependent on the Gauge condition (which is preserved when you transform)

4-Vectors and Fields

Field components transform according to:

$\vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \beta \times \vec{B}_{\perp})$

$\vec{B}'_{\perp} = \gamma(\vec{B}_{\perp} - \beta \times \vec{E}_{\perp})$

↑ besides a γ-factor, this is the same as the rest frame

$\vec{E}'_{\parallel} = \vec{E}_{\parallel}, \quad \vec{B}'_{\parallel} = \vec{B}_{\parallel}$

ex: The Gyrotron

$\vec{E} = 0, \ \vec{B} = B_0 \hat{z}$

$$\frac{d\vec{p}}{dt} = \frac{q}{c}\vec{v} \times \vec{B} \qquad \gamma = (1 - v^2/c^2)^{-1/2}, \ \vec{p} = m\gamma\vec{v}$$

$\hookrightarrow \gamma = (1 + p^2/m^2c^2)^{1/2}$

γ is a constant!

$$\frac{d\gamma}{dt} = \frac{d\gamma}{d\vec{p}} \cdot \frac{d\vec{p}}{dt}$$

$$= \frac{\vec{p}}{m^2c^2\gamma}\left(\frac{q}{c}\vec{v} \times \vec{B}\right) = 0$$

↓ plugging in for $\vec{p}$ ↓

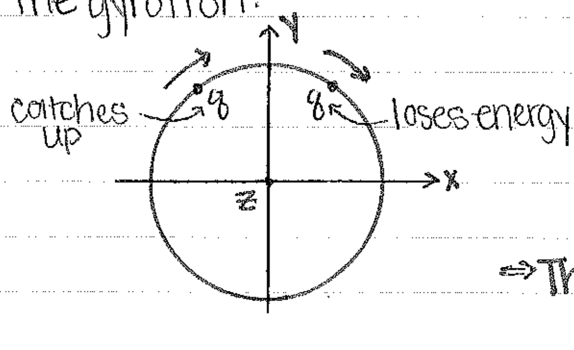
$$\frac{d}{dt}\gamma m\vec{v} = \frac{q}{c}\vec{v} \times \vec{B}$$

$$m\gamma\frac{d\vec{v}}{dt} = \omega_B \vec{v} \times \vec{B}_0 \Rightarrow \omega_B = \frac{qB_0}{mc\gamma}$$

larger energy → slower gyration

The Gyration Frequency

The gyrotron:



*Charges on such loops often in time tend to form clumps from displacement currents

→ the clumps will then radiate

⇒ This is known as the Negative Mass Effect

NOW: Add $\vec{E}(\vec{x},t) = \mathrm{Re}\{\hat{\vec{E}} e^{i\phi}\}$

↳ resulting effect ↴

γ is no longer constant

$$\frac{d\vec{p}}{dt} = q\left(\vec{E}_y + \frac{\vec{v} \times \vec{B}_z}{c}\right)$$

Want to separate $\vec{v}$ into gyration velocity and drift velocity

→ Define a frame velocity, $\vec{u}(\vec{x}) = u\hat{x}$ such that there is no $\vec{E}$

(we can then use our prior solution)

$$E_y' = \gamma_u \underbrace{\left(E_0 - \frac{u}{c} B_0\right)}_{\text{we want this} \rightarrow 0} \rightarrow \frac{u}{c} = \frac{E_0}{B_0}$$

(!!) This only works for $E_0/B_0 < 1$

If $E_0/B_0 > 1$

$$B_z' = \gamma\left(B_z - \frac{u}{c} E_0\right)$$