Lecture 23 – Special Relativity

04/26/16

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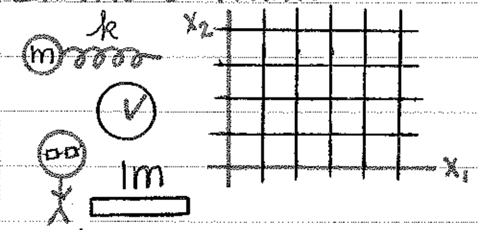
1. Covariance
2. Equivalence of Inertial Frames
3. Lorentz Transformation

1. Covariance

"Go forth & discover Newton's Laws"

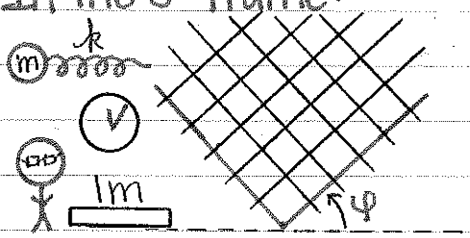
In the S-frame:



$$F_1 = m\frac{d^2x_1}{dt^2}, \quad F_2 = m\frac{d^2x_2}{dt^2}$$

↓ rotate the reference frame ↓

In the S'-frame:



In cartesian coordinates, the equations must look the same to be covariant!
(because space is isotropic)

$$F_1' = m\frac{d^2x_1'}{dt'^2}, \quad F_2' = m\frac{d^2x_2'}{dt'^2}$$

*the equations look the same!

These frames are related by some transformation.

Transformation of coordinates:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}}_{R = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$x_k' = R_{kl}\, x_l$$

$$x_l = \underbrace{R_{lk}}\, x_k'$$

where the return transformation is a transpose

Properties of R:

$$x' = Rx$$

$$x = R^T x'$$

$Rx = \underbrace{RR^T}_{RR^T = \mathbb{1}} x' = x'$

So for $RR^T = 1$

$R_{km} R^T_{ml} = \delta_{kl}$, $R_{km} R_{lm} = \delta_{kl}$

Return transformation over two matrices must sum over the inner indices

$\Rightarrow$ If we know the equations in the S-frame, we can deduce the equations in the S'-frame by transformation

Newton's Equations in S

Starting with the S-frame:

$m\ddot{x}_k = F_k$

$\rightarrow$ to get the equations in S', hit them with R

$R_{lk} \ddot{x}_k = R_{lk} \frac{F_k}{m}$

$m\ddot{x}'_{kl} = \underbrace{R_{lk} F_k}$ (S'-frame)

$$R_{lk} F_k = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{matrix} F_1 \cos(\varphi) - F_2 \sin(\varphi) \\ F_1 \sin(\varphi) + F_2 \cos(\varphi) \end{matrix}$$

But this is NOT covariant

$\rightarrow$ We must expand our definition to include vectors in our theory construct

To broaden covariance, introduce quantities called vectors whereby x_k is a vector and transforms like $x'_k = R_{kl} x_l$

Allow: All other quantities that also transform like vectors

↳ creates manifest covariance (k index the same on both sides)

- declare F_k to have the same transformation properties as x'_k

hit with rotation to get other equation

$F'_k = R_{kl} F_l$

in S': $\underbrace{m\ddot{x}'_l = F'}_{\text{now covariant!}}$

In General: All laws of nature must be relations between 3-vectors (and tensors)

→

→ **Manifest Covariance** ← (visually obvious covariance)

$$m\ddot{x}_k = F_k$$
$\downarrow$ apply R_{lk} $\downarrow$
$$mR_{lk}\ddot{x}_k = R_{lk}F_k$$
$$m\ddot{x}'_l = F'_l$$

✓ same form, etc.

This is a Galilean Transformation

Important to take to 4-Vectors?

Gradient $\vec{\nabla}$ is a vector:

Why? Because it transforms as one

$$f[x_k(x'_l)]\,,\quad x_k = R_{lk}x'_l$$

return transformation over inner index

$$\frac{\partial f}{\partial x'_l} = \underbrace{\frac{\partial x_k}{\partial x'_l}}\frac{\partial f}{\partial x_k}$$

Where we are summing over k

$$\frac{\partial x_k}{\partial x'_l} = R_{lk}$$

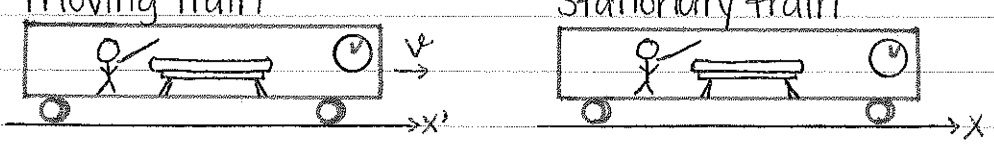
$$\rightarrow \frac{\partial}{\partial x'_l} = R_{lk}\frac{\partial}{\partial x_k}$$

Transforms just like a vector, so we may treat it as one!

2. <u>Equivalence of Inertial Frames</u>

⇒ leads to the idea of covariance in equations under inertial frame conditions

moving train　　　　stationary train



"If you were playing Pool, your strategy and technique would not change depending on what train you were on, so you know the equations of motion in both frames must be the same!"

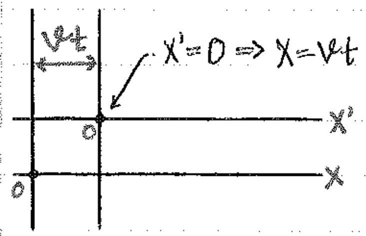
We know:

$$m\frac{d^2x}{dt^2} = F \rightarrow \text{Galilean Transformation} \rightarrow$$

We think we know:

$$x' = x - vt$$
$$t' = t$$

→



$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

$$\Rightarrow x = x' + vt'$$
$$t = t'$$

$$\frac{dx}{dt} = \frac{dx'}{dt} + v \longrightarrow \frac{d^2x}{dt^2} = \frac{d^2x'}{dt^2} = \frac{d^2x'}{dt'^2}$$

$dt = dt'$

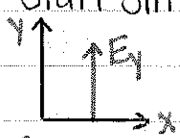
So in the Primed Frame:

$$m\frac{d^2x'}{dt'^2} = F' \ , \ \left(\frac{F}{m}\right)' = \left(\frac{F}{m}\right)$$

What about Maxwell's Equations?

→ Knowing covariance of 3-vectors, check covariance with respect to inertial frames

*Start simple: EM plane wave propagation



$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)E_y = 0$$ the lab equation

$f(x,t) \longrightarrow x(x',t'), \ t(t',x')$

$$\underbrace{\frac{\partial f}{\partial t} = \frac{\partial x'}{\partial t}\frac{\partial f}{\partial x'} + \frac{\partial t'}{\partial t}\frac{\partial f}{\partial t'}}_{\text{Jacobians}}$$

$$\frac{\partial f}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial f}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial f}{\partial t'}$$

↓ apply forward Galilean Transformation ↓

$$\frac{\partial x'}{\partial t} = -v \ , \ \frac{\partial t'}{\partial t} = 1 \ , \ \frac{\partial x'}{\partial x} = 1 \ , \ \frac{\partial t'}{\partial x} = 0$$

$$\Rightarrow \begin{cases} \partial_t \rightarrow \partial_{t'} - v\partial_{x'} \\ \partial_x \rightarrow \partial_{x'} \end{cases}$$

So in the S'-frame:

$$\left[\left(\partial_{t'} - v\partial_{x'}\right)^2 - c^2\partial_{x'}^2\right]E_y = 0$$

*Definitely NOT covariant!

extra $v\partial_{x'}$ term irreducible

for a propagating wave:

$e^{ikx-i\omega t}$, Fourier Transform $\partial_t \to -i\omega$, $\partial_x \to ik$

$-\omega^2 + c^2k^2 = 0 \Rightarrow \omega = \pm ck$ waves in either direction

$$\frac{\omega}{k} = -c, \quad \frac{\omega}{k} = +c$$

for the expected transform:

(trial plane wave solution that would work if actually covariant)

$e^{ikx'-i\omega t'}$ ↓ apply S'-equation

$-(+\cancel{i}\omega + \cancel{i}kv)^2 + c^2k^2 = 0$

$(\omega + kv)^2 = c^2k^2 \Rightarrow \omega + kv = \pm ck$

$$\longrightarrow \omega = \begin{cases} (c-v)k & (c-v) \\ (c+v)k & (c+v) \end{cases}$$

But we know this is not the case!

EITHER: Maxwell's equations are wrong

- Galilean transformation is wrong

OR our definition of covariance is wrong

Einstein - Demand that Maxwell's equations be covariant and find the transformation that does this (the brute force method)

Let there be a transformation that satisfies

$$\left.\begin{aligned} x_0' &= ax_0 + bx_1 \\ x_1' &= bx_0 + ax_1 \end{aligned}\right\} \text{where we've assumed a symmetric matrix}$$

↑ a linear transformation (homogeneity of space & time)

$$\left(\frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2}\right)E_y = 0 \quad \text{(S1)}$$

↑ Maxwell's equations in the lab frame that we know to be correct, written in x_0, x_1, coordinates (which has obvious symmetry)

Start with (S1) & transform the PDE using the given transformation:

$$\frac{\partial f}{\partial x_0} = a\frac{\partial f}{\partial x_0'} + b\frac{\partial f}{\partial x_1'}, \quad \frac{\partial f}{\partial x_1} = b\frac{\partial f}{\partial x_0'} + a\frac{\partial f}{\partial x_1'}$$

↓ plug in these intermediate partials ↓

$$[(a\partial_0' + b\partial_1')^2 - (b\partial_0' + a\partial_1')^2]E_y = 0$$

$a^2\partial_0'^2 + b^2\partial_1'^2 - b^2\partial_0'^2 - a^2\partial_1'^2 = 0$

cross-terms cancel out

$[(a^2-b^2)\partial_0'^2 - (a^2-b^2)\partial_1'^2]E_y = 0$

$\hookrightarrow$ compare to the lab-frame equation

$(\partial_0^2 - \partial_1^2)E_y = 0$

$\Rightarrow$ Condition for covariance: $(a^2-b^2)=1$

Alternative observation:

for $x_1' = 0 \rightarrow bx_0 + ax_1 = 0$

$bct + a\cancel{x} = 0$

$x = vt$

$bc + av = 0$

$\hookrightarrow b = -a(v/c) \rightarrow$ plugging in

$a^2 - a^2(v^2/c^2) = 1$

$a^2\left(1 - \frac{v^2}{c^2}\right) = 1$

$a^2 = \frac{1}{1 - v^2/c^2}$ ← the gamma factor!

This yields the

3. <u>Lorentz Transformation</u>

$$\underbrace{\begin{pmatrix} a & b \\ b & a \end{pmatrix} = a\begin{pmatrix} 1 & b/a \\ b/a & 1 \end{pmatrix}}$$

$$\Rightarrow \Lambda = \frac{1}{\sqrt{1-\beta^2}}\begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix}, \quad \beta = \frac{v}{c}$$

*reduces to Galilean transformation in the limit $v \to 0$

such that

$$\begin{pmatrix} x_0' \\ x_1' \end{pmatrix} = \Lambda \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$ ← covariant!

What happens if you apply Lorentz Transformation to Newton?

It doesn't work! Something is wrong with Newton's Laws!

∴ Newton's equations must be made to be covariant under Lorentz Transformation

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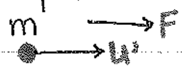
Einstein's prescription:

$$m\frac{du'}{dt'} = F$$



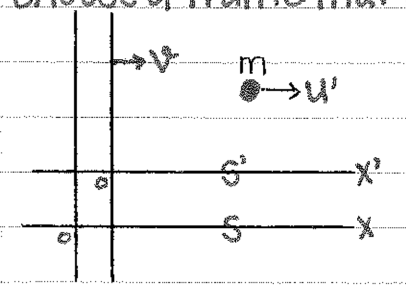
Where $m\,du' = F\,dt'$

$$\frac{dx'}{dt'} = u'$$

$\Rightarrow$ This is correct <u>provided</u> it is measured for $u' \to 0$

$dx' \to 0$ also

Choose a frame that is co-moving with the mass $(c=1)$



Choose the S'-frame such that $v = u'$

$\Rightarrow$ the above equations hold in the S'-frame if this is true

v independent of $u'(t')$ because it is co-moving

for the Lorentz Transformation:

$$x'(t) = \Gamma(x(t) - vt) \qquad \text{where } \Gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$t'(t) = \Gamma(t - vx(t))$$

↓ taking derivatives ↓

$$dx' = \Gamma(dx - v\,dt)$$

$$\hookrightarrow dx' = \Gamma(u - v)\,dt \quad \left(\frac{dx}{dt} = u,\ dx = u\,dt\right)$$

$$dt' = \Gamma(dt - v\,dx)$$

$$\hookrightarrow dt' = \Gamma(1 - vu)\,dt$$

need to find $\frac{du'}{dt'}$ ⌋ i.e., want to convert from the $v = u$ moving frame to the $u' \to 0$ lab frame

$$u' = \frac{dx'}{dt'} = \left(\frac{u - v}{1 - vu}\right)$$

$$du' = \frac{du}{1 - vu} + \frac{(u - v)}{(1 - vu)^2} + (+v\,du)$$

$$= \frac{du[(1 - \cancel{vu}) + (\cancel{u} - v)v]}{(1 - vu)^2} = \frac{du(1 - v^2)}{(1 - vu)^2}$$

↓ plugging in and manipulating ↓

$$\frac{du'}{dt'} = \frac{du}{dt}\frac{(1 - v^2)}{(1 - vu)^3\Gamma}$$

$$\Rightarrow \therefore \left[\frac{(1 - v^2)}{(1 - vu)^3\Gamma}\right]\frac{du}{dt} = \frac{F}{m} \leftarrow \text{New equation (lab frame)}$$

$u' \to 0,\ v \to u$

↓ applying provisions ↓

$$\gamma^2 = \frac{1}{1-u^2}, \quad \left[\frac{(1-v^2)}{(1-vu)^3\Gamma}\right] = \frac{\cancel{(1-u^2)}}{(1-u^2)^{3/2}\gamma}$$

$$= \frac{\gamma^4}{\gamma} = \gamma^3$$

This yields:

Modified Newton – Newton covariant under Lorentz transformation

$$m\gamma^3\frac{du}{dt} = F$$

$$\gamma^2 = (1-u^2)^{-1}, \quad \gamma^2 - (\gamma u)^2 = 1$$

$$\cancel{2}\gamma\dot{\gamma} = \cancel{2}(\gamma u)(\dot{\gamma u})$$

$$u\dot{u} = \frac{1}{\gamma^3}\dot{\gamma}$$

$$\dot{u} = \frac{(\dot{\gamma u})}{\gamma^3} \longrightarrow \frac{d}{dt}\underbrace{(\gamma m a) = F}_{\text{visually confirmable covariance}}$$

To get $E = mc^2$ from this, use the Work-Energy Theorem:

$$dW = F\,dx = Fu\,dt$$

$$mu(\dot{\gamma u}) = \frac{dW}{dt}$$

where $\gamma\dot{\gamma} = (\gamma u)(\dot{\gamma u}) \rightarrow \dot{\gamma} = u(\dot{\gamma u})$

$$m\frac{d\gamma}{dt} = \frac{dW}{dt}$$

$$\gamma m = \underbrace{(\gamma m)_0}_{\text{by units, this must be the rest energy}} + W$$

$$\gamma mc^2 = \underbrace{mc^2 + W}_{=\frac{E}{\gamma}} \checkmark$$