Lecture 6 – Taylor Expansion 09/17/15

First, a useful identity:

for $t \neq 1$

$$\frac{1}{1-t} = \sum_{n=0}^{N-1} t^n + \frac{t^N}{1-t}$$

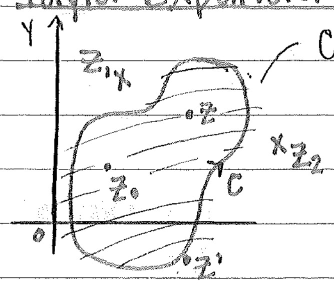
Check: $$1 \overset{?}{=} \sum_{0}^{N-1} t^n(1-t) + t^N$$

$$= (1 + \cancel{t} + \cancel{t^2} + \ldots + \cancel{t^{N-1}}) - (\cancel{t} + \cancel{t^2} + \ldots + t^N) + t^N$$

$$= 1 - t^N + t^N$$

$$= 1 \checkmark$$

Taylor Expansion around z_0



Contour of analytic region covering z & z_0 with value z' on the contour that changes with iteration

Proof: Start with Cauchy's Integral Formula

$$2\pi i f(z) = \oint_C \frac{dz' f(z')}{\underbrace{(z'-z)}}$$

$$(z'-z) = (z'-z_0) - (z-z_0)$$

$$(z'-z)^{-1} = \left[(z'-z_0)\left(1 - \frac{(z-z_0)}{(z'-z_0)}\right)\right]^{-1}$$

$$= \frac{1}{(z'-z_0)} \left\{ \sum_{n=0}^{N-1} \left[\frac{(z-z_0)}{(z'-z_0)}\right]^n + \frac{\left(\frac{z-z_0}{z'-z_0}\right)^N}{\left(1 - \left(\frac{z-z_0}{z'-z_0}\right)\right)} \right\}$$

$$2\pi i f(z) = \oint_C \frac{dz' f(z')}{(z'-z)} = \oint_C \left\{ \frac{1}{(z'-z_0)} \left[\sum_{n=0}^{N-1} \left(\frac{z-z_0}{z'-z_0}\right)^n + \underbrace{\frac{\left(\frac{z-z_0}{z'-z_0}\right)^N}{1 - \left(\frac{z-z_0}{z'-z_0}\right)}}_{\text{remainder}} \right] \right\}$$

$$= \sum_{n=0}^{N-1} (z-z_0)^n \underbrace{\oint_C \frac{dz' f(z')}{(z'-z_0)^{n+1}}} + \text{remainder}$$

Remember: $$CIF^{(n)} = 2\pi i f^{(n)}(z_0) = \oint_C \frac{dz' f(z')}{(z'+z_0)^{n+1}}$$

$\rightarrow$

$$2\pi i f(z) = \sum_{n=0}^{N-1} (z-z_0)^n \frac{2\pi i f^{(n)}(z_0)}{n!} + \text{remainder}$$

Remainder:

$$\left| \oint_C \frac{dz' f(z')}{\underbrace{(z'-z_0)\ \ 1-\left(\frac{z-z_0}{z'-z_0}\right)}_{(z'-z)}} \left(\frac{z-z_0}{z'-z_0}\right)^N \right| \le \oint \frac{|dz'||f(z')|}{|(z'-z)|} \frac{|z-z_0|^N}{|z'-z_0|^N}$$

as $N \to \infty$, the remainder $\to 0$

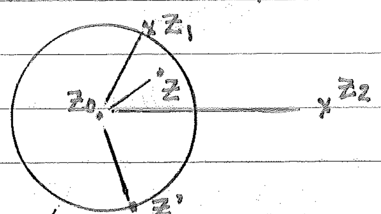
– Use absolute value:

$|z_1 z_2| = |z_1||z_2|$

$|z_1 + z_2| \le |z_1| + |z_2|$ ← Triangle inequality

$$\le \underbrace{|f(z')|_{max}}_{\text{maximum value on the contour}} \oint_{C,\ N\to\infty} \frac{|dz'|}{|z'-z|} \underbrace{\left|\frac{z-z_0}{z'-z_0}\right|^N}_{\lim_{N\to\infty} a^N \to 0 \text{ if } a<1 \text{ (small)}}$$

To take z to its maximum value is to deform the contour outwards



Required for this contour:

$|z-z_0| < |z'-z_0|$ ← Radius of Convergence

make contour over smallest distance (circular)

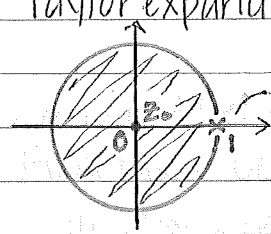
$$\Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n \quad \text{for } |z-z_0| < |z'-z_0|$$

Taylor Series for Complex Space!

Moral: Nearest singularity determines radius of convergence

Example: $f(z) = \frac{1}{1-z}$

Taylor expand around $z_0 = 0$



Radius of convergence determined by singularity @ $z=1$, $|z|<1$

$f(z_0) = f(0) = 1$

$f'(z_0) = \left.\dfrac{+1}{(1-z^2)}\right|_{z=0} = 1$

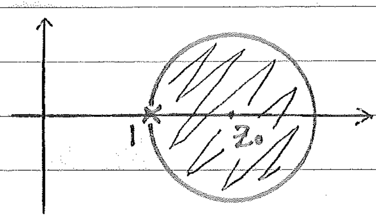
$f''(z_0) = \left.\dfrac{+2}{(1-z)^3}\right|_{z=0} = 2$

$\rightarrow f(z) = 1 + z + \dfrac{2z^2}{2!} + \ldots$

$\Rightarrow (1-z)^{-1} = 1 + z + z^2 + z^3 + \ldots, \underbrace{|z| < 1}$

Must give radius of convergence for complete solution

Example: $f(z) = \dfrac{1}{1-z}$, expand about $z_0 = 2$



Radius of Convergence:
$|z-2| < 1$

Quick Way -

$\dfrac{1}{1-z} = \dfrac{1}{\underbrace{1-(z-2)-2}_{\text{compensates for shift}}} = \dfrac{-1}{1+(z-2)}$ — let $(z-2) = \Delta z$ and use result from above

$= -1[1 - \Delta z + \Delta z^2 - \Delta z^3 \ldots]$

$= -1[1 - (z-2) + (z-2)^2 - (z-2)^3 \ldots]$

Example: $f(z) = e^z$, expand about $z_0 = 0$

$f(0) = 1$

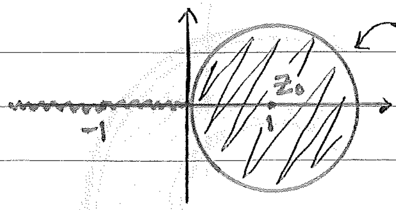
$f'(0) = 1 \quad \Rightarrow \quad f(z) = \sum_{n=0}^{\infty} \dfrac{z^n}{n!}$

$f''(0) = 1 \qquad |z| < \infty$ — no singularities; entire domain is analytic

Example: $f(z) = z^{1/2}$

Find taylor series for $z_0 = 1$

$\rightarrow$ This function is multi-valued, must place branch cut!



Branch cut defines radius of convergence here $|z-1|<1$

$f(1)=1$

$f'(1)=\frac{1}{2}z^{-1/2}\big|_{z=1}=1/2$

$f''(1)=-\frac{1}{4}z^{-3/2}\big|_{z=1}=-1/4$

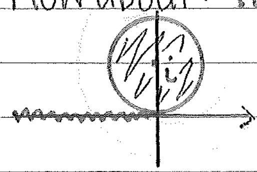
$\Rightarrow z^{1/2}=1+\frac{1}{2}(z-1)-\frac{1}{8}(z-1)^2+\ldots$

$|z-1|<1$

What if: Asked to find Taylor Series about $z_0=0$?

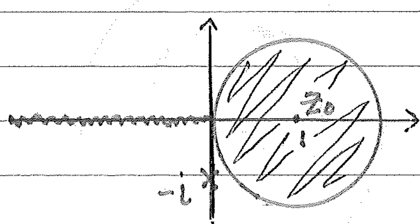
Doesn't exist! The Radius of Convergence would be 0!

How about: Taylor Series about $z_0=i$?



This works!
(ROC still defined by branch cut)

Example: $f(z)=\frac{z^{1/2}}{z+i}$ about $z_0=1$

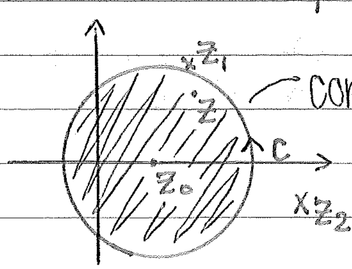


This works too!
(Does not contain the branch cut or the singularity)

Laurent Expansion

What if there is a singularity between z_0 and z?

Familiar case - Taylor Expansion R.O.C.

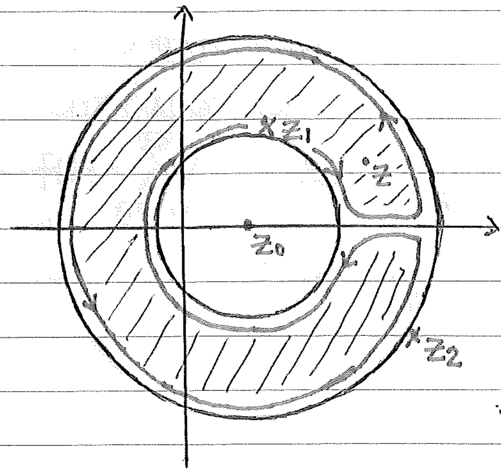


contour radius just inside distance to nearest singularity

R.O.C.: $|z-z_0|<|z-z_1|$

$\rightarrow$ Now, shift z to lie in the region between $\times z_1$ and $\times z_2$

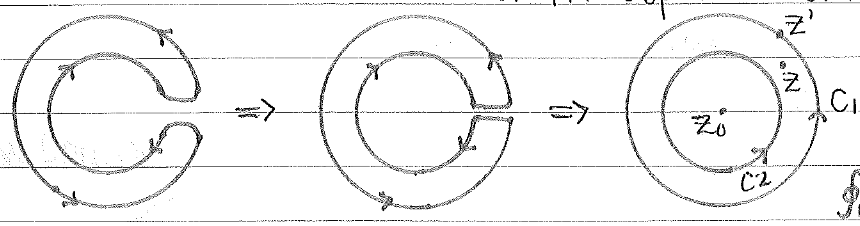
$\longrightarrow$



This requires a Laurent Expansion!

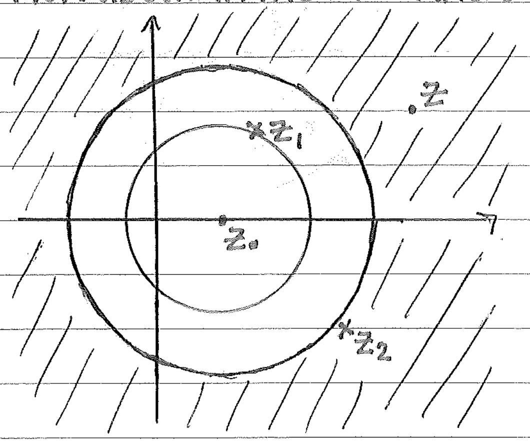
Radius of convergence now has two bounding relationships
$|z_1 - z_0| < |z - z_0| < |z_2 - z_0|$

You can also approximate this as two separate contours.



$\oint_C = \oint_{C_1} - \oint_{C_2}$

→ How about in the case where z lies beyond $^x z_2$?



New radius of convergence becomes bounded by $^x z_2$ and ∞

$|z_2 - z_0| < |z - z_0| < \infty$

In the case where $^x z_n$ lies between z_0 and z:

$$2\pi i f(z) = \underbrace{\oint_{C_1} \frac{dz' f(z')}{(z'-z_0)} \sum_{n=0}^{\infty}}_{z'-z \text{ expansion}} \left(\frac{z-z_0}{z'-z_0}\right) + \oint_{C_2} \frac{dz' f(z')}{(z-z_0)} \sum_{n=0}^{\infty} \left(\frac{z'-z_0}{z-z_0}\right)^n$$

$$z' - z = \underbrace{(z'-z_0)}_{\substack{\text{Bigger} \\ \text{in } C_1}} - \underbrace{(z-z_0)}_{\substack{\text{Bigger} \\ \text{in } C_2}}$$

← expand about this in C_2

$$= \sum_{n=0}^{\infty} (z-z_0)^n \underbrace{\oint_{C_1} \frac{dz' f(z')}{(z'-z_0)^{n+1}}}_{\text{NOT C.I.F.*}} + \sum_{m=1}^{-\infty} \frac{1}{(z-z_0)} \oint_{C_2} dz' f(z') (z'-z_0)^m$$

*f(z) non-analytic inside

Laurent Expansion:

$$\Rightarrow f(z) = \sum_{n=-\infty}^{+\infty} A_n (z - z_0)^n$$

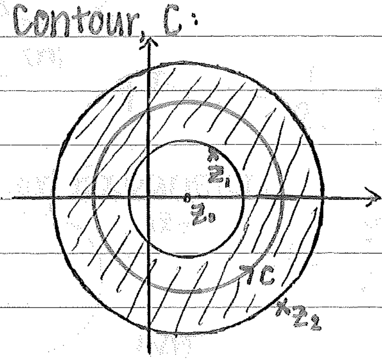
$$\hookrightarrow A_n(z_0) = \frac{1}{2\pi i} \oint_C \frac{dz' f(z')}{(z'-z_0)^{n+1}}$$

z' dissappears out with the integral

new C

This is NOT the derivative because it encloses a singularity! You must actually compute this integral (or convert into an appropriate form for a Taylor expansion)

New Contour, C:



Taylor Expansion ($A_n(z_0)$ form)

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n$$

$$\hookrightarrow a_n(z_0) = \frac{f^{(n)}(z_0)}{n!}$$