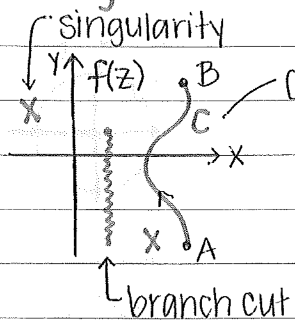
Lecture 4 – Complex Integration 09/10/15

Integration



Curve C from A→B; y(x) or x(y)

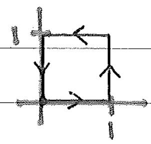
Find $I(A,B,C) = {}_A\int_C^B dz\, f(z)$

$z = x+iy \rightarrow dz = dx + i\,dy$

$f = u + iv$

$$I = \int (dx + i\,dy)(u + iv)$$
$$= \int (u\,dx - v\,dy) + i\int (v\,dx + u\,dy)$$

Direct integration example: (brute force method)



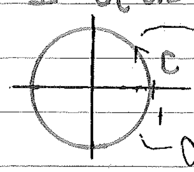
$\oint dz\, z$

$$\int dz\, z = \rightarrow + \uparrow + \leftarrow + \downarrow$$

$$\int dz\, z = {}_0\int^1 dx\, x + i\,{}_0\int^1 dy(1+iy) + {}_1\int^0 dx(x+i) + {}_1\int^0 i\,dy\, iy$$
$$= \cancel{\tfrac12} + \underbrace{i(1}_{i}\underbrace{+i\tfrac12)}_{-\tfrac12} + \cancel{-\tfrac12 - i} + \cancel{\tfrac12}$$
$$= 0$$

Example: Parameterization

$I = \oint_C dz\, z^n$, by direct integration (n-integer)



polar coordinates more useful here

$z = re^{i\theta} = e^{i\theta}$

$dz = i\,d\theta\, e^{i\theta}$

contour is the unit circle

$$I = {}_0\int^{2\pi} i\,d\theta\, e^{i\theta}(e^{i\theta})^n = i\,{}_0\int^{2\pi} d\theta\, e^{i(n+1)\theta}$$
$$= i\,{}_0\int^{2\pi} d\theta \Big[\underbrace{\cos[(n+1)\theta]}_{\text{periodic} \rightarrow \text{ oscillates to } 0} + \underbrace{i\sin[(n+1)\theta]}_{\text{periodic} \rightarrow \text{ oscillates an even \# of times to } 0}\Big]$$

for $n+1 \neq 0$

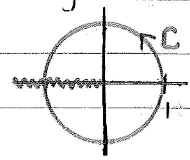
$I = 0$

for $n+1 = 0$, $\cos(0) = 1$, $\sin(0) = 0$

$I = i\,{}_0\int^{2\pi} d\theta = 2\pi i$

$I = \oint_c dz\, z^{\nu}$ (ν not an integer)

e.g. $\nu = 1/2$

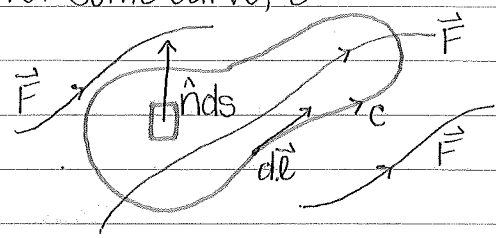


Integration is not defined for a multi-valued function without a branch cut / knowing what branch you're on

* You can integrate through the branch cut once it is placed and the contour has been defined

Review: Vector Analysis Theorem

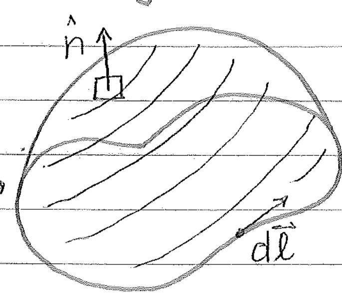
For some curve, C



▷ Stoke's Theorem: (general 3D statement)

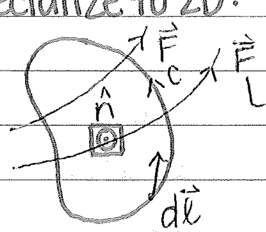
$$\int_S \vec{\nabla} \times \vec{F} \cdot \hat{n}\, ds \equiv \oint_c d\vec{l} \cdot \vec{F}$$

right-hand rule



The 3D surface integral → is over "the bubble"

Specialize to 2D:



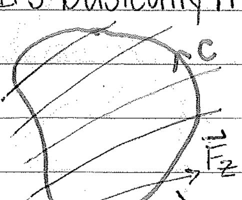
Portion of $\vec{F}$ in the plane

$$\int_S dx\,dy\,(\partial_x F_y - \partial_y F_x) = \oint_c (dx\,F_x + dy\,F_y)$$

→ Applied to complex functions, this becomes Cauchy's Theorem

Cauchy's Theorem

"It's basically magic!"



$\oint_c dz\, f(z) = 0$ if $f(z)$ is analytic at all z inside the curve, C.

dashed region → $\vec{F}_z$ is analytic

Example: $I = \int(u\,dx - v\,dy) + i\int(v\,dx + u\,dy)$

$I = \oint_C [dx\,u + dy(-v)] + i\oint_C [dx\,v + dy\,u]$

→ convert to a surface integral using the 2D Stoke's Theorem:

$$I_{sur.} = \oint_C dxdy\underbrace{\left[\frac{\partial}{\partial x}(-v) - \frac{\partial}{\partial y}(u)\right]}_{=0} + i\oint_C dxdy\underbrace{\left[\frac{\partial}{\partial x}u - \frac{\partial}{\partial y}v\right]}_{=0}$$

by Cauchy-Riemann

f(z) is analytic inside C

analytic if Cauchy-Riemann satisfied

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

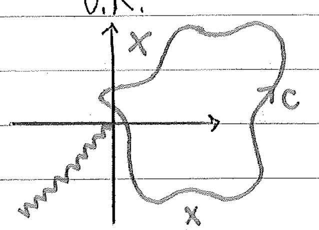
$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$I_{sur.} = \oint_C dxdy(0) + i\oint_C dxdy(0)$

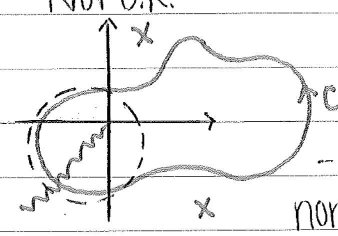
$= 0$ Cauchy's Theorem satisfied

*Cauchy's Theorem cannot be used for integrations that cross a branch cut or include a singularity

O.K.



Not O.K.



- cannot contain a non-analytic region!

Stoke's Theorem only holds if $\partial_x F_y$ and $\partial_y F_x$ exist and they do not at a branch cut/singularity.

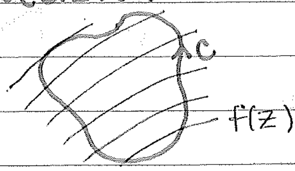
$\Rightarrow \partial_x F_y, \partial_y F_x$ must exist $\forall \vec{x} \underbrace{\in C}_{\text{inside the curve}}$

$\Rightarrow \partial_x F_y, \partial_y F_x$ must be continuous

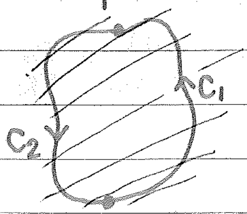
- derivatives must also be continuous within the curve

Implications of Cauchy's Theorem

① $\oint_C dz\,f(z) = 0$ if f is analytic everywhere inside C



② Cauchy's Theorem for multi-part curves



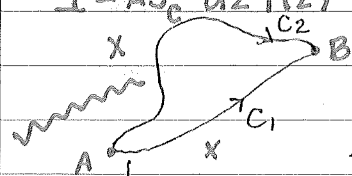
$\int_{C_1} dz f(z) + \int_{(-C_2)} dz f(z) = 0$

$\rightarrow \int_{C_1} - \int_{C_2} = 0$

$\rightarrow \int_{C_1} = \int_{C_2}$ as long as it is analytic inside

③ Path independence

$I = {}_A\int_C^B dz\, f(z)$



can deform C_1 into C_2

Integral (A,B) is independent of path (provided the enclosed region is analytic)

$\rightarrow$ We can deform contours so long as we do not cause them to cross any non-analytic regions (branch cuts, singularities)

$\int_{C_1} = \int_{C_2}$

↖ where C_2 does not cross non-analiticities

How to use Anti-Derivatives

First, prove the fundamental theorem of calculus

In real space:

if $f(x) \equiv {}_a\int^x dx' F(x')$ is true

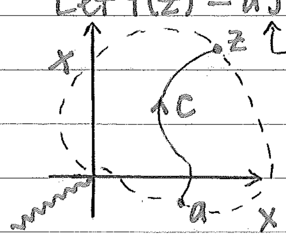
$\Rightarrow$ df/dx exists AND $df/dx = F(x)$

e.g. $F = x^2$, $f = x^3/3 + C$ } we can just say this with no further work as a consequence of the fundamental theorem

Now, generalize this for complex functions

In the complex plane:

Let $f(z) \equiv {}_a\int^z dz' F(z')$



↖ must define path!

$f(z) \equiv {}_a\int_C^z dz' F(z')$, C on an analytic path

$\rightarrow$ but C is deformable (with provisions)

C must stay in the simply-connected region, that is, the region where all paths (which are all equal) are completely within the analytic region of $f(z)$

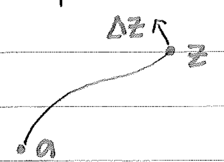
① Does df/dz exist?

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

exists if this limit holds

We can make this statement because paths are equivalent

$$\frac{df}{dz} = \frac{{}_a\int^{z+\Delta z} - {}_a\int^{z}}{\Delta z} = {}_z\int^{z+\Delta z} \frac{dz' F(z')}{\Delta z}$$



as $\Delta z \to 0$, $F(z') \to F(z)$

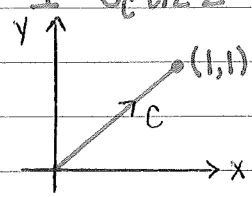
$$= F(z) \frac{{}_z\int^{z+\Delta z} dz'}{\Delta z} \quad \text{numerator} = \Delta z$$

$$= F(z)$$

$\Longrightarrow$ $df/dz = F(z)$ exists and anti-derivatives work for the simply connected domain

Example:

$I = \int_c dz\, z$



① First, by $\underbrace{\text{brute force}}_{\text{i.e. directly}}$:

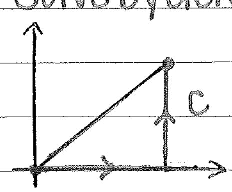
Parameterize: $z = x + iy = x + ix = x(1+i)$

$dz = dx(1+i)$

$$I = {}_0\int^1 dx(1+i)x(1+i)$$

$$= \tfrac{1}{2}(1+i)^2 = i$$

② Solve by deforming the contour



$$I = {}_0\int^1 dx\, x + i\, {}_0\int^1 dy(1+iy)$$

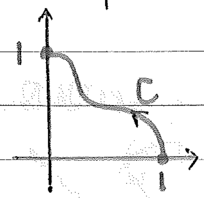
$$= 1/2 + i - 1/2 = i$$ ← path independence proven

③ Use antiderivatives

$$\int_{(0,0)}^{(1,1)} dz\, z = \left[\frac{z^2}{2}\right]_{(0,0)}^{(1,1)} = \frac{(1+i)^2}{2} = i$$ ← same result by all three methods

contour does not enter into integral because there are no non-analytic regions

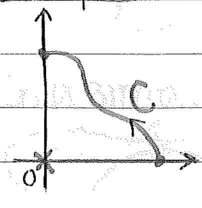
Example:



Find $\int_C dz\, z$

$\Rightarrow \frac{z^2}{2}\Big|_{1,0}^{0,1} = \frac{-1}{2} - \frac{1}{2} = -1$ ← contour impacts the result!

Example:



Find $\int_C dz/z$

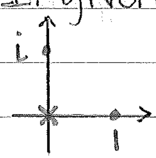
in the analytic domain $= [\ln(z)]_1^i$ └ $z = re^{i\theta}$

$= [\underbrace{\ln(r) + i\theta}_{@\, i=0,\ \theta=\pi/2}]_1^i$

$= i\pi/2 - i(\theta=0) = i\pi/2$

This function is only well-defined because we were given this path.

If given:



Find $\int_1^i dz/z$

You ~~cannot~~ use antiderivatives here because you don't know which path to take!