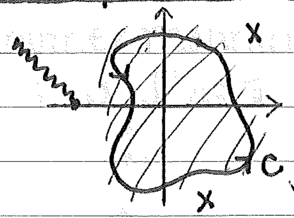
Lecture 5 - Existence & Analiticity of Derivatives 09|15|15

HW 1 solutions posted later today

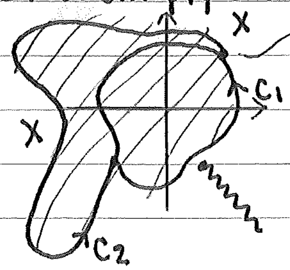
Cauchy's Theorem



$\oint_C dz\, f(z) = 0$

C is a simply connected region

In a simply connected region, you can deform the path

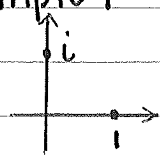


endpoints remain fixed

$\oint_{C_1} = \oint_{C_2} \Rightarrow$ path independence over the analytic region

* You can make a path such that it crosses itself → cancel-out crossover region

example 1:



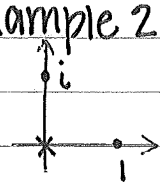
Find $\int_1^i dz\, z$

single-valued, no singularities

$\Rightarrow$ No need to define curve

$\int_1^i dz\, z = \left[\frac{1}{2}z^2\right]_1^i = -\frac{1}{2} - \frac{1}{2} = -1$

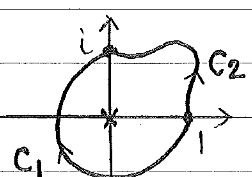
example 2:



Find $\int_1^i dz\, \frac{1}{z}$

nonsensical!

There exists a singularity → you must specify a curve!



$\Rightarrow$ Find $\int_{C\,1}^{\,i} dz\, \frac{1}{z}$

We can do this using anti-derivatives

for C_1: $I = \ln(z)\big|_1^i$ $\quad -2\pi < \theta < 0$ or $0 < \theta < 2\pi$ } same answer

$z = re^{i\theta}$

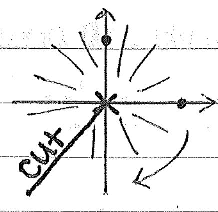
$= \underbrace{\ln(r)\big|_1^i}_{0} + \underbrace{i\theta\big|_1^i}_{-i3\pi/2 - 0} = -3\pi i/2$

For C_2: $I = 0 + i\theta\Big|_1^i$

$= i\pi/2 - i(0) = i\pi/2$

Another meaningful statement:

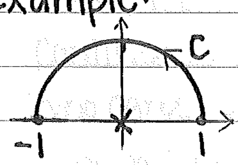
Find $\int_1^i dz \frac{1}{z}$ in the cut plane



Cut defines the plane of the simply-connected domain. It stretches from $z=0 \rightarrow z=\infty$

- Contour chosen cannot cross the cut
- Removes ambiguity as far as what theta values to use as the endpoints

Example:



Find $\int_C dz \frac{1}{z}$

Remember:

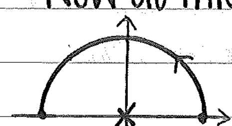
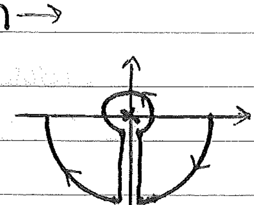
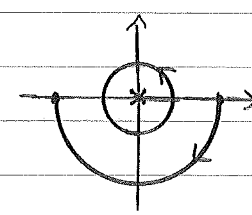
$$\oint_C dz \frac{1}{z^n} = \begin{cases} 2\pi i, & n = 1 \\ 0, & n \neq 1 \end{cases}$$

n = integer

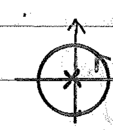
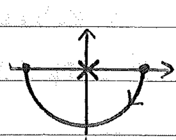
* for a counter-clockwise contour

$\int_C dz \frac{1}{z} = i\theta\Big|_0^\pi = i\pi$

→ Now do this by deformation →

 = by Cauchy's Theorem =  = 

bring so close as to [approximately] overlap

=  + 

$$= \underbrace{2\pi i}_{\text{as given above}} + \underbrace{i\theta\Big|_{\theta=0}^{\theta=-\pi}}_{\text{by anti-derivative}} = 2\pi i - \pi i = \pi i$$

✓ Same as before! (as expected for deformation within the simply-connected plane)

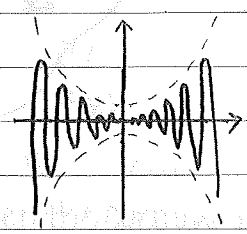
Goursat Theorem

Same statement as Cauchy's Theorem, but with a less restrictive proof.

① To prove Cauchy's theorem, we used Stoke's theorem

② BUT, Stoke's theorem assumes that U_x, V_x, etc., are $\underbrace{\text{analytic}}_{\text{in real space}}$ and are continuous*

* We can show that for $f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$



← derivatives exist, are analytic, but not continuous

③ We need a proof of Cauchy's Theorem that assumes that $f'(z)$ exists, $f(z)$ is analytic, but $f'(z)$ is not necessarily continuous @ z (and in some neighborhood of z)

④ Who cares? Why do we need this?

→ We will show that for analytic $f(z) \Longleftrightarrow f'(z)$ exists & exists in some neighborhood, then $f'(z)$ is necessarily continuous

└ $f'(z)$ follows from analytic $f(z)$, so you don't want to prove something using a product of the solution

⑤ Who really cares?

→ Later we will show $\underbrace{f^{(n)}(z)}_{\text{all derivatives of } f(z) \text{ if } f \text{ is analytic}}$ exists

*Proof is NOT in Arfken, but can be found easily online

Bottom line for analytic $f(z)$:

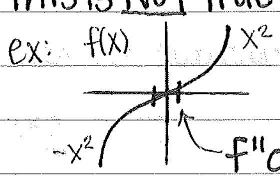
If $f(z)$ is analytic @ z

$\Rightarrow f^{(n)}(z)$ is analytic $\forall n$

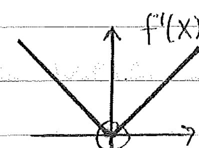
└ any derivative of analytic f exists and is analytic

This is NOT true for real $f(x)$

ex:



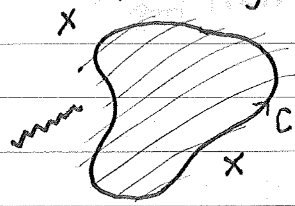
$f' = \begin{cases} 2x, & x > 0 \\ -2x, & x < 0 \end{cases}$



f'' does not exist @ $x = 0$!

no 2^{nd} derivative here

Cauchy's Integral Formula (C.I.F.)



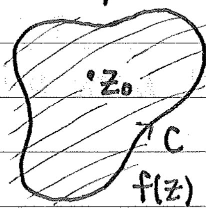
$f(z)$ – analytic inside C

Cauchy's [integral] theorem:

$$\oint_C dz\, f(z) = 0$$

Now say, for any z_0

└ where z_0 lies within the contour



∴ Cauchy's Integral Formula

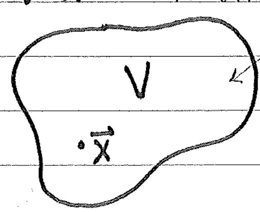
$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{dz\, f(z)}{(z-z_0)}$$

→ Why this is powerful ←

If given the contour integral for some unspecified function f and given f along the contour, then you can solve for what f is inside C, without ever needing to know f in the first place!

Later on, we will show that if

$\nabla^2 u = 0$ (for a harmonic function inside the volume)



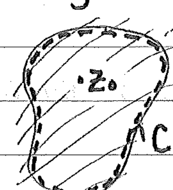
then we can show

$$\underbrace{u(\vec{x})}_{u \text{ inside}} = \int_S d\vec{s} \cdot \vec{\nabla} G\, \underbrace{\vec{u}(\vec{x})}_{u \text{ on surface}}$$

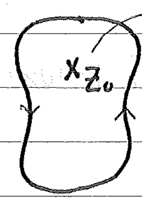
surface integral — Green's function

→ analogous to Cauchy's Integral Formula

Proving Cauchy's Integral Formula

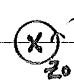


f known on the contour



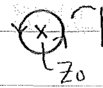
singularity @ $z = z_0$

→ deform contour inwards until almost completely on z_0



in the limit $z \to z_0$ $\quad \oint_C \longrightarrow \frac{1}{2\pi i} \oint_C \frac{dz\, f(z\to z_0)}{(z-z_0)}$ – pull this out of integral in limit

$$\lim \frac{1}{2\pi i} f(z_0) \underbrace{\oint_C \frac{dz}{(z-z_0)}}_{=2\pi i;\ \text{known}}$$



previously, z_0 was $= 0$

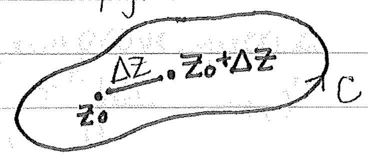
$$= f(z_0) \checkmark$$

↑ C.I.F.$^{(0)}$

C.I.F.$^{(n)}$ for all derivatives

What about df/dz_0?

i.e. Does information on the boundary give information inside for the derivative?



$$\frac{df}{dz_0} \equiv \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$= \frac{1}{\Delta z}\frac{1}{2\pi i}\oint \frac{dz\, f(z)}{z-(z_0+\Delta z)} - \frac{1}{\Delta z}\frac{1}{2\pi i}\oint \frac{dz f(z)}{(z-z_0)}$$

$$\lim_{\Delta z \to 0} = \frac{1}{\cancel{\Delta z}}\frac{1}{2\pi i}\oint dz\, f(z) \underbrace{\left[\frac{1}{z-(z_0+\Delta z)} - \frac{1}{(z-z_0)}\right]}_{\text{give common denominator}}$$

$$\frac{(\cancel{z}-\cancel{z_0}) - \cancel{z} + (\cancel{z_0} + \Delta z)}{[z-(z_0+\Delta z)][(z-z_0)]} \Rightarrow \frac{\cancel{\Delta z}}{[z-(z_0 \underbrace{+\Delta z}_{\text{must approach } 0})](z-z_0)}$$

$$\underbrace{\frac{df}{dz_0} \equiv f'(z_0) = \frac{1}{2\pi i}\oint \frac{dz\, f(z)}{(z-z_0)^2}}_{\text{C.I.F.}^{(1)}}$$

In general:

$$\text{C.I.F.}^{(n)} = \frac{n!}{2\pi i}\int \frac{dz\, f(z)}{(z-z_0)^{(n+1)}} = f^{(n)}(z_0)$$

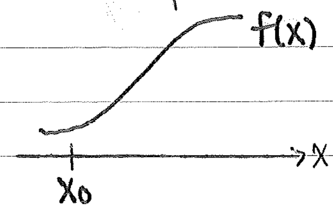
Cauchy's Integral Formula works for all derivatives

$\Rightarrow$ All derivatives exist inside & they are all analytic

→

Taylor expansion in complex space

In real space:



Given x_0, we can show

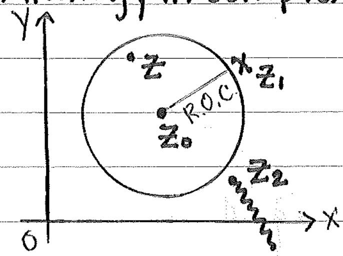
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n,$$

provided $|x-x_0|<1$

radius of convergence (R.O.C.)

→ assumes all derivatives $f^{(n)}(x_0)$ exist

Analogy in complex space:



later we will prove, given z_0:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

where f(z) in general has singularities & branch cuts ⇒ changes R.O.C.

Radius of convergence is defined by the distance to just within the closest singularity of branch cut point

$$|z-z_0| < |z_1-z_0|$$