Lecture 3 – Complex Analiticity

09/08/15

- Problem Set #1 Posted online
  (due next Tuesday)
- TA coordinates posted online
- Topics for the next few weeks

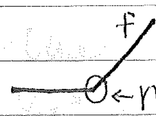
① Complex functions

  differentiation, integration, etc.

② Ordinary Differential Equations

Real Analysis

- $x, f(x)$, single-valued
- $\frac{df}{dx}$ exists
  - in the limit $\Delta x \to 0$
  - must be independent of path



← none of this

- Integration:

$$\int_a^b dx\, f(x) \equiv \sum_k (\Delta x)_k f(x_k)$$

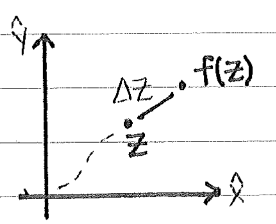
Complex Analysis

- $z, f(z)$, single- or multi-valued

  Use Branch Cuts to make a multi-valued function single-valued

  → You decide where to put the Branch Cut

- Differentiate $f(z)$:



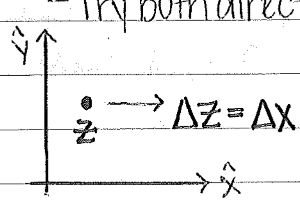
$$\frac{df}{dz} \equiv \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

Δz ← could be anything

$\frac{df}{dz}$ exists @ z IFF the value is independent of the path (same in Δx and Δy)

Under what conditions does this exist?

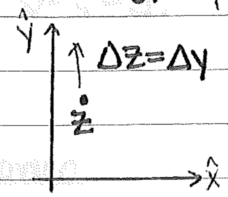
– Try both directions ⇒ $z = x + iy$, $\Delta z = \Delta x + i\Delta y$



What if the change was entirely in x?

$$\frac{df}{dz} \equiv \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$ ~ $f = u(x,y) + i v(x,y)$

$$= \frac{\partial f}{\partial x} = \boxed{\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}}$$



And if the change was entirely in y?

must be equal!

$$\frac{df}{dz} = \lim_{\Delta y \to 0} \frac{\Delta f}{i \Delta y} = \frac{1}{i} \frac{\partial f}{\partial y}$$

$$= \frac{1}{i}\left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}\right) = \boxed{-i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}}$$

⇨ **Cauchy-Riemann Conditions for existence**

$u_x = v_y$ ~ derivatives w.r.t.

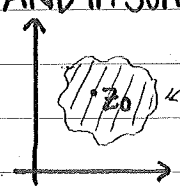
$u_y = -v_x$

(works both ways: df/dz exists ⟺ Cauchy-Riemann satisfied)

Further Definition:

① $f(z)$ differentiable @ z_0 IFF ⟺ Cauchy-Riemann satisfied @ z_0

② $f(z)$ is <u>analytic</u> @ z_0 IFF ⟺ $f(z)$ is differentiable @ z_0 AND in some neighborhood of z_0



← some neighborhood around z_0 where $f(z)$ is still differentiable

→ We are constraining the types of allowable complex analytic functions

Note the following if Cauchy-Riemann satisfied:

$u_x = v_y \Rightarrow u_{xx} = v_{yx}$

$u_y = -v_x \Rightarrow u_{yy} = -v_{xy}$

these two cancel because you can flip the order of differentiation for nice analytic functions

⇨ $u_{xx} + u_{yy} = 0$

└ second derivative usually means curvature ($f_{xx} = 0$ → straight line/flat plane)

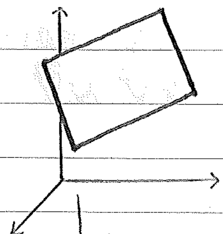
$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\Rightarrow \left.\begin{array}{l} \nabla^2 u = 0 \\ \nabla^2 v = 0 \end{array}\right\} \text{u and v can only be harmonic functions}$$

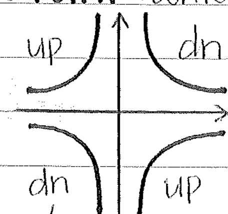
2-dimensional curvature must be 0
(must be flat planes or saddle points)

Allowed:

Flat Plane

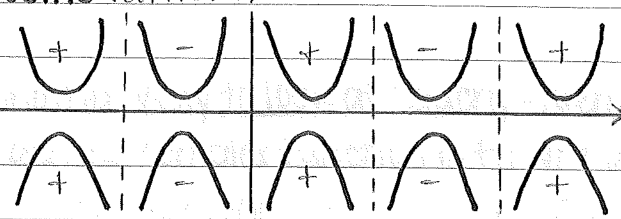


Saddle Point (contour)



Any angle or tilt okay

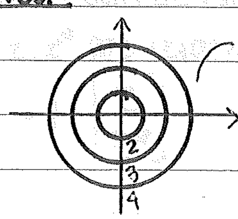
Cosine (contour)



- oscillatory in x
- evanescent in y

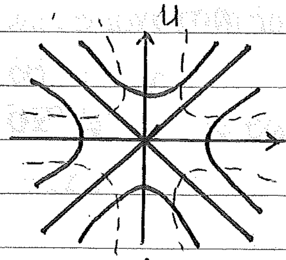
Not Allowed:

Bowl
(contour)

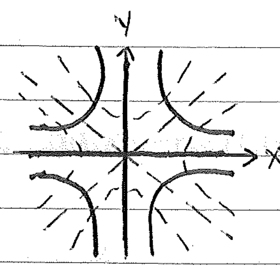


second derivative $\neq 0$ here

ex: $z^2 = \underbrace{(x^2 - y^2)}_{u} + i\underbrace{2xy}_{v}$



u



v

→ combine for conformal mapping

$\nabla^2 u = \nabla^2(x^2-y^2) = 2-2=0$
$\nabla^2 v = \nabla^2(xy) = 0$
$\Rightarrow$ Cauchy-Riemann satisfied for this entire region, so it is differentiable for this entire region.
* If a single-valued function is analytic, it cannot have a maximum

Note, also for Cauchy-Riemann:
$\vec{\nabla} u \cdot \vec{\nabla} v = 0$
Proof: $\vec{\nabla} u = \hat{i} u_x + \hat{j} u_y$
$$\vec{\nabla} u \cdot \vec{\nabla} v = (\hat{i} u_x + \hat{j} u_y)\cdot(\hat{i} v_x + \hat{j} v_y)$$
$$= \underbrace{u_x v_x}_{u_x = v_y} + \underbrace{u_y v_y}_{u_y = -v_x}$$
$$= v_y v_x + (-v_x) v_y$$
$$= 0$$

<u>Another way to look at Cauchy-Riemann</u>
A general complex function is $f(x,y) = u(x,y) + iv(x,y)$
└ not necessarily analytic, mind you
Recall, $f(x,y)$ can be written $f(x,y) = f(z,z^*)$
- x,y and z,z^* are related by a coordinate transformation
$$\begin{bmatrix} x \\ y \end{bmatrix} = M \begin{bmatrix} z \\ z^* \end{bmatrix}$$
└ some matrix

→ $f(x,y)$ is not a function of z and z^* but $f(z,z^*)$ is a function of x and y
We can show that for analiticity:
$$\left(\frac{\partial f}{\partial z^*}\right)_z = 0$$ is equivalent to Cauchy-Riemann
i.e. $f(z,z^*) \rightarrow f(z)$ if analytic
└ $f(z)$ being analytic does <u>not</u> guarantee $f(z^*)$ is too
e.g. z^2 is analytic but z^{*2} is not
$z^2 = (x^2-y^2) + i2xy$

$z^{*2} = (x^2 - y^2) - i2xy \Rightarrow u_x = 2x,\ v_y = -2x$

$u_x \neq v_y \rightarrow$ not analytic

<u>One more example</u>:

Consider $f(z, z^*) = |z|^2$

$\llcorner |z|^2 = zz^*$ (completely real)

Is this analytic? No.

$|z|^2 = u + iv$ — $v = 0$, cauchy-Riemann fails

$u = r^2,\ v = 0$

$= x^2 + y^2$ derivatives do not match → not analytic

Also, the level curves of u is $\underbrace{\text{a bowl}}_{\text{plotted earlier}}$ → second derivative > 0 not allowed!

exception: @ $(x,y) = z = 0$

$u_x = u_y = v = 0$

$\llcorner$ the function is differentiable @ $z=0$, but in the neighborhood around 0, this is no longer satisfied and therefore the function is still not analytic

Remember: differentiable $\neq$ analytic, necessarily

*can also do Cauchy-Riemann in polar

<u>Some Analytic Functions</u>

e.g.: consider $\frac{d}{dz}(z^2) = \frac{\partial}{\partial x}(z^2)$

$$= 2z\frac{\partial z}{\partial x} = 2z$$

in the y-direction?

$$\frac{d}{dz}(z^2) = \frac{\partial}{i\partial y}(z^2)$$

$$= \frac{2z}{i}\underbrace{\frac{\partial z}{\partial y}}_{i} = 2z$$

Analytic - same in both directions

e.g.: consider e^z → check analiticity

$$\frac{d}{dz}e^z = \frac{\partial}{\partial x}e^z = e^z$$

$\stackrel{?}{=} \frac{\partial}{i\partial y} e^{z} = \frac{e^{x}}{\cancel{i}} e^{iy} \cdot \cancel{i} = e^{x+iy} = e^{z}$

Analytic $\forall_z$

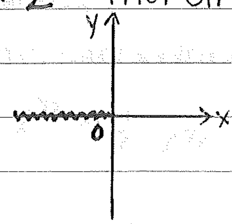
└ everywhere in z; the whole function

Careful!: z^n is analytic for all z, n = integer, except for $n<0$

$1/z^{|n|}$ not defined @ $z=0$

not defined → point of non-analiticity

Consider: $z^{1/2}$ (not single-valued → needs a branch cut)

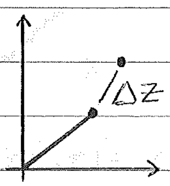


for $-\pi \le \theta < \pi$ ⌐ start exactly on $-\pi$, go all the way around, but can't exactly reach $+\pi$

$\frac{d}{dz} z^{1/2}$ → more convenient in $[r, \theta]$ coordinates

$\Rightarrow z^{1/2} = r^{1/2} e^{i\theta/2}$

$\frac{d}{dz} z^{1/2} = \frac{d}{dz}\left(r^{1/2} e^{i\theta/2}\right)$



$z = re^{i\theta}$

$\Delta z = \Delta r e^{i\theta} + re^{i\theta} i\Delta\theta$

└ can change r or θ

$\frac{df}{dz} = \frac{\Delta z^{1/2}}{e^{i\theta}\Delta r} = \frac{1}{e^{i\theta}} \frac{1}{2} \frac{1}{r^{1/2}} e^{i\theta/2}$

multiply by $(r^{1/2}/r^{1/2})$

$= \frac{z^{1/2}}{2z}$ ← they're equal! even though approached in different ways

$\frac{df}{dz} = \frac{\Delta z^{2}}{iz\Delta\theta} = \frac{1}{\cancel{i}z} r^{1/2} e^{i\theta/2} \frac{\cancel{i}}{2} = \frac{z^{1/2}}{2z}$

But what about the branch cut?

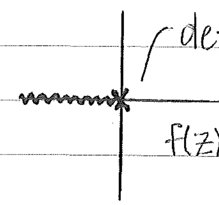
There's a discontinuity (a jump) in Δv and a cusp in Δu on the branch cut

→ the function is analytic everywhere in θ except for on the branch cut

Consider: $\ln(z)$ (multi-valued)

$z = re^{i\theta}$

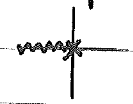
$\ln(z) \equiv \underbrace{\ln(r)}_{u} + \underbrace{i\theta}_{v}$



$r \neq 0$ – this function not defined (not analytic) @ $z = 0$

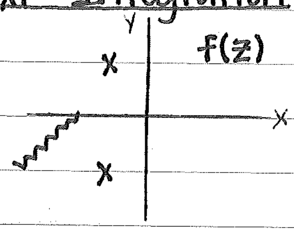
Once you define your singularity (x) and make the function single-valued (〰 Branch Cut), you can then show that $\ln(z)$ is analytic everywhere except @ $z=0$ and on the branch cut.

$\frac{d}{dz} z^{1/2} = \frac{1}{2}\frac{1}{z^{1/2}}$



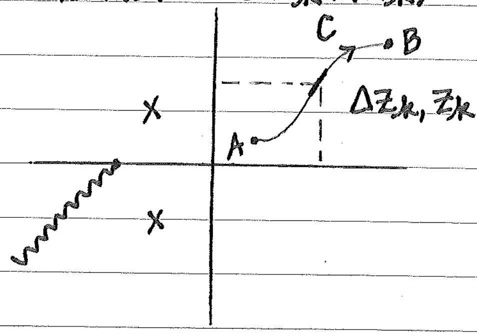
$\frac{d}{dz} z^{\nu} = \nu z^{\nu - 1}$

Next - Integration



When given a complex function, identify/map out any singular points and include branch cuts as given if multi-valued

$\int dz\, f(z) \equiv \sum \Delta z_k f(z_k)$



$I(A,B,C) = {}_A\int_C^B \underbrace{dz\, f(z)} \equiv \lim_{\Delta z \to 0} \sum_k \Delta z_k f(z_k)$

Complex function – dependent on A, B & C

Where the function is analytic everywhere along C from A → B

$z_k = x_k + iy_k \to \Delta z_k = \Delta x_k + i\Delta y_k$

$f(z_k) = u(x,y)_k + iv(x,y)_k$

$\Delta z_k f(z_k) = \underbrace{(\Delta x + i\Delta y)(u + iv)}_{\text{Must be calculated at each point}}$