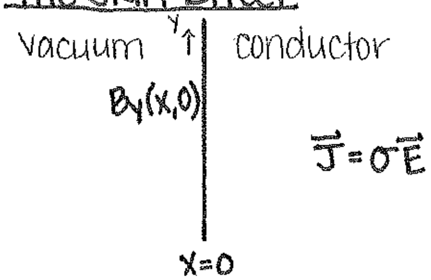
Lecture 14 - Applications of Changing Magnetic Fields 03/22/16

1. The Skin Effect
2. Magnetic Dynamo (Faraday's Law)
3. Displacement Current
4. Scalar and Vector Potentials

1. The Skin Effect



- combine Faraday's, Ampere's, and Ohm's Laws

$$\frac{\partial\vec{B}}{\partial t} = -\nabla\times\vec{E} \quad , \quad \vec{J} = \frac{1}{\mu_0}\nabla\times\vec{B} \quad , \quad \vec{J}=\sigma\vec{E}$$

Together, these yield the diffusion equation

$$\frac{\partial\vec{B}}{\partial t} = \frac{1}{\mu_0\sigma}\nabla^2\vec{B}$$

the effective diffusion coefficient

$$d^2 \propto \frac{1}{\mu_0\sigma}t$$

d = depth of penetration (in skin)

for the above geometry:

$$\vec{B} = \hat{y}\,B_y(x,t)$$

$$B_y(x,t) = \mathrm{Re}\{\hat{B}_y(x)\,e^{-i\omega t}\}$$

↓ apply the diffusion equation ↓

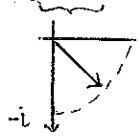
$$\frac{d^2}{dx^2}\hat{B}_y(x) = -i\omega\mu_0\sigma\,\hat{B}_y(t)$$

$$\hat{B}_y(x) = \exp(-\mathbb{K}x)$$

choose the solution such that $\mathrm{Re}\{\mathbb{K}\}>0 \rightarrow$ prevents blow-up at origin

$$\mathbb{K}^2 = -i\omega\mu_0\sigma$$

$$\mathbb{K} = \pm\{\sqrt{-i}\sqrt{\omega\mu_0\sigma}\}$$



$$\sqrt{-i} = \frac{1-i}{\sqrt{2}}$$

Apply constraints on $\mathbb{K}_2$

$$\mathbb{K} = (1-i)\underbrace{\sqrt{\frac{\omega\mu_0\sigma}{2}}}$$

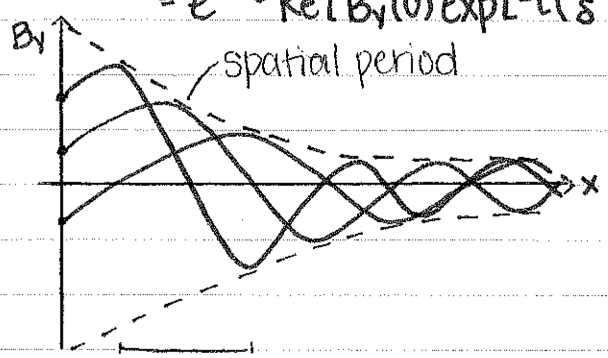
$$\delta = \text{skin depth} = \sqrt{\frac{2}{\omega\mu_0\sigma}}$$

We can use this to now write:

$$B_y(x,t) = \text{Re}\{\hat{B}_y(0)\exp[-i\omega t - \underbrace{(1-i)x/\delta}]\}$$

creates an envelope of $e^{-x/\delta}$

$$= e^{-x/\delta}\,\text{Re}\{\hat{B}_y(0)\exp[-i(\tfrac{x}{\delta} - \omega t)]\}$$



δ determines the attenuation rate.

$2\pi\delta$, the approximate distance between peaks
(changes slightly due to attenuation)

What about the other fields?

$B_y = \mu H_y$, $J_z = \frac{\partial}{\partial x} H_y$, $E_z = \frac{1}{\sigma} J_z$

$$\vec{E}_z(x,t) = \text{Re}\{\hat{E}_z(0)\exp[-i\omega t - (1-i)x/\delta]\}$$

negative sign guarantees power flow into the conductor

$$\hat{E}_z(0)\left[\frac{V}{m}\right] = -\frac{(1-i)}{\sigma\delta}$$

$\equiv$ surface impedance

The impedance of a layer of thickness δ

$$Z_s = \frac{(1-i)}{\sigma\delta}$$

We can now completely characterize conductors using surface impedance!

Some numbers:

Material	Conductivity	Skin Depth at... 60 Hz	1 MHz	1 GHz
Copper	5.8×10^{7} S/m	8.53 mm	0.066 mm	0.0021 mm
Sea Water	4 S/m	32 m	0.25 m	—

polarization & displacement current larger than conduction current here

Power flow is given by the Poynting Flux

$\vec{S}=\vec{E}\times\vec{H}$ — we will return to this

Surface impedance (Alternate form)

$$Z_S=\frac{1-i}{2}\overbrace{\sqrt{\frac{\mu_0}{\varepsilon_0}}}^{\text{impedance of free space} \sim 377\,\Omega}\underbrace{\frac{\omega\delta}{c}}_{=2\pi\delta/\lambda \ll 1}$$

λ: the vacuum wavelength

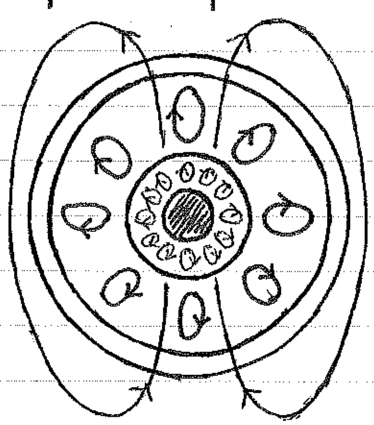
from this we can see that surface impedance ≪ free space impedance for a conductor.

2. Magnetic Dynamo

The conductor is now moving! — we now consider velocity effects

$$\frac{\partial\vec{B}}{\partial t}=-\nabla\times\vec{E}\ ,\quad \vec{J}=\frac{1}{\mu_0}\nabla\times\vec{B}\ ,\quad \vec{J}=\sigma(\vec{E}+\underline{\vec{v}}\times\vec{B})$$

The Dynamo explains why Earth$_\oplus$ has a magnetic field.



If you start with a magnetic field already present:

$d^2=\frac{1}{\mu\sigma}t\ ,\quad d=R_\oplus$

$t=$ decay time $\simeq$ 15 kyr — less than the age of the Earth!

→ this must not be the whole story...

A dynamo allows us to show that even if we start with no $\vec{B}$-field, one will build up & sustain due to convection.

With a weak starting $\vec{B}$-field (weak as not to influence flow)...

The kinematic dynamo → Does the magnetic field grow or decay in time? (i.e., is no $\vec{B}$-field stable?)

→ combining our equations ↴

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \frac{1}{\sigma \mu_0}(\nabla \times \vec{B}) + \nabla \times (\vec{v} \times \vec{B})$$

$$\nabla \times (\vec{v} \times \vec{B}) = \vec{B}\cdot\nabla\vec{v} + \underbrace{(\nabla\cdot\vec{B})}_{=0 \text{ by definition}}\vec{v} - \vec{v}\cdot\nabla\vec{B} - \vec{B}\underbrace{\nabla\cdot\vec{v}}_{=0}$$

→ the flow is INCOMPRESSIBLE

Aside: **Continuity of Materials**

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$$

$$\underbrace{\frac{\partial \rho}{\partial t} + \vec{v}\cdot\nabla\rho}_{} + \rho\nabla\cdot\vec{v} = 0$$

this is a convective derivative → the rate of change of ρ if you move along the medium (move with a fluid element)

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \vec{v}\cdot\nabla\rho \text{ must} = 0$$

($\vec{v}$: $d\vec{x}/dt$)

If you're following a fluid element, this states that you cannot compress the flow.

$$\frac{\partial \vec{B}}{\partial t} = \frac{1}{\sigma\mu_0}\nabla^2\vec{B}$$

→ plugging in ↴

$$\underbrace{\frac{\partial}{\partial t}\vec{B} + \vec{v}\cdot\nabla\vec{B}}_{\text{the change in } \vec{B} \text{ while following the rate of flow}} = \vec{B}\cdot\nabla\vec{v} + \underbrace{\frac{1}{\mu\sigma}\nabla^2\vec{B}}_{\text{typically a damping term}}$$

diffusion of $\vec{B}$ due to resistivity

→ **The solution must be dependent on this term concerning the non-uniformity of $\vec{v}$**

Suppose $\vec{v}$ = constant

$\vec{B}\cdot\nabla\vec{v}=0$

$$\frac{\partial}{\partial t}\vec{B}+\vec{v}\cdot\nabla\vec{B}=\frac{1}{\mu\sigma}\nabla^2\vec{B}$$

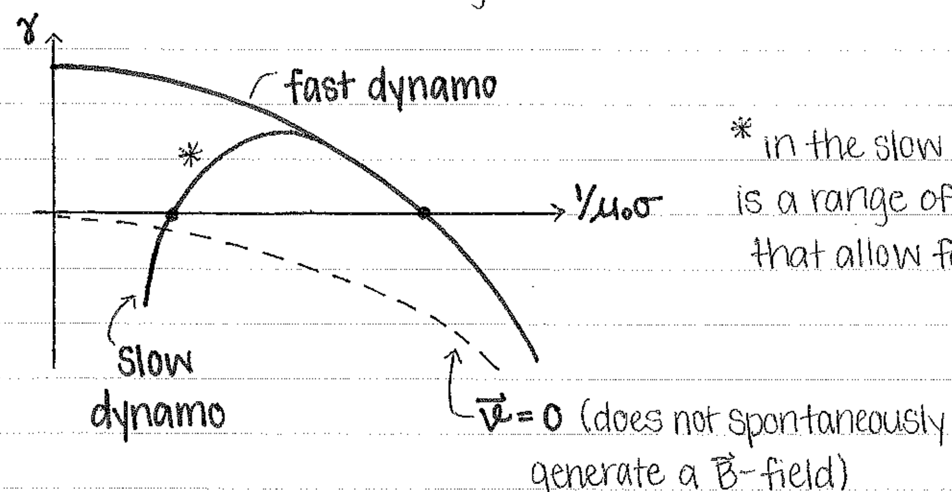
solution of the form:

$$(\gamma+i\vec{k}\cdot\vec{v})=\frac{-k^2}{\mu\sigma}$$

the Doppler shift! This gives the frequency of decay (damping rate)

$$\gamma=\underbrace{-i\vec{k}\cdot\vec{v}}_{}-\frac{k^2}{\mu\sigma}$$, a non-additive term!

γ must be >0 for field growth



* in the slow dynamo, there is a range of conductivities that allow for growth

Effects of $\vec{v}$: What kind of $\vec{v}(\vec{x},t)$ is needed?

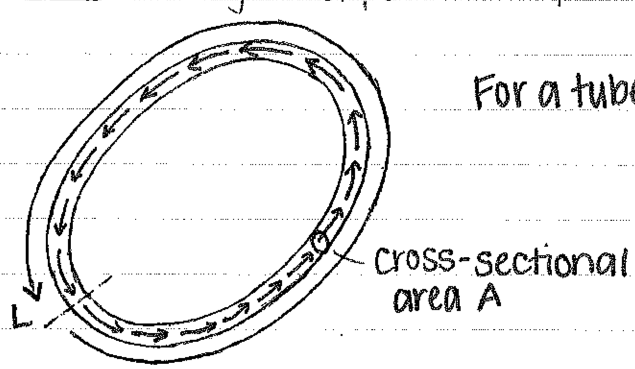
$\vec{v}(\vec{x},t)$ should cause

→ stretching

→ twisting

→ folding

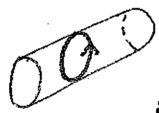
<u>then</u> $\vec{B}$ can grow exponentially



For a tube of fluid, length L

→ Amount of flux passing through A:

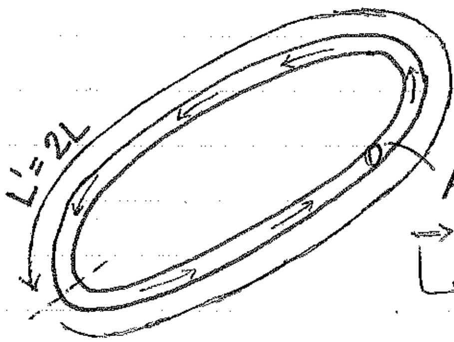
$$\psi=\int dA\,\hat{n}\cdot\vec{B}$$



$\oint(\vec{E}+\vec{v}\times\vec{B})\cdot d\vec{l}=0$

i.e., the conductivity is large enough that the loop of integration is moving with $\vec{B}$, so this never changes.

① $\vec{v}$ stretches the loop



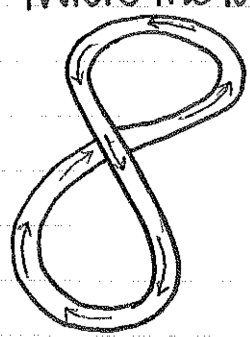
due to incompressibility

$A'=A/2$

$\rightarrow \psi'=\psi$, required ($V=A'L'=AL=\text{constant}$)

$\hookrightarrow$ the $\vec{B}$-field must have doubled when the loop was stretched! $B'=2B$

② $\vec{v}$ twists the loop

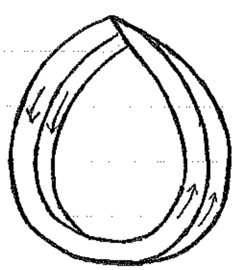


L' and A' unchanged

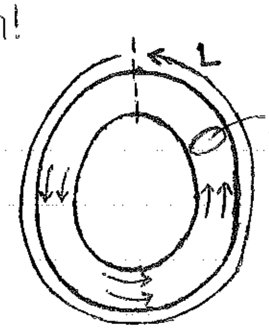
$\hookrightarrow \psi'=\psi$, $B'=2B$ (still)

③ $\vec{v}$ folds the loop

Arrows of fluid flow in the twist will align!



→ apply diffusion →



$\psi''=2\psi$

$B'=2B$ (!)

We have returned to our original geometry, but our magnetic field has doubled!

→ this is how stretching, twisting, and folding allow $\vec{B}$ to grow exponentially in time.

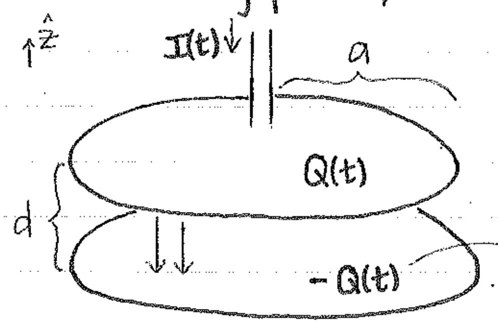
⟶

3. Displacement Current

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \; ; \; \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \; ; \; \nabla \cdot \vec{B} = 0 \; ; \; \nabla \times \vec{B} = \mu_0 \left(\vec{J} + \underbrace{\frac{\partial}{\partial t} \varepsilon_0 \vec{E}}_{} \right)$$

displacement "current"
needed to satisfy
continuity of charge

ex. Conducting plates, fed by I:



*assume $a \gg d$ such that

building up charge due to $I(t)$

For the electrostatic condition, we have

$E_z(r,t)$, $B_\theta(r,t)$

- apply Ampere's Law with Displacement Current

$$(\nabla \times \vec{B})_z = \mu_0 \varepsilon_0 \frac{\partial E_z}{\partial t}$$

$$\frac{1}{r}\frac{\partial}{\partial r} r B_\theta = \underbrace{\mu_0 \varepsilon_0 \frac{\partial E_z}{\partial t}}_{= \text{a constant, } E_z(r,t) = E_z(0,t)}$$

$$B_\theta(r,t) = \frac{r}{2} \varepsilon_0 \mu_0 \frac{\partial E_z}{\partial t}$$

We can now apply Faraday's Law using the approximation of B_θ in the case that E_z = constant

$$-\frac{\partial B_\theta}{\partial t} = (\nabla \times \vec{E})$$

$$= -\frac{\partial}{\partial r} E_z \text{ (for statics)}$$

$$E_z(r) = E_z(0) + \int_0^r dr' \frac{\partial}{\partial t} B_\theta(r',t)$$

$$= E_z(0,t) + \int_0^r dr' \frac{r'}{2} \varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2} E_z(0,t)$$

$$= E_z(0,t) + \underbrace{\varepsilon_0 \mu_0 \frac{r^2}{4} \frac{\partial^2}{\partial t^2} E_z(0,t)}_{\text{correction term to the static condition}}$$

$$E_z(r) = E_0\left(1 - \frac{\omega^2 r^2}{4}\underbrace{\varepsilon_0\mu_0}_{\varepsilon_0\mu_0 = \frac{1}{c^2}}\right) \text{, minimum @ } r = a$$

$f = \omega/2\pi$

$\varepsilon_0\mu_0 = \frac{1}{c^2}$; $\lambda = c/f$

→ From this we know that statics applies for

$\frac{1}{4}\omega^2 a^2 \varepsilon_0 \mu_0 \ll 1$

$= \left(\frac{\pi a}{\lambda}\right)^2 \ll 1 \Rightarrow$ the static approximation is valid when your wavelength is large compared to the radius of your conductor

What if we did not start with the static condition?

$$\frac{1}{r}\frac{\partial}{\partial r} r B_\theta = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} E_z(r,t) \ ; \quad \frac{\partial B_\theta}{\partial r} = \frac{\partial}{\partial r} E_z(r,t)$$

↓ combining for an equation purely in E_z ↓

yields the wave equation in 2D cylindrical coordinates

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial E_z}{\partial r} = \varepsilon_0 \mu_0 \frac{\partial^2 E_z}{\partial t^2}$$

- assume sinusoidal oscillations

$$E_z(r,t) = \mathrm{Re}\{\hat{E}_z(r) e^{-i\omega t}\}$$

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} \hat{E}_z(r) + \frac{\omega^2}{c^2}\hat{E}_z(r) = 0 \rightarrow$$ the Bessel function (with dependence on ω)

Solutions of the form:

$$E_z(r) = E_0 \underbrace{J_0\left(\frac{\omega}{c} r\right)}_{(\text{bessel})_0 \text{ function}}$$

$$\frac{1}{x}\frac{\partial}{\partial x} + \frac{\partial}{\partial x} J_0(x) + J_0(x) = 0$$

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m}}{(m!)^2}$$

where for small x, we may expand ↴

$$\approx 1 - \underbrace{\tfrac{1}{4}x^2}_{} + \mathcal{O}(x^4)$$

the correction term we found!

4. Scalar & Vector Potentials

$\nabla \cdot \vec{B} = 0$, $\vec{B} = \nabla \times \vec{A}(\vec{x},t)$

↳ $\vec{B}$ must be the curl of $\vec{A}(\vec{x},t)$, where $\vec{A}$ is a 3-dimensional vector

$$\vec{A} = \begin{pmatrix} A_x(\vec{x},t) \\ A_y(\vec{x},t) \\ A_z(\vec{x},t) \end{pmatrix} \rightarrow \text{3 independent functions}$$

but since $\nabla \cdot \vec{B} = 0$, $\vec{B}$ can only have 2 independent functions

So if we have $B_x(\vec{x},t)$, $B_y(\vec{x},t)$

$$B_z = -\left[\frac{\partial}{\partial x} B_x + \frac{\partial}{\partial y} B_y\right]$$

This disparity in the number of independent functions means we can make up a scalar rule about $\vec{A}$

⟹ This is the origin of the Gauge condition

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t} \nabla \times \vec{A} \rightarrow \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

$$\therefore \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \phi \rightarrow \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \phi$$

We must still satisfy Poisson & Ampere's laws

P: $\nabla^2 \phi + \frac{\partial}{\partial t} \nabla \cdot \vec{A} = -\rho/\varepsilon_0$

A: $\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{A} - \nabla\left(\nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial}{\partial t} \frac{\partial \phi}{\partial t}\right) = -\mu_0 \vec{J}$

} coupled equations

Choices for our scalar rule on $\vec{A}$: *must be wary of causality here

① $\nabla \cdot \vec{A} = 0$, Coulomb's Gauge

where $$\phi(\vec{x},t) = \frac{1}{4\pi\varepsilon_0} \int \frac{d^3x' \rho(\vec{x}',t)}{|\vec{x} - \vec{x}'|}$$

② $\phi(x,t) = 0 \rightarrow \vec{E} = -\partial \vec{A}/\partial t$

③ $\nabla_\perp \cdot \vec{A} = 0$

④ $\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = \frac{-\rho}{\varepsilon_0}$, Lorentz Gauge

→ causes scalar & vector potentials to be independent