Lecture 17 - Behavior of Electromagnetic Plane Waves

03/31/16

1. Review of Plane Waves
2. Dispersion
3. Polarization
4. Dissipation

1. Review of Plane Waves

$\vec{E}(\vec{x},t) = \mathrm{Re}\{\hat{\vec{E}}\exp(i\vec{k}\vec{x} - i\omega t)\}$
$\vec{H}(\vec{x},t) = \mathrm{Re}\{\hat{\vec{H}}\exp(i\vec{k}\vec{x} - i\omega t)\}$ > both solutions to Maxwell's equations → also solutions to the Helmholtz eq.

where $\vec{E} = -\nabla\phi$, $\hat{\vec{E}} = -i\vec{k}\hat{\phi}$

then from Faraday's Law and Ampere's Law (w/ Displacement current):

$\vec{k}\times\hat{\vec{E}} = \omega\mu\hat{\vec{H}}$ (Faraday)
$\vec{k}\times\hat{\vec{H}} = -\omega\varepsilon\hat{\vec{E}}$ (Ampere) $\Rightarrow$ implicitly satisfies $\nabla\cdot\vec{H} = 0 = \vec{k}\cdot\vec{H}$, $\nabla\cdot\vec{D} = 0 = \vec{k}\cdot\varepsilon\vec{E}$

What if $\varepsilon(\omega)$?

$$\vec{k}\cdot(\vec{k}\times\hat{\vec{H}}) = 0 = \underbrace{-\omega\varepsilon\vec{k}\cdot\hat{\vec{E}}}_{\text{one value here must be 0}}$$

Typically, $\vec{k}\perp\hat{\vec{E}}$

*note: e.g., for plasma waves, there is a class of waves for which $\varepsilon(\omega) = 0 \Rightarrow \vec{k}\parallel\hat{\vec{E}}$ in that case

$$\underbrace{\vec{k}\times(\vec{k}\times\hat{\vec{E}})}_{=\vec{k}(\vec{k}\cdot\hat{\vec{E}}) - k^2\hat{\vec{E}}} = \omega\mu(\vec{k}\times\hat{\vec{H}})$$
$$= -\omega^2\varepsilon\mu\hat{\vec{E}}$$

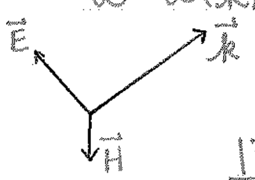
→ combining these

This all yields the dispersion relation

$\omega^2\varepsilon\mu = k^2$

└→ $\omega = \omega(\vec{k})$ *recall: $\lambda = c/f$; $f = c/\lambda$; $2\pi f = c\,2\pi/\lambda$

$\Rightarrow \omega = c|\vec{k}|$, the dispersion relation for vacuum



$\dfrac{|\hat{\vec{E}}|}{|\hat{\vec{H}}|} = \sqrt{\dfrac{\mu}{\varepsilon}} = \eta$, the intrinsic impedance of the medium

where $\eta_0 = \sqrt{\mu_0/\varepsilon_0} \sim 377\,\Omega$
└ the impedance of free space

→

2. Dispersion

Recall from the simple model of Polarizability:

k

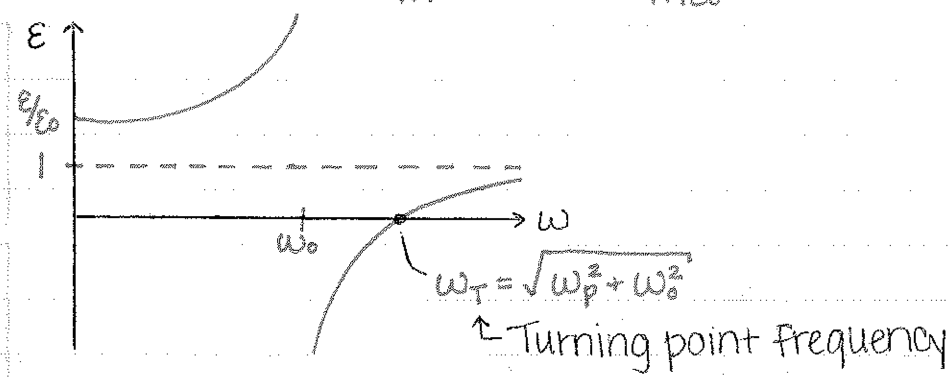
$$\varepsilon(\omega) = \varepsilon_0\left[1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2}\right]$$

where

$$\omega_0^2 = \frac{k_{sp}}{m} \quad ; \quad \omega_p^2 = \frac{N e^2}{m\varepsilon_0}$$

density

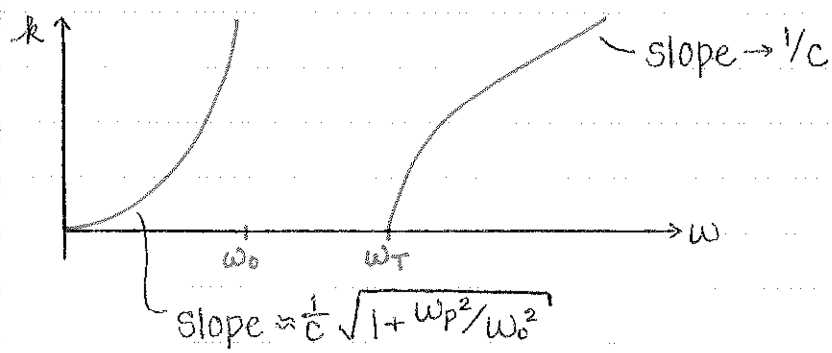
electron charge



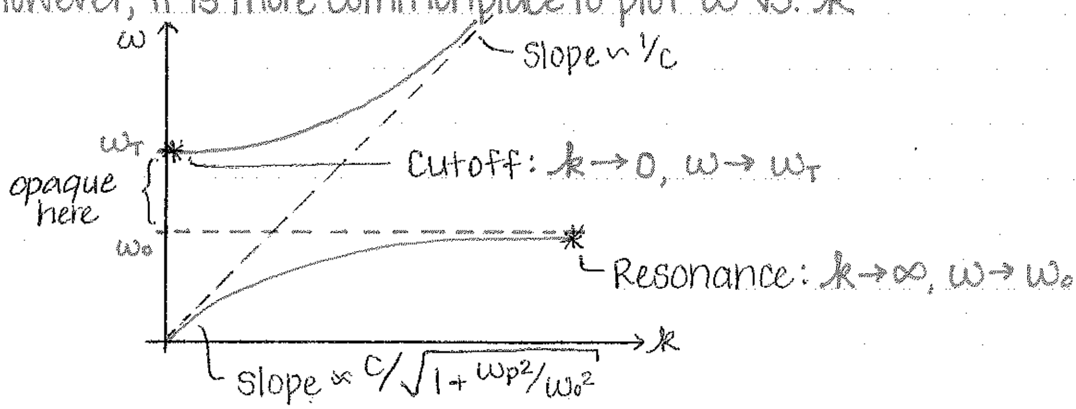
It is more informative for us to look at $k(\omega)$ here instead

→ assuming $\mu = \mu_0$?

$$k = \frac{\omega}{c}\sqrt{\frac{\varepsilon}{\varepsilon_0}}$$



However, it is more commonplace to plot ω vs. k



Why do we care about slope?

group velocity

$v_g = \frac{\partial \omega}{\partial k} = \frac{c}{\sqrt{1+k^2c^2/\omega_p^2}} < c$ group velocity must always be $< c$ as it carries both information & energy

$v_p = \frac{\omega}{k} = \frac{c}{\sqrt{\varepsilon/\varepsilon_0}}$ can be $> c$

phase velocity

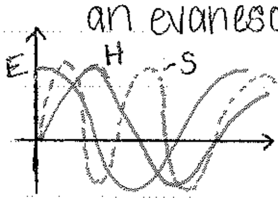
For $k = \pm\sqrt{\omega^2 \varepsilon \mu} = \pm i\alpha$

$E(z,t) = \mathrm{Re}\{\hat{E} e^{(\pm i\alpha)z - i\omega t}\}$

$\Rightarrow e^{-\alpha z}$, the evanescent solution for $E(z,t)$

$\Rightarrow$ this solution decays in space but not in time. The fields penetrate the medium, but power does not

- no average power into the medium is the definition of an evanescent solution



the Poynting Flux, $E \times H$, oscillates at 2x the frequency in evanescent conditions (E & H 90° out of phase)

So for the Poynting Flux in this case

$S = (\vec{E} \times \vec{H})$ where we know there is a magnetic field

$= \underbrace{\tfrac{1}{2}\mathrm{Re}\{\eta\}}_{\text{this can be zero here!}} |\vec{H}|^2$

$\frac{|\hat{E}|}{|\hat{H}|} = \eta = \sqrt{\frac{\mu}{\varepsilon}}$

if this is wholly non-real, $\vec{E} \perp \vec{H}$

$\rightarrow \mathrm{Re}\{\eta\} = 0$ for $\hat{E} \perp \hat{H}$

$\Rightarrow S = 0$

3. Polarization

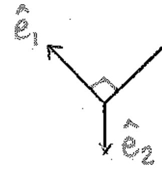
unit vector in direction of propagation

$\hat{n} \parallel \vec{k}$

$\vec{k} = k\hat{n}$

$\rightarrow$

→ Define two more unit vectors, $\hat{e}_1$ & $\hat{e}_2$



$\hat{e}_1 \perp \hat{n}$
$\hat{e}_2 = \hat{n} \times \hat{e}_1$

The electric field is perpendicular to $\hat{n}$
→ $\vec{E}$ has components only in $\hat{e}_1$ & $\hat{e}_2$

$\hat{\vec{E}} = \hat{E}_1 \hat{e}_1 + \hat{E}_2 \hat{e}_2$

$\left\{ \begin{array}{l} \hat{E}_1 = |\hat{E}_1| e^{i\varphi_1} \\ \hat{E}_2 = |\hat{E}_2| e^{i\varphi_2} \end{array} \right.$ > the most general solutions to Maxwell's equations (where φ_1 & φ_2 are the phases of the complex amplitudes)

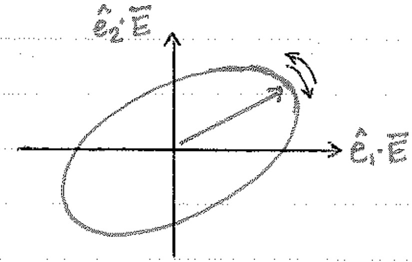
Full field equation:

$$\vec{E}(\vec{x},t) = \frac{1}{2}\left\{(\hat{E}_1\hat{e}_1 + \hat{E}_2\hat{e}_2)\exp[i(\vec{k}\cdot\vec{x} - \omega t)] + \text{complex conjugate}\right\}$$

└ E sweeps out an ellipse

$\hat{e}_1 \cdot \vec{E}_1 = Re\{\hat{E}_1 e^{i\vec{k}\cdot\vec{x} - i\omega t}\} \overset{=}{x=0} |E_1| \cos(\varphi_1 - \omega t)$

$\hat{e}_2 \cdot \vec{E}_2 = Re\{\hat{E}_2 e^{i\vec{k}\cdot\vec{x} - i\omega t}\} = |E_2| \cos(\varphi_2 - \omega t)$



↑ sinusoidal in both directions

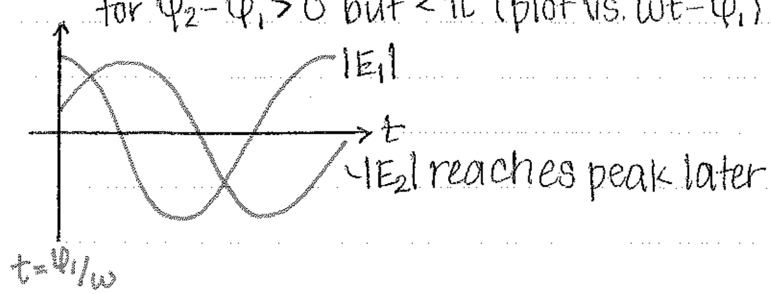
Yields an ellipse for $\varphi_1 \neq \varphi_2$

⇅ direction:

$|E_2| \cos(\omega t - \varphi_1 - (\varphi_2 - \varphi_1))$

$|E_1| \cos(\omega t - \varphi_1)$

for $\varphi_2 - \varphi_1 > 0$ but $< \pi$ (plot vs. $\omega t - \varphi_1$)



• for $\varphi_2 - \varphi_1 > 0$
$|E_1|$ @ peak, $|E_2|$ still rising
⇒ Left-Handed Polarization (CCW rotation)

• for $\varphi_2 - \varphi_1 < 0$
$|E_2|$ @ peak, $|E_1|$ still rising
⇒ Right-Handed Polarization (CW rotation)

* This is known as Linear Decomposition

- for $\varphi_1 = \varphi_2$

$\Rightarrow$ This is linear polarization; forms a degenerate ellipse

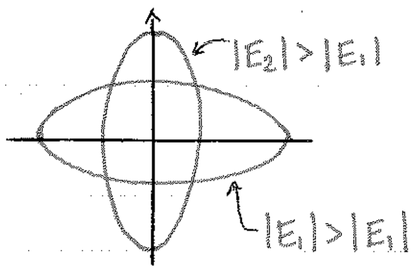
What about a specific relationship between φ_1 & φ_2?

$\varphi_2 = \varphi_1 \pm \pi/2$ – 90° out of phase

→ ellipse orientation extremum

$\hat{e}_1 \cdot \bar{E}(x=0,t) = E_1 = |\hat{E}_1| \cos(\omega t - \varphi_1)$

$\hat{e}_2 \cdot \bar{E}(x=0,t) = E_2 = |\hat{E}_2| \cos(\omega t - \varphi_1 - \underbrace{(\varphi_2 - \varphi_1)}_{\mp \pi/2})$



$\cos(\omega t - \varphi_1 \mp \pi/2) = \pm \sin(\omega t - \varphi_1)$

Polarization direction:

$\varphi_2 = \varphi_1 + \pi/2 \rightarrow$ LHP

$\varphi_2 = \varphi_1 - \pi/2 \rightarrow$ RHP

The field may also be decomposed into two circularly polarized waves.

circular polarization

$\hat{\hat{E}} = \hat{C}_1(\hat{e}_1 + i\hat{e}_2) + \hat{C}_2(\hat{e}_1 - i\hat{e}_2)$

where these C's specify the relations between circularly polarized waves.

$\hat{E}_1 = (\hat{C}_1 + \hat{C}_2) \Longrightarrow \hat{C}_1 = \frac{1}{2}(\hat{\hat{E}}_1 - i\hat{\hat{E}}_2)$

$\hat{E}_2 = i(\hat{C}_1 - \hat{C}_2) \quad\quad \hat{C}_2 = \frac{1}{2}(\hat{\hat{E}}_1 + i\hat{\hat{E}}_2)$

i adds $e^{i\pi/2}$ to the real part

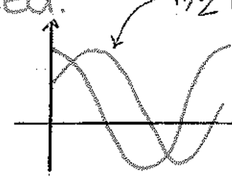
Which one is Right-Handed Polarized?

→ if we only had $\hat{C}_1$

$\hat{e}_1 \cdot \bar{E} = |\hat{C}_1| \cos(\omega t - \varphi_{C_1})$

$\hat{e}_2 \cdot \bar{E} = |\hat{C}_1| \cos(\pi/2 + \varphi_{C_1} - \omega t)$

$= |\hat{C}_1| \cos(\omega t - \varphi_{C_1} - \pi/2)$



$\Longrightarrow \hat{E}_2$ is RHP

4. Dissipation

Adding friction: $d\bar{v}/dt = -2\nu\bar{v}$ coefficient of friction

$\varepsilon(\omega) = \varepsilon_0\left(1 - \frac{\omega_p^2}{(\omega^2 - \omega_0^2)}\right)$ want to replace the denominator

$$\omega^2 \to \omega(\omega + 2i\nu), \quad \omega = -2i\nu$$

$$d\vec{v}/dt = -2\nu\vec{v}$$

$$= -i\omega v + 2\nu\vec{v}$$

$$d\vec{v}/dt = -i(\omega + 2i\nu)\vec{v}$$

We can see from this that our ε will now be complex

$$\varepsilon(\omega) = \varepsilon_r + i\varepsilon_i$$

then recall

$$k = \omega\sqrt{\mu_0 \varepsilon} = \omega\sqrt{\mu_0 \varepsilon_0}\sqrt{\varepsilon/\varepsilon_0} \qquad \frac{\varepsilon}{\varepsilon_0} = \frac{\varepsilon_r}{\varepsilon_0} + i\frac{\varepsilon_i}{\varepsilon_0}$$

α measures decay length

$\to k = \beta + i\alpha$, to match forms with $\varepsilon(\omega)$

β measures the wavelength

*note: both α & β can be functions of ω themselves

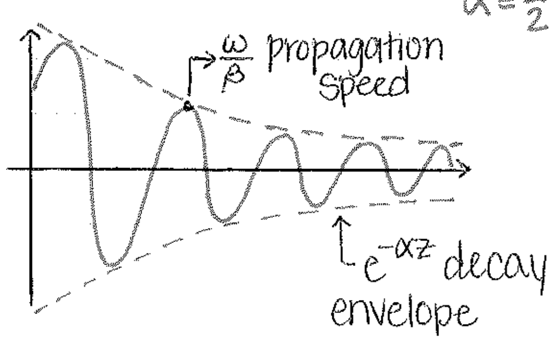
$$e^{ikz} \to e^{i\beta z - \alpha z}$$

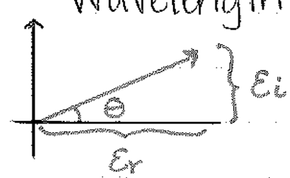
$$E(z,t) = \mathrm{Re}\{\hat{E}\, e^{i\beta z - i\omega t} e^{-\alpha z}\}$$

$$k = \omega\sqrt{\mu_0 \varepsilon_0}\sqrt{\frac{\varepsilon_r}{\varepsilon_0}}\left(1 + \underbrace{\frac{i}{2}\frac{\varepsilon_i}{\varepsilon_r}}\right)$$

$$\alpha = \frac{1}{2}\left(\frac{\varepsilon_i}{\varepsilon_r}\right)\beta$$

Loss Tangent $\to$ how much the wave will decay in one wavelength





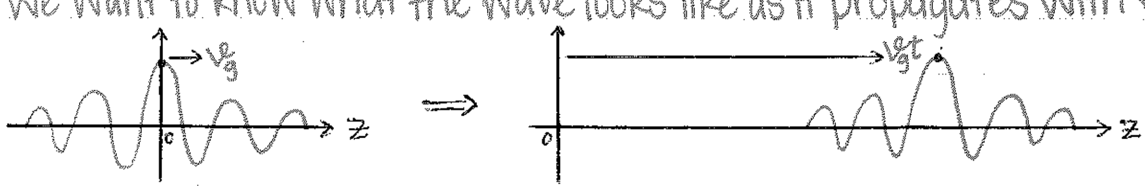
$$\tan(\theta) = \frac{\varepsilon_i}{\varepsilon_r}$$

Often, $\varepsilon_i < \varepsilon_r$

Pulse Propagation

(a somewhat initial value problem)

We want to know what the wave looks like as it propagates with v_g



Now suppose you know the temporal form of the wave @ $z=0$. How does it compare to the temporal form some time later? (prediction @ z)

→ Use the dispersion relation!

the rate of change of each spectral component's phase in space

$\omega(k) \rightarrow k(\omega)$

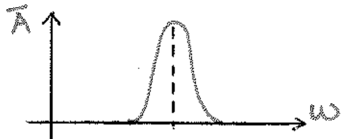
generic form: Wave Amplitude, $\alpha << \beta$

$$A(z,t) = \mathrm{Re}\{\overbrace{\hat{A}(z,t)\exp[ik_0 z - i\omega t]}^{\text{envelope of travel}}\}$$

$$\hat{A}(0,t) = \int \frac{d\hat{\omega}}{2\pi}\bar{A}(\hat{\omega})e^{-i\hat{\omega}t}$$

the time-dependence @ $z=0$

$\omega = \omega_0 + \hat{\omega}$



ω_0 - the central frequency

*What happens to the pulse depends on its frequency content!

Then, with respect to $\hat{\omega}$,

$$A(z,t) = \mathrm{Re}\left\{\int \frac{d\hat{\omega}}{2\pi} e^{-i(\hat{\omega}+\omega_0)t} \underbrace{e^{ik(\hat{\omega}+\omega_0)z}}_{\text{Taylor expand here so we can do this integral}} \bar{A}(\hat{\omega})\right\}$$

Fourier Transform w.r.t. $\hat{\omega}$ gives us a spectrum of components of wave propagation at different v-values

Phase Lag: $\phi(\omega_0+\hat{\omega}) = k(\omega+\omega_0)z$

$$\phi[\omega_0+\hat{\omega}] \simeq \phi(\omega_0) + \frac{\partial\phi}{\partial\omega_0}\hat{\omega} + \frac{1}{2}\frac{\partial^2\phi}{\partial\omega_0^2}\hat{\omega}^2 + \ldots$$

$$\rightarrow A(z,t) = \mathrm{Re}\left\{\underbrace{e^{i[k(\omega_0)z-\omega_0 t]}}_{\text{the carrier}} \underbrace{\int \frac{d\hat{\omega}}{2\pi}\bar{A}(\hat{\omega})e^{-i\hat{\omega}(\underbrace{t-\phi'}_{\tau})}}_{\text{equivalent to } A(0,\underbrace{t-\phi'}_{=\tau\text{, the time delay}})} e^{i\hat{\omega}^2\phi''/2}\right\}$$

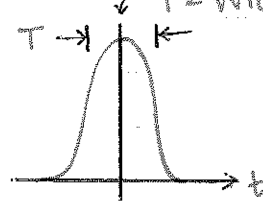
$\Rightarrow \phi'(\omega_0)$ is known as the phase delay

$$\frac{\partial\phi}{\partial\omega_0} = \left.\frac{\partial k}{\partial\omega_0}\right|_z = \frac{z}{v_g}$$

What is the effect of the third term?

$$\text{Suppose } \hat{A}(z=0,t) = A_0 \overbrace{\exp[-t^2/T^2]}^{\text{a Gaussian}}$$

T = width of the pulse in time



then in time space...

$$\bar{A}(\hat{\omega}) = \underbrace{\int_{-\infty}^{\infty} dt\, e^{i\hat{\omega}t} A_0 e^{-t^2/T^2}}_{\text{a Gaussian integral}}$$

$-t^2/T^2 + i\hat{\omega}t$ ↓ complete the square

$-\frac{1}{T^2}(t^2 - i\hat{\omega}tT^2)$

$-\frac{1}{T}\left[\left(t - \frac{i}{2}\hat{\omega}T^2\right)^2 - \frac{1}{4}\hat{\omega}^2T^2\right]$

↳ plugging in ↓

$$\bar{A}(\hat{\omega}) = A_0 e^{-\frac{1}{4}\hat{\omega}^2T^2} \underbrace{\int_{-\infty}^{\infty} dt \exp\left[-\frac{1}{T^2}\left(t - i\frac{\hat{\omega}T^2}{2}\right)^2\right]}_{= T\sqrt{2\pi}}$$

$$= A_0 T\sqrt{2\pi}\, e^{-\frac{1}{4}\hat{\omega}^2T^2}$$

Then, combining this all to solve for $\hat{A}$ in terms of z & t:

$$\hat{A}(z,t) = \frac{A_0 T}{\sqrt{\pi}} \int d\hat{\omega} \exp\left[-i\hat{\omega}\tau + \frac{i\phi''}{2}\hat{\omega}^2 - \frac{1}{4}\hat{\omega}^2T^2\right]$$

let $T_s^2 \equiv (T^2 - 2i\phi'')$ the spreading time

→ completing the square and integrating as before ↓

$$\hat{A}(z,\tau) = A_0 \frac{T}{T_s} \exp\left[-\frac{\tau^2}{T_s^2}\right]$$

Reality check: If you go to $z=0$, then $\phi \to 0$ and $T_s^2 \to T^2$

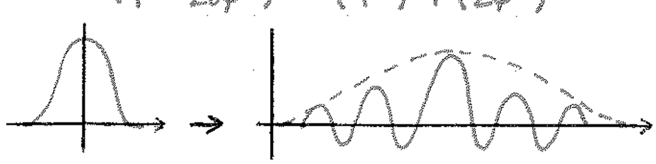
So what's happening to this pulse in time?

break T_s into constituent parts

$$\frac{1}{T_s^2} = \mathrm{Re}\left\{\frac{1}{T_s^2}\right\} + i\,\mathrm{Im}\left\{\frac{1}{T_s^2}\right\}$$

$$= \frac{1}{(T^2 - 2i\phi'')^2} = \frac{(T^2 + 2i\phi'')}{(T^2)^2 + (2\phi'')^2}$$

→ The instantaneous frequency, Ω, is changing in time!



$$\Omega = \omega_0 + \frac{\partial}{\partial\tau}\mathrm{Im}\left\{\frac{1}{T_s^2}\right\}$$

$$\Omega(\tau) = -\frac{\partial}{\partial\tau}\underbrace{\angle A}_{\text{phase of } A}$$

The arriving pulse is wider and contains "chirps," as the group velocity is a function of frequency.