Lecture 13 - Time-Variable Magnetics

03/10/16

1. Magnetostatics Problems
2. Faraday's Law
3. Moving Loops
4. Self-Inductance
5. The Skin Effect

1. Magnetostatics Problems

Relevant magnetostatics equations:

$\nabla \cdot \vec{B} = 0$

$\nabla \times \vec{H} = \vec{J}_{free} = 0$ for our problems here

case #1   case #2   magnetization; given

$\vec{B} = \mu \vec{H}$ OR $\vec{B} = \mu_0(\vec{H} + \vec{M})$

$\vec{H} = -\nabla \phi_m$

for case 1: $\nabla \cdot (\mu \nabla \phi_m) = 0$

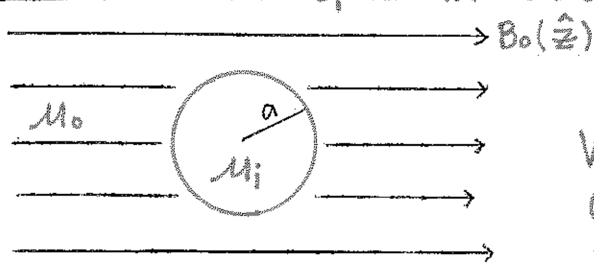
for case 2: $\nabla \mu_0 (-\nabla \phi_m + \vec{M}) = 0$

↓ can write as poisson equation ↓

$\nabla^2 \phi_m = \nabla \cdot \vec{M}$

*Warning! $\vec{M}$ can change frequently within materials so this equation may be hard to work with

Ex. 1: Permeable sphere in a constant $\vec{B}$-field (a case #1 problem)



We want to calculate the strength of the magnetic field induced inside the sphere.

Outside sphere: $\nabla^2 \phi_m = 0$

$$\phi_m = \cos(\theta)\left[\frac{-B_0}{\mu_0} r + \frac{m}{4\pi r^2}\right]$$

the magnetic moment

Inside sphere: $\nabla^2 \phi_m = 0$

$$\phi_m = \cos(\theta)[-rH_i]$$

$\frac{1}{r^2}$ term dropped so as not to blow up at the origin

→

Boundary Conditions:

- Continuity of ϕ @ $r=a$

$$H_i = \frac{B_0}{\mu_0} - \frac{m}{4\pi a^3}$$

unknowns

- Continuity of B_n — normal component of field

$$\mu_i \frac{\partial \phi}{\partial r}\bigg|_{a-0} = \mu_0 \frac{\partial \phi}{\partial r}\bigg|_{a+0}$$

→ Solve for unknowns

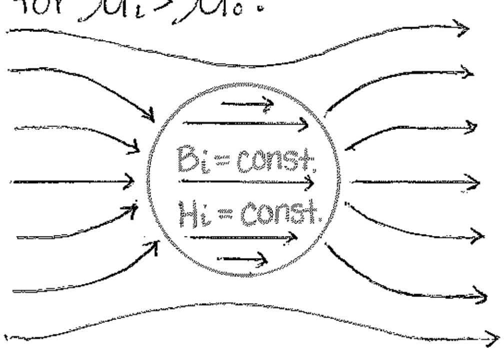
$$m = \frac{4\pi a^3}{3}\boxed{\left(\frac{\mu_i - \mu_0}{\mu_i \mu_0}\right)B_0} \text{ — } M_0$$

limit: max $m = \frac{4\pi a^3}{3}B_0$ when $\mu_i \to \infty$, same as a conducting sphere

$$H_i = \frac{B_0}{\mu_i}\left[1 + \frac{2}{3}\frac{\mu_i - \mu_0}{\mu_i}\right] \to B_i = \mu_0\left[H_i + \overline{M}\right]$$

$$\hookrightarrow \overline{M} = \frac{B_i}{\mu_0} - H_i$$

for $\mu_i > \mu_0$:



Generally,

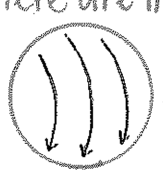
$$B_i = B_0\left[1 + \frac{2}{3}\frac{\mu_i - \mu_0}{\mu_i}\right]$$

In the limit $\mu_i \to \infty$,

$$B_i = B_0\left[1 + \frac{2}{3}\right]$$

Why is B_i stronger than B_0?

→ there are induced currents on the surface of the sphere



— Induced currents travel azimuthally around the sphere

How do we calculate these induced currents?

→ return to $\overline{M}$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

$$\hookrightarrow \vec{J} = \vec{J}_{free} + \underbrace{\vec{J}_{ind}}_{\vec{J}_{ind} = \nabla \times \overline{M}}$$

use cylindrical coordinates ($\vec{B}_0$ in $\hat{z}$-direction)

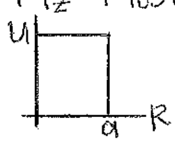
$$\vec{J}_{ind} = \nabla \times \underbrace{\hat{z} M_0}_{=\vec{M}}$$

$$\begin{cases} J_{r_i} = \frac{1}{r}\frac{\partial}{\partial \varphi} M_z - \frac{\partial}{\partial z} \cancelto{0}{M_\varphi} = 0 \\ J_z = \frac{1}{r}\frac{\partial}{\partial r} \cancelto{0}{M_\varphi} - \frac{1}{r}\frac{\partial}{\partial \varphi} \cancelto{0}{M_r} = 0 \\ \boxed{J_{\varphi_i} = \frac{\partial}{\partial z} \cancelto{0}{M_r} - \frac{\partial}{\partial r} M_z} \end{cases}$$

work with this equation

$M_z = M_0 \mathcal{U}(R)$ for $R = \sqrt{r^2 + z^2} < a$

$$\mathcal{U}(R) = \begin{cases} 1, & R < a \\ 0, & R > a \end{cases}$$



$$\frac{du}{dR} = -\delta(R - a)$$

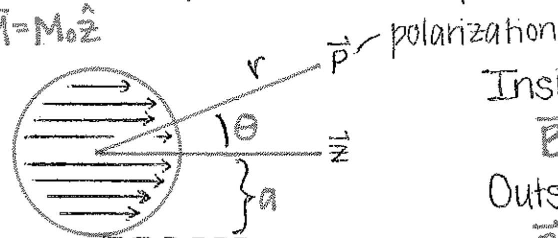
$$\frac{\partial M_z}{\partial r} = M_0 \frac{du}{dR} \underbrace{\frac{dR}{dr}}_{=\frac{r}{R}}$$

$$= -M_0 \delta(R-a) \frac{r}{R}$$

Ex. 2: Uniformly Magnetized Sphere (B_0 now subject to change)

→ Continuity of $\vec{B}_{normal}$ is expressed differentially here

$\vec{M} = M_0 \hat{z}$



Inside sphere:

$$\vec{B} = \mu_0[-\nabla \phi_m + \vec{M}_0]$$

Outside sphere:

$$\vec{B} = \mu_0[-\nabla \phi_m]$$

Continuity of $\vec{B}_n$ (all terms $\propto \cos(\theta)$)

$$\underbrace{\frac{-\partial \phi_m}{\partial r}\bigg|_{a+0}}_{\text{outside}} = \left(\underbrace{\frac{-\partial \phi_m}{\partial r}\bigg|_{a-0}}_{\text{inside}} + M_0 \hat{z} \right)$$

$\hat{r}_i \cos(\theta) = \hat{z}$

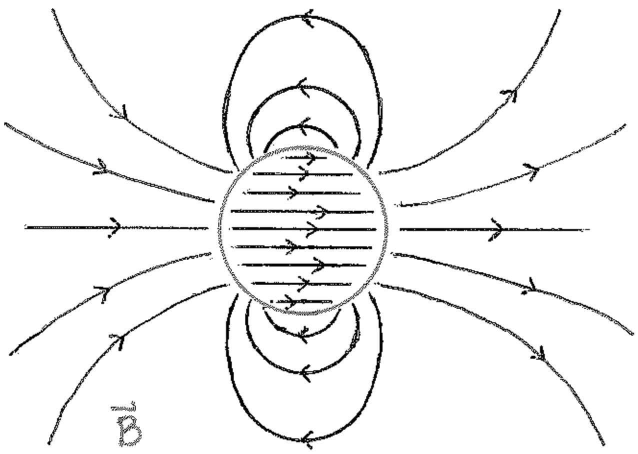
→ Inside the sphere (constant)

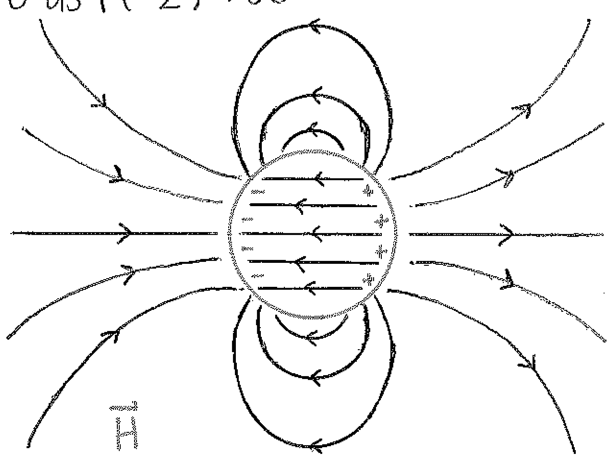
$$H_i = \frac{-m}{4\pi a^3} = \frac{-1}{3} M_0 \quad , \quad B_i = \frac{2}{3} \mu_0 M_0$$

*note: H_i & B_i in opposite directions

→ Outside the sphere (variable)

$H_z = \frac{2}{3} M_0 \frac{a^3}{z^3}$, $B_z = \frac{2}{3} \mu_0 M_0 \frac{a^3}{z^3}$ – there's no applied field, so $B_z \to 0$ as $r (= \hat{z}) \to \infty$





Now, abandoning statics in favor of time-derivatives...

2. <u>Faraday's Law</u>

- Faraday's Law:

  Induced $\vec{E}$ from changing $\vec{B}$

  $$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

- Ampere's Law:

  Induced $\vec{B}$ from changing $\vec{E}$

  Maxwell's fix to Ampere's eq.

  $$\nabla \times \vec{B} = \mu_0 \left[\vec{J} + \underbrace{\varepsilon_0 \frac{\partial \vec{E}}{\partial t}} \right]$$

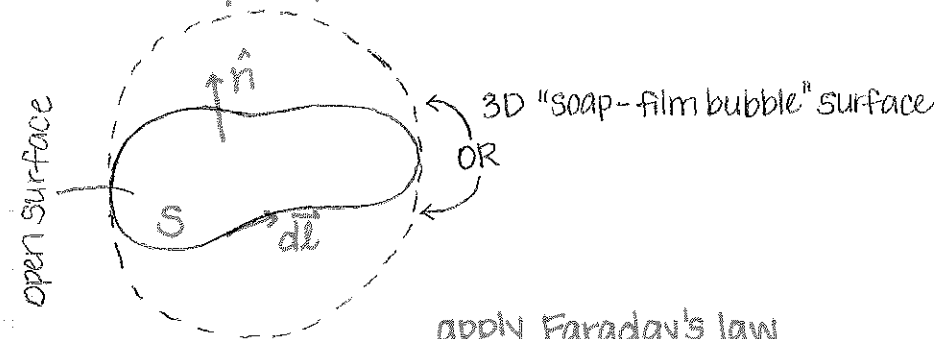
  displacement "current" → the presence of displacement current means that a changing electric field causes a magnetic field, even without a current (the converse of Faraday's law)

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \cdot \vec{J} + \varepsilon_0 \frac{\partial}{\partial t} \underbrace{(\nabla \cdot \vec{E})}_{\nabla \cdot \vec{E} = \rho/\varepsilon_0} = 0$$

– the new continuity equation

Stationary Loops:



$$\int_S da\, \hat{n}\cdot \nabla\times\vec{E} = \int_l d\vec{l}\cdot\vec{E} = -\int_S da\,\hat{n}\cdot\frac{\partial \vec{B}}{\partial t}$$

apply Faraday's law

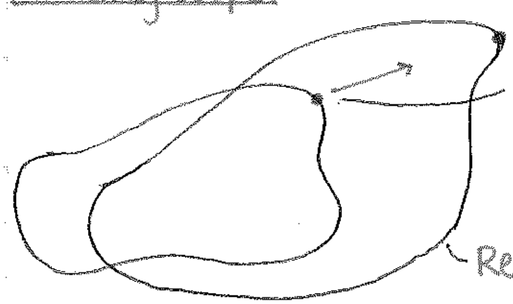
apply Stoke's law

define magnetic surface flux
$\Psi = \int_S da\,\hat{n}\cdot\vec{B}$ (=F in Jackson)

$= -\frac{\partial}{\partial t}\Psi$ ← the time-rate-change of magnetic flux through a surface

*We know this solution/relation works for all surfaces with the same perimeter because the divergence of $\vec{B}$ equals 0 ($\nabla\cdot\vec{B}=0$) and the integral over a closed surface (the "soap bubble" on both sides) is always equal to 0.

3. Moving Loops



We examine this point, moving with velocity $\vec{v}(x_0,t)$

Rest of loop (surface) moves with it

We want to Calculate $\frac{d\Psi}{dt} = ?$

Where $\Psi = \int_S dA\,\hat{n}\cdot\vec{B}$ ← integral over the moving surface

→ Want to find the change between the first loop and the second, where the overlap is the common area

→

↙ integral around perimeter of loop

$$\frac{d\Psi}{dt} = \underbrace{\overbrace{\int_S dA\,\hat{n}\cdot\frac{\partial\vec{B}}{\partial t}}^{\text{common area}}}_{\substack{\text{contribution from}\\ \text{the changing}\\ \text{magnetic field}}} - \underbrace{\int_C \vec{d\ell}\cdot\vec{v}\times\vec{B}}_{\substack{\text{contribution from}\\ \text{moving/changing}\\ \text{shape of loop}}}$$

can switch

$$\int_C \vec{B}\cdot\vec{v}\times\vec{d\ell}$$

$$= \int_C \underbrace{\vec{B}\times\vec{v}}_{\text{reorder}}\cdot\vec{d\ell}$$

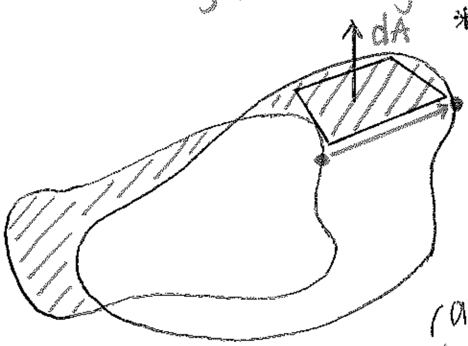
$$= -\int_C \vec{d\ell}\cdot(\vec{v}\times\vec{B})$$

From this, we know that the first contribution to the calculation is the "common area" of the flux.

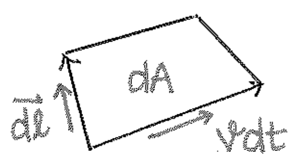
→ We need to use the definition of a derivative to inspect the differential change in area over time Δt and find the common area

$$\frac{d\Psi}{dt} = \lim_{\Delta t\to 0}\frac{\Psi(t+\Delta t)-\Psi(t)}{\Delta t}$$

Examining the change in this area...



* We will focus on the second term in the derivative as the common area is the overlap



$$dA = |\vec{d\ell}\times\vec{v}|\,dt$$

↙ applying the derivative definition

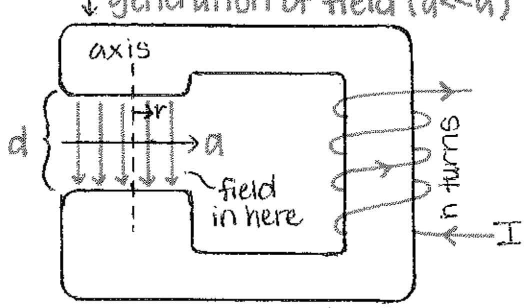
$$\rightarrow \frac{d\Psi}{dt} = \frac{\int \vec{B}\cdot(\vec{v}\Delta t\times\vec{d\ell})}{\Delta t}$$

$$\frac{d\Psi}{dt} = \int_S dA\,\hat{n}\cdot\frac{\partial\vec{B}}{\partial t} - \int_C \vec{d\ell}\cdot\vec{v}\times\vec{B} = \underbrace{-\int_C \vec{d\ell}\cdot\left[\vec{E}+\vec{v}\times\vec{B}\right]}_{\text{works for any loop}} = EMF$$

← electro-motive force

Suppose we have a time-dependent field in $\hat{z}$:

↙ generation of field ($d \ll a$)



$$\vec{B}(r,t) = \hat{z}\begin{cases} B_0(t), & r<a \\ 0, & r>a \end{cases}$$

Assume all fields are axis-symmetric (with respect to θ)

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

0 due to symmetries

$$\frac{1}{r}\frac{\partial}{\partial r} r E_\theta - \cancel{\frac{1}{r}\frac{\partial}{\partial \theta} E_r} = -\frac{\partial}{\partial t}\begin{cases} B_0(t) & , r<a \\ 0 & , r>a \end{cases}$$

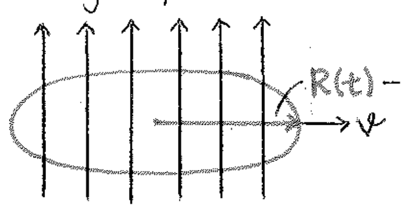
$$\oint_C \vec{E}\cdot d\vec{l} = -\frac{d\psi}{dt}$$

$$2\pi r \underbrace{E_\theta}_{E_\theta \to 0 \text{ as } r \to 0} = -\int_S da \frac{\partial B_0}{\partial t}$$

*Note: To apply Stoke's theorem, make sure you have $\vec{E}$ & $\vec{B}$ defined in the same reference frame

$$\rightarrow E_\theta(r) = -\frac{1}{r}\frac{\partial B_0}{\partial t}\begin{cases} r^2/2 & , r<a \\ a^2/2 & , r>a \end{cases}$$

What is the time-rate-change in magnetic flux through a moving loop within this field?



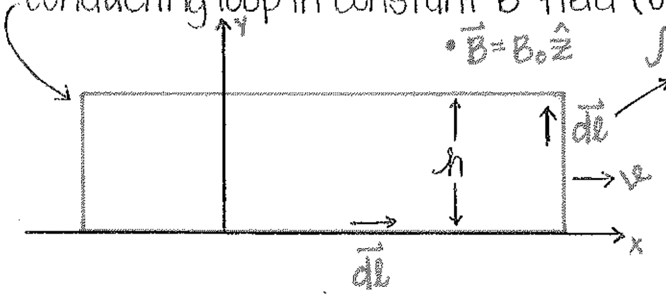
$R(t)$ – variable moving loop within field (radius variable in time)

$$\psi = \int_S da\, \hat{n}\cdot\vec{B} = \pi R^2 B_0(t)$$

$$\hookrightarrow \frac{d\psi}{dt} = \underbrace{\pi R^2 \frac{\partial B_0}{\partial t}}_{\text{contribution from the changing magnetic field}} + \underbrace{2\pi R \frac{dR}{dt} B_0}_{\text{contribution from the changing radius/shape of the loop}}$$

Thought Problems:

conducting loop in constant $\vec{B}$-field (out of page)



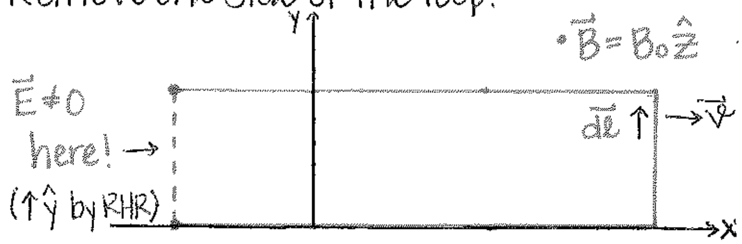
$\int d\vec{l}\cdot\vec{v}\times\vec{B} = -H v B_0$

$\vec{v}\times\vec{B} \downarrow$ –y-direction (on this side) by the Right-Hand-Rule

$\psi = \int_S da\, \hat{n}\cdot\vec{B}\,\hat{z}$

$\Longrightarrow \psi = B_0 H v t$

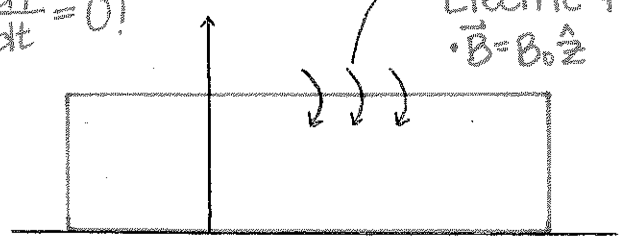
Remove one side of the loop.



$\vec{v} \times \vec{B}$ still ↓ $-\hat{y}$ by RHR
→ $\vec{E}$ must be ↑ in the $+y$-direction here such that $\vec{E} + \vec{v} \times \vec{B} = 0$ to satisfy the non-closed loop (this moves charges to create a field on the open end)

→ Now, close the switch ($\vec{E}$-fields active)

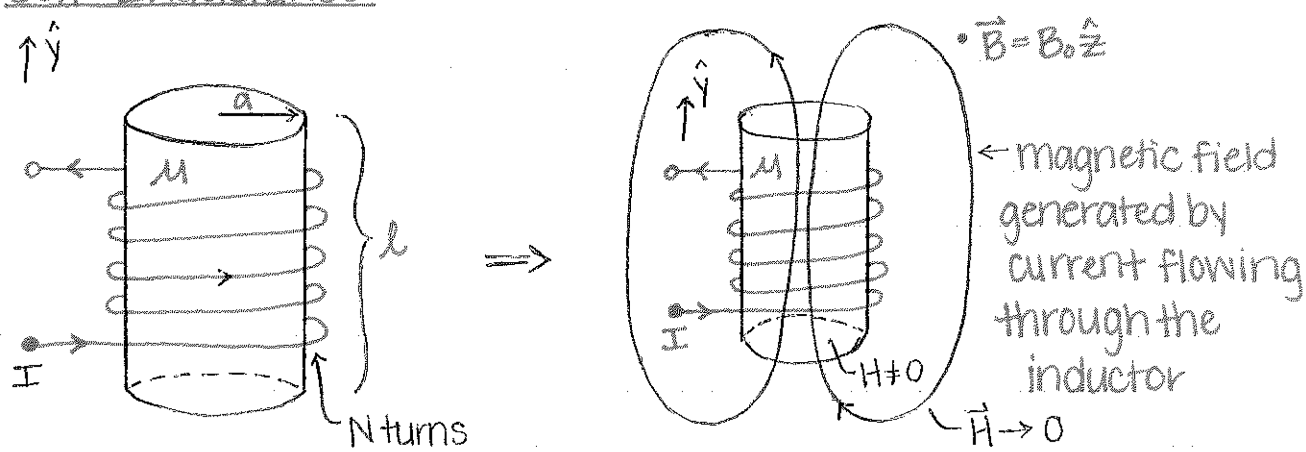
$\frac{d\Psi}{dt} = 0!$



Electric fields create an induced current in the conductor, then since $(\vec{E} + \vec{v} \times \vec{B})$ must $= 0$ under the above conditions,

$$\frac{d\Psi}{dt} = -\int_c \vec{dl} \cdot \left[\vec{E} + \vec{v} \times \vec{B}\right] = 0$$

4. <u>Self-Inductance</u>



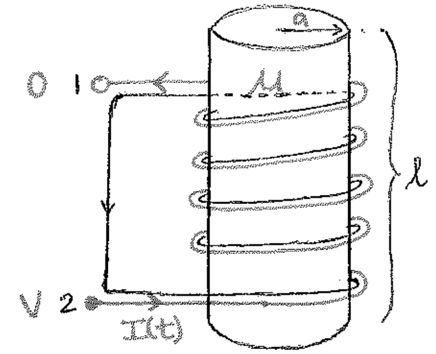
$\Psi = \pi a^2 B = \pi a^2 \mu H$

$$\oint \vec{H} \cdot \vec{dl} = \underbrace{I_{free}}_{\text{free current} \to \text{applied to coil}}$$

$lH = IN \to H = \frac{IN}{l}$

→

Now, what if $I = I(t)$?



→ Draw the loop to be integrated over such that it follows the coils

$$\psi = \frac{\pi a^2 \mu N}{\ell} I$$

$$\oint \vec{E} \cdot d\vec{\ell} = -\underbrace{\boxed{\frac{\pi a^2 \mu N^2}{\ell}}}_{\equiv L,\ \text{inductance}} \frac{dI}{dt}$$

$$\int_1^2 d\vec{\ell} \cdot \vec{E} = \underbrace{-L\frac{dI}{dt}}$$

$V = L\frac{dI}{dt}$, Ohm's law for Inductors

$$V = -\int_1^2 \vec{E} \cdot d\vec{\ell}$$

definition of V with respect to terminals 1 & 2 will take care of this neg.

5. The Skin Effect

So far we have...

Faraday's law: $\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}$

Ampere's Law: $\vec{J} = \frac{1}{\mu_0} \nabla \times \vec{B}$

Ohm's Law: $\sigma \vec{E} = \vec{J}$

→ Want to combine these into one differential equation

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \underbrace{\vec{E}}_{=\frac{\vec{J}}{\sigma}} = -\nabla \times \left(\frac{\vec{J}}{\sigma}\right) = -\nabla \times \frac{1}{\sigma \mu_0}(\nabla \times \vec{B})$$

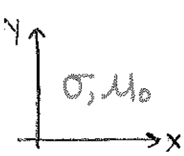
BAC CAB expansion

$$= -\frac{1}{\sigma \mu_0}\left[\nabla(\cancel{\nabla \cdot \vec{B}}) - \nabla^2 \vec{B}\right]$$

0

$$\Rightarrow \frac{\partial \vec{B}}{\partial t} = \frac{1}{\sigma \mu_0} \nabla^2 \vec{B}$$ or equivalently ($\vec{B} = \nabla \times \vec{A}$) $$\nabla^2 \vec{A} = \mu \sigma \frac{\partial \vec{A}}{\partial t},$$ the diffusion equation

*Note: Magnetic fields diffuse into conductors if they vary slowly. Vary them quickly and current will just sit on the conductor's surface.



In vacuum,

$B_y(x=0,t) = \mathrm{Re}\{\hat{B}_y(0) e^{-i\omega t}\}$ ← Spatially constant but time-varying magnetic field

- Apply the diffusion equation:

$$\frac{\partial B_y}{\partial t} = \frac{1}{\mu_0 \sigma} \frac{\partial^2}{\partial x^2} B_y(x,t)$$

↓ phaser notation ↓

$$-i\omega \hat{B}_y(x) = \frac{1}{\mu_0\sigma}\frac{\partial^2}{\partial x^2}\hat{B}_y(x)$$

$$\longrightarrow -i\omega = \frac{1}{\mu_0\sigma}\mathbb{k}^2$$

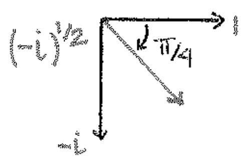
such that the solution is of the form

$$\hat{B}_y(x) = \exp(-\mathbb{k}x)$$

→ Solve for $\mathbb{k}$

$$\mathbb{k}^2 = (-i)\,\omega\mu_0\sigma$$

$$\mathbb{k} = \pm(-i)^{1/2}\sqrt{\omega\mu_0\sigma}$$



$$(-i)^{1/2} = \frac{(1+i)}{\sqrt{2}}$$

$$\rightarrow \mathbb{k} = \pm(1+i)\underbrace{\sqrt{\frac{\mu_0\sigma\omega}{2}}}_{\text{dimensions of inverse length}} = \pm\frac{(1+i)}{\delta}$$

$$\Longrightarrow \delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$ the skin depth (characteristic of the medium and the frequency)

The magnetic field will fall off exponentially in x, with a spatial oscillation being confined mainly to a depth less than the skin depth, δ.