Lecture 15 - Retarded Potentials & Conservation

03/25/16

1. Retarded Potentials
   - Green's Function for the 3D Wave Equation
2. Conservation of Energy & Momentum

1. Retarded Potentials

Recall: There is a disparity in the number of independent functions between the field and the potential.

The disparity in independent functions means we can make up a scalar rule about the potential

$\Rightarrow$ this is the origin of the Gauge condition

for $(\vec{A}, \varphi)$, four variables

$$\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0$$, the Lorentz Gauge

$$\hookrightarrow \begin{Bmatrix} \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} \\ \nabla^2 \varphi - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = -\frac{\rho}{\varepsilon_0} \end{Bmatrix}$$

*We have previously encountered Coulomb's Gauge

$$\nabla \cdot \vec{A} = 0$$

We can check these for accuracy in the static limit:

for the static condition $\frac{\partial}{\partial t} = 0$

$$\vec{A} = \frac{\mu_0}{4\pi}\int d^3x' \frac{J(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

$$\varphi = \frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

both equations as previously encountered in electro- and magnetostatics!

Green's Function for the 3D Wave Equation

$$\underbrace{\nabla^2 \psi(\vec{x},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(\vec{x},t)}_{\text{3D wave equation}} = -4\pi \overbrace{f(\vec{x},t)}^{\text{arbitrary source function localized in space}}$$

$\rightarrow$ We want a Green's function solution such that we can determine φ, or any component of $\vec{A}$, for any arbitrary source function.

$\longrightarrow$

$$\psi(\vec{x},t) = \int d^3x' \int dt' \, f(\vec{x}',t')\, G(\vec{x},t;\vec{x}',t')$$

$$\hookrightarrow \nabla^2 G - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} G = -4\pi\,\delta(\vec{x}-\vec{x}')\,\delta(t-t')$$

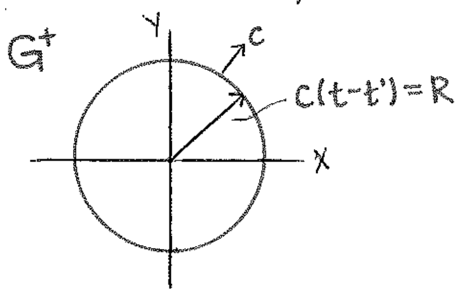
Yields two solutions → G^+ & G^-

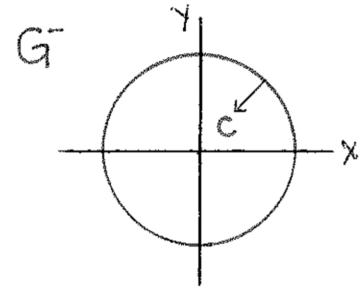
$$G^+ = \begin{cases} \dfrac{\delta(t-t'-R/c)}{R} & , t>t' \\ 0 & , \text{otherwise} \end{cases}$$

Where $R = |\vec{x}-\vec{x}'|$
↳ the distance between the observation & source points

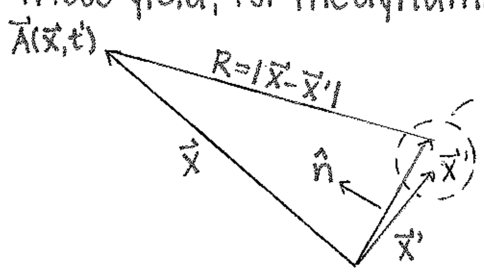
$$G^- = \begin{cases} \dfrac{\delta(t-t'+\overbrace{R/c}^{\text{travel time}})}{R} & , t<t' \\ 0 & , \text{otherwise} \end{cases}$$

Satisfies time-reversal (which we will later show does not hold)





These yield, for the dynamic condition ($\frac{\partial}{\partial t} \neq 0$)



where $\hat{n}$ points in the direction of the observation point

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\vec{J}(\vec{x}', t'=t-R/c)}{|\vec{x}-\vec{x}'|}$$

$$\varphi(\vec{x},t) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{\rho(\vec{x}', t'=t-R/c)}{|\vec{x}-\vec{x}'|}$$

Potentials are "retarded" in time by a factor of R/c, the travel time between source and observation points.

Alternative derivation of this Green's function:

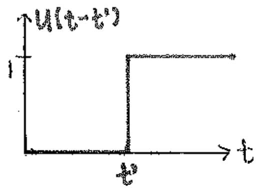
Want to solve

$$\nabla^2 G - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} G = -4\pi\,\delta(\vec{x}-\vec{x}')\,\overbrace{\delta(t-t')}^{\text{delta function in time is very problematic to work with}}$$

→ Rewrite in terms of a unit step function ↓

$$\nabla^2 G_u - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} G_u = -4\pi\,\delta(\vec{x}-\vec{x}')\,U(t-t')$$

Where



Assume $G_u = 0$ for $t < t'$

and

$$\nabla^2 \frac{\partial}{\partial t} G_u - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{\partial G_u}{\partial t}\right) = -4\pi\,\delta(\vec{x}-\vec{x}')\,\underbrace{\frac{\partial U(t-t')}{\partial t}}_{\delta(t-t')}$$

to regain the delta function

then shift the step function to zero

let $\tau = t - t'$, $\underbrace{R = |\vec{x}-\vec{x}'|}$

solution is spherically symmetric here where the source has been shifted to the origin

$\delta(R) = 4\pi\,\delta(\vec{x}-\vec{x}')$

$$\frac{1}{R^2}\frac{\partial}{\partial R} R^2 \frac{\partial G_u}{\partial R} - \frac{\partial^2}{\partial t^2} G_u = -\frac{\delta(R)}{R^2} U(\tau)$$

- as $\tau \to \infty$, only terms in R remain

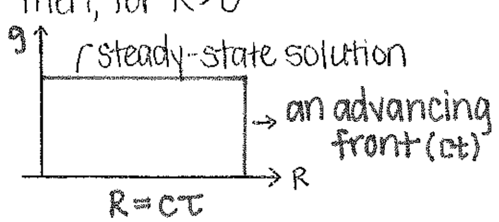
$\Rightarrow G_u = \frac{1}{R}$

$= \frac{g(R,\tau)}{R}$

must solve here now

$$\frac{\partial^2}{\partial R^2} g(R,\tau) - \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2} g(R,\tau) = \frac{-\delta(R)}{R} U(\tau)$$

Then, for $R > 0$



two solutions: g^+ and g^-

$$G = \underbrace{g_+(R - c\tau)}_{\text{outgoing solution}} + \cancel{g_-(R + c\tau)} \to 0$$

, no incoming wave solution

Then, putting it all together yields

$$\frac{\partial^2}{\partial R^2} g_+ - \frac{1}{c^2}\frac{\partial^2 g_+}{\partial \tau^2} = \frac{-\delta(R)}{R}$$

→ the steady-state solution:

$$g_+ = g_{out} = \begin{cases} 1 , & R - c\tau < 0 \\ 0 , & R - c\tau > 0 \end{cases}$$

$$G_u = \frac{1}{R} U(c\tau - R) = \frac{1}{R} U(\tau - R/c)$$

$$\Rightarrow \frac{\partial}{\partial \tau} G_u = \frac{1}{R}\delta(\tau - \frac{R}{c}) = G$$

ex. Current density oscillating sinusoidally in time

$$\vec{J}(\vec{x},t) = \mathrm{Re}\{\hat{\vec{J}}(\vec{x})e^{-i\omega t}\} = \frac{1}{2}[\underbrace{\hat{\vec{J}} e^{-i\omega t}}_{} + \hat{\vec{J}}^* e^{i\omega t}]$$

↓ yields ↓ use this to find phaser scalar potential

$$\vec{A}(\vec{x},t) = \mathrm{Re}\{\hat{\vec{A}}(\vec{x})e^{-i\omega t}\} = \frac{1}{2}[\hat{\vec{A}} e^{-i\omega t} + \hat{\vec{A}}^* e^{i\omega t}]$$

– plugging in for Vector potential...

$$\hat{\vec{A}}(\vec{x})e^{-i\omega t} = \frac{\mu_0}{4\pi}\int d^3x' \frac{\hat{\vec{J}}(\vec{x}')e^{-i\omega t'}}{|\vec{x}-\vec{x}'|}$$

$$t' = t - R/c ; \quad e^{-i\omega t'} \rightarrow e^{-i\omega t} e^{i\omega R/c}$$

$$\hat{\vec{A}}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\hat{\vec{J}}(\vec{x}')\exp(i\frac{\omega}{c}|\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|} \quad \text{for } \vec{x} > \vec{x}'$$

What if we were very far away? ($\vec{x} \gg \vec{x}'$)

$$|\vec{x}-\vec{x}'| \simeq \underbrace{|\vec{x}|}_{\text{pulls out of integral}}$$

$$\hat{\vec{A}}(\vec{x}) = \frac{\mu_0}{4\pi|\vec{x}|}\int d^3x' \hat{\vec{J}}(\vec{x}')\exp(i\frac{\omega}{c}\underbrace{|\vec{x}-\vec{x}'|}_{})$$

because this is part of the exponential, you cannot make the same simplification here

– instead, for the exponent...

$$|\vec{x}-\vec{x}'| = \sqrt{\vec{x}^2 + \vec{x}'^2 - 2\vec{x}\cdot\vec{x}'}$$

$$= x\sqrt{1 - \underbrace{\frac{2\vec{x}\cdot\vec{x}'}{x^2}}_{\text{this term} \ll 1}}$$

→

$$\simeq x\left(1-\frac{\vec{x}\cdot\vec{x}'}{x^2}\right) \rightarrow \text{redistribute}$$

$$= x - \frac{\vec{x}\cdot\vec{x}'}{x} \Rightarrow \hat{n} \equiv \vec{x}/|\vec{x}|$$

$$\simeq |\vec{x}| - \hat{n}\cdot\vec{x}' + \ldots$$ higher-order terms from the square root appx.

Together, this yields (for $\vec{x} \gg \vec{x}'$)

$$\hat{\vec{A}}(\vec{x}) = \frac{\mu_0}{4\pi|\vec{x}|} e^{i\overset{\omega/c}{k}|\vec{x}|} \int d^3x' \, \hat{\vec{J}}(\vec{x}') \exp(-i\vec{k}\cdot\vec{x}')$$

where $\vec{k} = \hat{n}\frac{\omega}{c}$

2. <u>Conservation of Energy & Momentum</u>

→ Macroscopic media conservation laws ←

Vector Equations

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\cdot\vec{D} = \rho_{free}$$

$$\nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{H} = \vec{J}_{free} + \frac{\partial\vec{D}}{\partial t}$$

Linear Constituative Relationships

$$\left.\begin{aligned}\vec{D} &= \varepsilon\vec{E} \\ \vec{B} &= \mu\vec{H}\end{aligned}\right\} \varepsilon \,\&\, \mu = \text{constants}$$

where we have made the assumption that the medium is isotropic, linear, instantaneous, and local in space

We can combine these using

Poynting's Theorem:

$$\underbrace{\vec{H}\cdot\nabla\times\vec{E} - \vec{E}\cdot\nabla\times\vec{H}}_{\nabla\cdot(\vec{E}\times\vec{H})} = -\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{E}\cdot\left(\vec{J}_{free} + \frac{\partial\vec{D}}{\partial t}\right)$$

(BAC CAB inverse expansion)

$$\nabla\cdot(\vec{E}\times\vec{H}) = \underbrace{-\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}}_{\longleftarrow \text{ move to the other side}} - \vec{E}\cdot\vec{J}_{free}$$

– first, rewriting $\vec{B}$ w.r.t. $\vec{H}$ and $\vec{D}$ w.r.t. $\vec{E}$

$$\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} = \frac{\partial}{\partial t}\left(\tfrac{1}{2}\mu|\vec{H}|^2\right) \; ; \quad \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} = \frac{\partial}{\partial t}\left(\tfrac{1}{2}\varepsilon|\vec{E}|^2\right)$$

→

$$\rightarrow \underbrace{\int_S dA\, \hat{n}\cdot(\vec{E}\times\vec{H})}_{\text{rate at which energy leaves through the surface}} + \underbrace{\frac{\partial}{\partial t}\int d^3x\, \frac{\varepsilon|\vec{E}|^2+\mu|\vec{H}|^2}{2}}_{\text{rate of change of stored energy}} = \underbrace{-\int d^3x\, \vec{J}_{free}\cdot\vec{E}}_{\text{rate of work done by } \vec{E} \text{ on } \vec{J}_{free}}$$

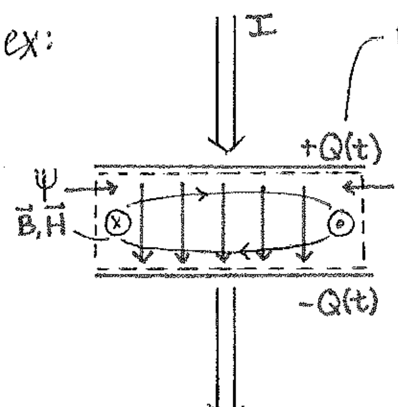
This is a statement on conservation of energy with the Poynting vector defining the surface over which the energy is leaving.

Let's focus on the surface integral:

Poynting flux $= \Psi = \int_S dA\, \hat{n}\cdot(\vec{E}\times\vec{H})$

└ only makes sense for a closed surface
$\vec{E}\times\vec{H}$ can only describe a flow of power for a closed surface.

ex:



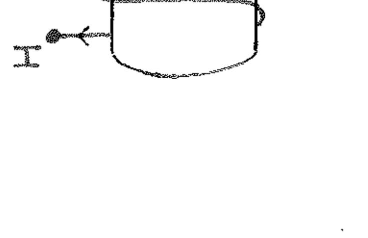
Parallel plates are charging → $\vec{E}$ field building up ↓

Poynting flux

No free charge → there must be a Poynting flux coming in from the sides (by the Right-Hand-Rule)

I -- same amount of current leaving

ex:



Energy is building up & being stored in the magnetic field

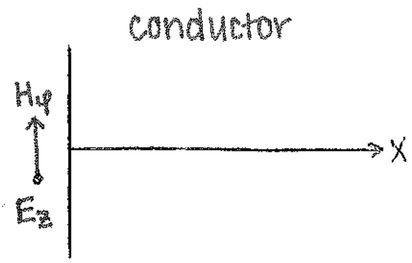
$\Psi = \int_S dA\, \hat{n}\cdot\vec{B}$

$\oint \vec{E}\cdot d\vec{l} = -\frac{\partial \Psi}{\partial t}$

→

Applying the Poynting Flux...

<u>The Skin Effect</u>:



$$H_y(0,t) = \tfrac{1}{2}\left(\hat{H}_y(0)e^{-i\omega t} + \hat{H}_y^*(0)e^{i\omega t}\right)$$

constant in time

doubly oscillatory

$$E_z = \tfrac{1}{2}\left(-Z_s\hat{H}_y(0)e^{-i\omega t} - Z_s^*\hat{H}_y^*(0)e^{i\omega t}\right)$$

└ surface impedance $= (1-i)\frac{1}{\sigma\delta}$

└ reactance originates from impedance (<0)

$$S_x = (\vec{E}\times\vec{H})_x = -E_z H_y$$
$$= -\tfrac{1}{4}\left(-Z_s\hat{H}_y e^{-i\omega t} - Z_s^*\hat{H}_y^* e^{i\omega t}\right)\left(\hat{H}_y e^{-i\omega t} + \hat{H}_y^* e^{i\omega t}\right)$$

$$\underbrace{\langle S_x\rangle_t}_{\text{average over time}} = \tfrac{1}{4}\left\{Z_s^*|\hat{H}_y(0)|^2 + Z_s|\hat{H}_y(0)|^2\right\}$$
$$= \tfrac{1}{2}\,\underbrace{\mathrm{Re}\{Z_s\}}\,|\hat{H}_y(0)|^2$$

$\langle S_x\rangle$ must be real → $\mathrm{Re}\{Z_s\}$ gives how much power flows into the conductor

What about momentum?

"Nobody can stop us!" – Adil Hassam

for a blob of charge density

– free & induced charge here subject to $\vec{E}, \vec{B}$

the rate of change of momentum may be expressed as

$$\frac{d\vec{P}_{mech}}{dt} = \vec{F} = \int d^3x\,[\underbrace{\rho\vec{E} + \vec{J}\times\vec{B}}_{\text{these are known as "volume forces"}}]$$

→ Want to use Maxwell's equations to get rid of $\rho, \vec{J}$

$\rho = \nabla\cdot\vec{D}$, $\vec{J} = (\nabla\times\vec{H} - \frac{\partial}{\partial t}\vec{H})$

$$\frac{d}{dt}\vec{P}_{mech} = \int d^3x\left\{\vec{E}(\nabla\cdot\vec{D}) + \left(\nabla\times\vec{H} - \frac{\partial\vec{D}}{\partial t}\right)\times\vec{B}\right\}$$

↳ add $\frac{\partial B}{\partial t}$ to both sides to make a perfect time derivative

$$\frac{d}{dt}\left[\vec{P}_{mech} + \underbrace{\int d^3x\,\vec{D}\times\vec{B}}\right] = \int d^3x\left[\vec{E}\,\nabla\cdot\vec{D} + (\nabla\times\vec{H})\times\vec{B} + \underbrace{\vec{D}\times\frac{\partial\vec{B}}{\partial t}}\right]$$

$$\overbrace{\int d^3x\, \vec{D}\times\vec{B}} = \vec{P}_{fields} \qquad \overbrace{\vec{D}\times\frac{\partial \vec{B}}{\partial t}} = -\vec{D}\times(\nabla\times\vec{E})$$

If we assume vacuum relations...

$$\left.\begin{array}{l}\vec{D} = \varepsilon_0\vec{E} \\ \vec{B} = \mu_0\vec{H}\end{array}\right\} \begin{array}{l}\vec{E}\,\nabla\cdot\vec{E} - \vec{E}\times\vec{E}\times\vec{E} = \vec{E}\,\nabla\cdot\vec{E} + \vec{E}\cdot\nabla\vec{E} - \nabla\frac{1}{2}|\vec{E}|^2 \\ -\vec{H}\times(\nabla\times\vec{H}) = \vec{H}\,\nabla\cdot\vec{H} - \nabla\frac{1}{2}|\vec{H}|^2\end{array}$$

→ ρ and $\vec{J}$ are just free charges/currents in this case

DEFINE: $\overset{\equiv}{T} \equiv$ the electromagnetic stress tensor

→ the electromagnetic contribution to the stress-energy tensor that describes the flow of energy and momentum in spacetime.

unit tensor — like a pressure, but not guaranteed to be in the surface direction

$$\overset{\equiv}{T} = \underbrace{\varepsilon_0\vec{E}\vec{E}}_{\text{origin}\to\,\varepsilon_0[(\nabla\cdot\vec{E})\vec{E} + \vec{E}\cdot\nabla\vec{E}]} + \mu_0\vec{H}\vec{H} - \overset{\equiv}{1}\,\overbrace{\tfrac{1}{2}(\varepsilon_0|\vec{E}|^2 + \mu_0|\vec{H}|^2)}$$

$$\frac{d}{dt}\Big(\vec{P}_{mech} + \underbrace{\vec{P}_{field}}_{\substack{\int d^3x\,\vec{D}\times\vec{B} \\ \varepsilon_0\vec{E}\times\mu_0\vec{H}}}\Big) = \int d^3x\,\nabla\cdot\overset{\equiv}{T} = \underbrace{\int_S dA\,\hat{n}\cdot\overset{\equiv}{T}}_{\substack{\text{oriented such that} \\ \text{normal surface force} = \hat{n}\cdot\overset{\equiv}{T}\cdot\hat{n}}}$$

Normal Force Density:

At the surface of a conductor

$$\hat{n}\cdot\overset{\equiv}{T}\cdot\hat{n} = \varepsilon_0\underbrace{|\hat{n}\cdot\vec{E}|^2}_{=|\vec{E}|^2} + \mu_0\underbrace{|\hat{n}\cdot\vec{H}|^2}_{=0} - \tfrac{1}{2}(\varepsilon_0|\vec{E}|^2 + \mu_0|\vec{H}|^2) \quad \text{for a good conductor}$$

$$= -\tfrac{1}{2}\mu_0|\vec{H}|^2 + \tfrac{1}{2}\varepsilon_0|\vec{E}|^2$$

⇒ The force from $\vec{E}$ pulls on the surface while $\vec{B}$ pushes back on the conductor