Lecture 1 – Introduction & Fundamentals   01/28/16

*There are potentially two 3rd ed's of the book — double check that the homework problems are the correct ones! (Assigned & due Tuesdays)

Website on department server:

www.glue.umd.edu/~antonsen/PHYS606_S16

→ there will be handouts not covered in class posted here

Jackson Chapter I: everything that won't be covered in class

Electrostatics ($\partial/\partial t = 0$)

Coulomb's law for the $\vec{E}(\vec{x})$ – field from charge

assumed evolution

$\vec{E}(t) = Re\{\hat{\vec{E}}\, e^{-i\omega t}\}$

ω – frequency

Differential

$\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0$

$\vec{E} = -\vec{\nabla}\underbrace{\varphi}_{\text{electrostatic potential}}$

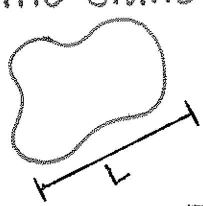
Gauss' Law

$\int_S dA\, n \cdot \vec{E} = Q/\varepsilon_0$

$\oint d\vec{l} \cdot \vec{E} = 0$

↑ Where knowing these two equations would allow us to derive Coulomb's law and Gauss' Law, etc.

*The "static" condition is relative!



L – typical length

T – typical time scale

→ Considered static if $T > L/c$ (c – speed of light)

ex. Fork in the microwave

For a wave, the static condition is written as

$\lambda = c/f \gg L$

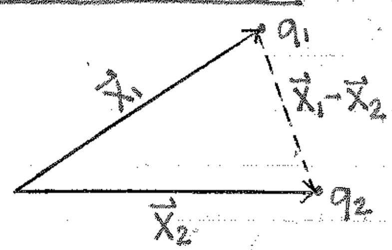
for a standard microwave: $f = 2.45$ GHz

$f = 2.45\text{ GHz} \sim 3\text{ GHz} \rightarrow T = 0.3\times10^{-9}$ sec.
This satisfies the static requirement: $T > L/c = 10^{-11}$ s

ex. Standard IR laser interacting with an atom
$\lambda = 1\mu m$, $L \sim a_0 = 5.3\times10^{-11}$ m
$\sim 10^{-6}$ m — the Bohr radius
$\lambda \gg L$ satisfied $\Rightarrow$ electrostatic conditions

*The electrostatic condition is valid even when fields are changing very rapidly if what you're talking about is small enough.

Coulomb's Law



$\vec{F}_{(12)}$ = force on q_1 due to q_2

$$\vec{F}_{(12)} = k\frac{q_1 q_2(\vec{x}_1-\vec{x}_2)}{|\vec{x}_1-\vec{x}_2|^3} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2(\vec{x}_1-\vec{x}_2)}{|\vec{x}_1-\vec{x}_2|^3}$$

takes care of repulsion/attraction signs

$$|\vec{F}_{(12)}| \propto \frac{1}{|\vec{x}_1-\vec{x}_2|^2} \propto q_1 q_2$$

Units: MKS-SI
- $\vec{F}$ – Newtons
- $\vec{x}$ – meters
- q – Coulombs; elementary charge $= 1.602\times10^{-19}$ C

↓ then, in these units ↓
$k = 8.988\times10^{9}\ N\cdot m^2/C^2$
$\hookrightarrow \varepsilon_0 = 1/4\pi k = 8.85\times10^{-12}$ F/m (Farads) (permittivity of free space)

We want a system of units where $k^{1/2}$ is absorbed into the definition of charge (more convenient to use)

Units: cgs-$\underbrace{\text{esu}}_{\text{electrostatic units}}$

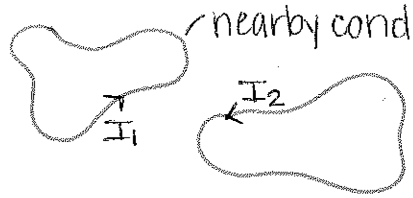
- q – stat coulombs

1 C $= 2.998\times10^{9}$ stat coulombs

But who says how much charge is in a coulomb?

→ We first set the Ampere ↓

Biot-Savart Law



$$\vec{F}_{12} = \frac{\mu_0 I_1 I_2}{4\pi} \iint \frac{d\vec{l}_1 \times [d\vec{l}_2 \times (\vec{x} - \vec{x}')]}{|\vec{x} - \vec{x}'|^2}$$

Ampere chosen such that $\underbrace{\mu_0}_{\text{permeability of free space}} = 4\pi \times 10^{-7}\ N/A^2$

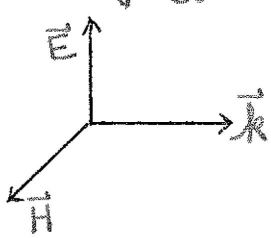
⇒ $\underbrace{Q}_{\text{charge of one Coulomb}} = 1\,\text{Ampere} \times 1\,\text{sec.}$

Relating ε_0 and μ_0:

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

but we can also use - Impedence of free space

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\ \Omega$$



$$\frac{|\vec{E}|}{|\vec{H}|} = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

Electric forces are generated in pairs...

$$\vec{F}_{(12)} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2 (\vec{x}_1 - \vec{x}_2)}{|\vec{x}_1 - \vec{x}_2|^3}$$

$$\vec{F}_1 = \sum_{i=2}^{n} \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_i (\vec{x}_1 - \vec{x}_i)}{|\vec{x}_1 - \vec{x}_i|^3}$$

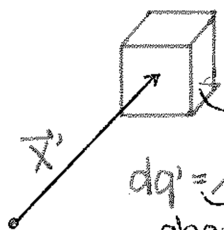
can pull q_1 out of sum → remainder is $\vec{E}$-field

$$\vec{F}_1 = q_1 \vec{E}_1(\vec{x}_1)$$

$$\hookrightarrow \vec{E}(\vec{x}_1) = \frac{1}{4\pi\varepsilon_0} \underbrace{\sum_{i=2}^{n}}_{\text{leave the charge you're trying to find the force on out of the sum}} \frac{q_i (\vec{x}_1 - \vec{x}_i)}{|\vec{x}_1 - \vec{x}_i|^2}$$

*BUT, this form does not allow you to include the self-field

of the particle of interest
→ yields a different answer depending on what charge you're trying to find the force on!
This problem goes away if you are dealing with a continuous charge distribution rather than point charges.
(we will weight the field by the charge density)



$dq' = \underbrace{\rho(\vec{x}')}_{\text{charge density}} d^3x'$

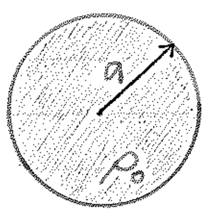
$$\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{\rho(\vec{x}')(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

└ this form handles the "not including self-charge" thing even if the denominator goes to zero since the numerator will go to zero faster.

ex. Applying Coulomb's Law to a simple problem
• Charge density spherically symmetric & uniform out to a radius 'a'

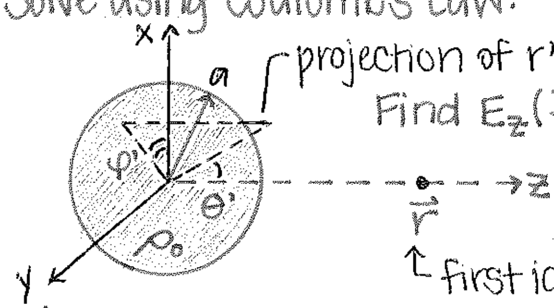
$$\rho(\vec{x}') = \rho(\vec{r}') = \begin{cases} \rho_0 , & |\vec{r}'| < a \\ 0 , & \text{otherwise} \end{cases}$$

└ spherical coordinates



Simplest solution?
Apply Gauss' Law!
$\varepsilon_0 \int_S dA\, \vec{n}\cdot\vec{E} = Q_{encl.}$
└ but this only works for very symmetric charges...

→ more generally ↴
Solve using Coulomb's Law:



┌ projection of r' onto the xy-plane
Find $E_z(\vec{z})$
└ other components cancel out

└ first identify the "point of observation"

• $\vec{r}$ is our observation point that lies along the z-axis
• $\vec{r}'$ is our source point and is variable
$\vec{x} = \vec{z}e_z = \vec{r}$

$d^3r' = r'^2 dr' d\varphi' \sin(\theta') d\theta'$

pull out

$$E_z = \frac{1}{4\pi\varepsilon_0} \int_0^a r'^2 dr' \int_0^\pi \sin(\theta') d\theta' \underbrace{\int_0^{2\pi} d\varphi'}_{2\pi} \frac{\rho_0 (\overbrace{\vec{e}_z \cdot \vec{r}}^{z} - \overbrace{\vec{e}_z \cdot \vec{r}'}^{r'\cos(\theta')})}{|\vec{r} - \vec{r}'|^3}$$

$$|\vec{r} - \vec{r}'|^2 = (\vec{r} - \vec{r}') \cdot (\vec{r} - \vec{r}') = |\vec{r}|^2 + |\vec{r}'|^2 - 2\vec{r}\vec{r}' = z^2 + r'^2 - 2r'z\cos(\theta')$$

$$= \frac{2\pi\rho_0}{4\pi\varepsilon_0} \int_0^a r'^2 dr' \underbrace{\int_0^\pi \sin(\theta') d\theta' \frac{z - r'\cos(\theta')}{[z^2 + r'^2 - 2r'z\cos(\theta')]^{3/2}}}_{I}$$

$$I = \int_0^\pi \sin(\theta') d\theta' \frac{z - r'\cos(\theta')}{[z^2 + r'^2 - 2r'z\cos(\theta')]^{3/2}} = \begin{cases} 2/z^2 , & r' < 0 \\ 0 , & r' > 0 \end{cases}$$

$$\rightarrow E_z(z) = \frac{2\pi\rho_0}{4\pi\varepsilon_0} \int_0^a r'^2 dr' \begin{cases} 2/z^2 , & r' < z \\ 0 , & r' > z \end{cases}$$

$$\Rightarrow E_z(z) = \frac{Q}{4\pi\varepsilon_0 z^2} \quad \text{where } Q = \begin{cases} 4/3\pi a^3 \rho_0 , & z > a \\ 4/3\pi z^3 \rho_0 , & z < a \end{cases}$$

└ you get this answer whether you say for $r' \leq z$ or $r' \geq z$ (on the surface of a hollow conductor, however, this is not true)

* For a spherical shell of charge, outside you get the same $\vec{E}$ as if all the charge were at a point located at the center. Inside the sphere, no matter where you are, the $\vec{E}$-field $= 0$ (cancels out)

– Always two answers to Coulomb's Law

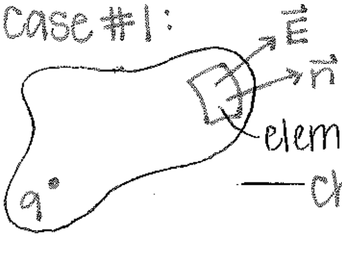
Gauss' Law – a direct consequence of Coulomb's Law

(how to go from Coulomb's law to the integral versions of the field equations)

$\varepsilon_0 \oint_S dA\, \vec{n} \cdot \vec{E} = \int_V d^3x\, \rho(\vec{x}) = Q_{encl.}$

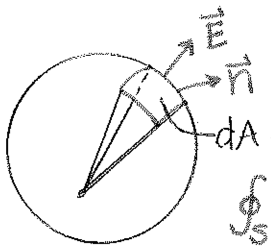
└ closed surface

• Case #1:



element of area, dA

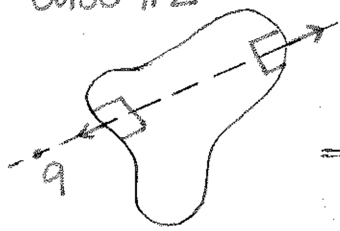
— choose a point on the edge →



dA subtends a solid angle, $d\Omega$

$$\oint_S dA\,\vec{n}\cdot\vec{E} = \int d\Omega \frac{q}{4\pi\varepsilon_0} = \frac{q}{\varepsilon_0}$$

- Case #2



No charge is enclosed
↳ the forces cancel out

$$\Rightarrow \int d\Omega\, q/4\pi\varepsilon_0 = \oint_S dA\,\vec{n}\cdot\vec{E} = 0$$

– We have shown here from Coulomb's Law that Gauss' Law applies.

To go from Gauss' Law to electrostatic potential, we must first define divergence.

$$\nabla\cdot\vec{B} \equiv \lim_{\Delta V\to 0} \frac{1}{\Delta V}\int_S dA\,\vec{n}\cdot\vec{B}$$

divergence tells about the amount of something leaving the Volume, V

Divergence theorem: (for any vector field, $\vec{B}$)

$$\int_S \underbrace{dA}_{\text{area}}\,\vec{n}\cdot\vec{B} = \int_V d^3x\,\nabla\cdot\vec{B}$$

└ the volume integral of this kind of derivative is just the field evaluated at the volume's endpoints (the surface)

this is akin to the fundamental theorem of calculus

$$\int_{x_1}^{x_2} dx\,\frac{df(x)}{dx} = f(x_2) - f(x_1)$$

More specifically, for $\vec{E}$:

$$\int_S dA\,\vec{n}\cdot\vec{E} = \int_V d^3x\,\nabla\cdot\vec{E} = \int_V d^3x\,\frac{\rho(x)}{\varepsilon_0}$$

↳ only if these are all equal does this hold for any field in any volume

<u>Re-expressing Coulomb's Law</u>

→ In terms of scalar potential

→ Utilizing the gradient

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\rho(x')\underbrace{\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}}$$

may be alternately expressed using the gradient

$$= -\nabla_x \frac{1}{|\vec{x}-\vec{x}'|}$$

$$\nabla_{\vec{x}} \frac{1}{|\vec{x}-\vec{x}'|} = \underbrace{\left(e_{\vec{x}}\frac{\partial}{\partial x} + e_{\vec{y}}\frac{\partial}{\partial y} + e_{\vec{z}}\frac{\partial}{\partial z}\right)}_{\nabla_{\vec{x}}}\underbrace{\frac{1}{[(x-x')^2+(y-y')^2+(z-z')^2]}}_{|\vec{x}-\vec{x}'|^{-1}}$$

$$= -\frac{e_{\vec{x}}(x-x') + e_{\vec{y}}(y-y') + e_{\vec{z}}(z-z')}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\rho(\vec{x}')\left(-\nabla_x \frac{1}{|\vec{x}-\vec{x}'|}\right)$$

$$\vec{E} = -\nabla_x \varphi(\vec{x})$$

$$\Longrightarrow \varphi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}$$