11/17/15

Lecture 22 - Partial Differential Equations

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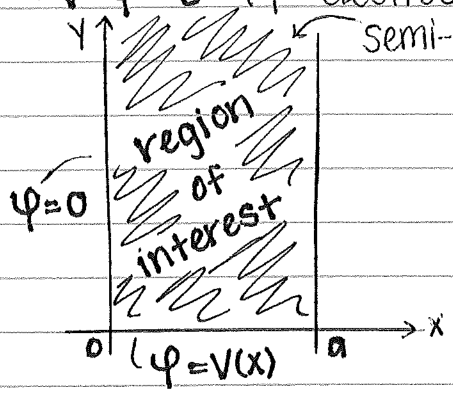
Today - PDEs:

- **Laplace in 2D Cartesian → Fourier series**
  (basics of separation of variables)
- **Diffusion equation**
- **Laplace in cylindrical → Bessel functions**

Laplace's Equation

$\nabla^2 \varphi = 0$ (φ - electrostatic potential)

semi-infinite rectangular region



Boundary conditions:

(1) $\varphi(x=0, y) = 0$

(2) $\varphi(x, y \to \infty) = 0$

(3) $\varphi(x=a, y) = 0$

(4) $\varphi(x, y=0) = V(x)$

Steps to achieving final solution:

① **Choose Cartesian coordinates, $\varphi(x,y)$**

suits rectangular region

$$\nabla^2 \varphi \to \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\varphi(x,y) = 0$$

② **Ansatz: Solution is separable**

$\varphi(x,y) = f(x)\, g(y)$

↳ plug into Laplace's equation & divide by the function $\varphi = fg$ →

$$\frac{1}{fg}\left[g\frac{d^2f}{dx^2} + f\frac{d^2g}{dy^2}\right] = 0$$

$$\underbrace{\frac{1}{f}\frac{d^2f}{dx^2} + \frac{1}{g}\frac{d^2g}{dy^2}}_{\text{only true if each of these is separately a constant}} = 0$$

→

③

(i) $\frac{1}{f}\frac{d^2f}{dx^2} = \underbrace{+k^2}_{\geq 0,\ k \text{ real}}$; $\frac{1}{g}\frac{d^2g}{dy^2} = -k^2$ — k = separation constant

(ii) $\frac{1}{f}\frac{d^2f}{dx^2} = -k^2$; $\frac{1}{g}\frac{d^2g}{dy^2} = +k^2$

Two solution options — must choose set that satisfies B.C.s

– for solution set (i):

$f \sim \begin{Bmatrix} e^{kx} \\ e^{-kx} \end{Bmatrix}$; $g \sim \begin{Bmatrix} \sin(ky) \\ \cos(ky) \end{Bmatrix}$

evanescent oscillatory

But does this work for our boundary conditions? No!!

- **fix y** (solutions must be separately constant)

$f(x) = Ae^{kx} + Be^{-kx}$

↑ want $f(0) = 0$

$\Rightarrow A = -B$

$f(a) = B(e^{-ka} - e^{ka})$

$\neq 0$, this cannot be satisfied for a non-trivial solution! (that is, for $B \neq 0$)

*This solution would be useful if semi-infinite in x instead of y

***Want x-dependence to be oscillatory → choose solution set (ii)**

– for solution set (ii):

$f \sim \begin{Bmatrix} \sin(kx) \\ \cos(kx) \end{Bmatrix}$; $g \sim \begin{Bmatrix} e^{ky} \\ e^{-ky} \end{Bmatrix}$

↑ dies as $y \to \infty$ as required

④ **We can further use boundary conditions to obtain a more specific solution.**

B.C.s:

(1) $\varphi = 0$ @ $x = 0$

$f(0) = 0$ (fix y)

$f(x) = C\sin(kx) + \cancel{D\cos(kx)}$

Must be dropped ($D=0$) so that f satisfies boundary condition (1)

(2) $\varphi \to 0$ as $y \to \infty$

$g(y \to \infty) = 0$ (fix x)

$g(y) = \cancel{Me^{+ky}} + Ne^{-ky}$

M=0 such that the evanescent solution dies as $y \to \infty$

⑤ **Use boundary condition (3), $\varphi(x=a, y) = 0$, to fix the separation constant k**

$f(a) = 0 \to \sin(ka) = 0$

$\Rightarrow k = n\pi/a$ $n \geq 1$, integer

⑥ **Find C:**

$$\varphi(x,y) = C\sin\left(\frac{n\pi x}{a}\right) e^{-\frac{n\pi}{a}y}$$

$\to$ use boundary condition (4) $\to$

$\varphi(x, y=0) = V(x)$

└ given

...but this implies the dependence on x is also a sine function (which will not usually be the case)

Must sum a set of solutions instead!

$$\varphi_n(x, y=0) = \underbrace{\sum_{n=1}^{\infty}}_{} C_n \sin\left(\frac{n\pi x}{a}\right) e^{-\frac{n\pi}{a}y}$$

* **Sum of sines is a solution to this PDE because Laplace's equation is <u>linear</u> & <u>homogeneous</u>** (but this would not work if the RHS of Laplace's equation $\neq 0$)

We must also check that summing these solutions still satisfies the boundary conditions

↳ because the solutions disappear at the boundaries, summing multiple will still satisfy boundary conditions (1), (2), and (3)

Find C_n's:

$$V(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right)$$

└ this is just expanding the function V in a fourier series

$\to$ **invert (and use orthogonality) to get C_n**

$\longrightarrow$

$$\int V(x)\sin\left(\frac{m\pi x}{a}\right)dx = \sum_{n=1}^{\infty}\int dx\,\underbrace{\sin\left(\frac{n\pi x}{a}\right)A_n\sin\left(\frac{mx\pi}{a}\right)}_{=0 \text{ unless } m=n}$$

$$= A_m\left(\frac{a}{2}\right)$$

⑦ Final solution

$$\varphi(x,y) = \sum_{m=1}^{\infty} A_m \sin\left(\frac{m\pi x}{a}\right) e^{-\frac{m\pi}{a}y}$$

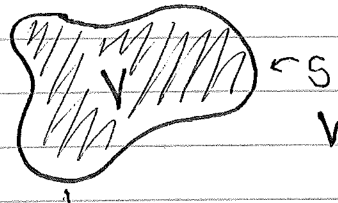
$$\hookrightarrow A_m = \frac{2}{a}\int dx \sin\left(\frac{m\pi}{a} V(x)\right)$$

Are there other solutions that are not of this form? No!

<u>Uniqueness Theorem</u>

for Poisson's equation

$$\nabla^2\varphi = \underbrace{-\rho(\bar{r})}_{\text{reduces to Laplace's equation when charge density } \rho=0}$$



Want φ for some volume V with boundary S

$\hookrightarrow \varphi$ on S is given $\equiv V(S)$

Proof: Ψ is a scalar function

$\Psi(S) = 0$ & $\nabla^2\Psi = 0$

Ψ vanishes on the boundary and satisfies Laplace's equation

$\rightarrow \Psi = 0$ everywhere in the volume

$$\int_V d\tau\, \underbrace{\bar{\nabla}\cdot\left[\Psi\bar{\nabla}\Psi\right]}_{\bar{\nabla}\cdot(f\bar{A}) = f\bar{\nabla}\cdot\bar{A} + \bar{\nabla}f\cdot\bar{A};\ f\to\Psi,\ \bar{A}\to\bar{\nabla}\Psi} \overset{\text{Gauss'}}{\underset{\text{Theorem}}{=}} \oint_S d\bar{\sigma}\cdot(\Psi\bar{\nabla}\Psi) \quad (1)$$

LHS of equation (1) $= \int_V d\tau\,|\bar{\nabla}\Psi|^2$

... but the RHS $= 0$ (Ψ vanishes on the boundary)

$\rightarrow \bar{\nabla}\Psi = 0$ everywhere inside V

$\Rightarrow \Psi =$ constant in V

$\hookrightarrow \Psi = 0$ on S $\rightarrow \Psi = 0$ in V

...back to Poisson's equation...

Suppose two solutions $\varphi_{1,2}$:

$$\nabla^2 \underbrace{(\varphi_1 - \varphi_2)}_{\varphi_1 - \varphi_2 \text{ like } \psi} = 0; \quad (\varphi_1 - \varphi_2) = 0 \text{ on } S$$

$\varphi_1 - \varphi_2$ must satisfy Laplace's equation

$\rightarrow \varphi_1 - \varphi_2 = \psi = 0$

$\rightarrow \varphi_1 = \varphi_2$, There is only one solution!

Diffusion Equation

temperature

$$\frac{\partial T(x,t)}{\partial t} = D \frac{\partial^2 T(x,t)}{\partial x^2}$$

by adding time, our boundary conditions become initial conditions!

Note: Schrödinger's equation is a slight variation on this

$$-i \frac{\partial}{\partial t} \psi \propto \nabla^2 \psi$$

$\rightarrow$ apply separation of variables $\rightarrow$

$T = f(x) g(t)$

$$\frac{1}{g}\frac{dg}{dt} = \frac{D}{f}\frac{d^2 f}{dx^2} \Rightarrow \frac{1}{f}\frac{d^2 f}{dx^2} = -k^2$$

Choose $-k^2$ solution because you don't want g to blow up as time progresses;

$$\frac{dg}{dt} = -k^2 D g$$

want $\underbrace{g(t)}_{\text{time cannot be oscillatory and diffusion implies decay}} \propto e^{-k^2 D t}$

$$f \sim \begin{Bmatrix} \sin(kx) \\ \cos(kx) \end{Bmatrix}$$

*We may need to keep the $k=0$ solution! ($g(t) \propto$ constant)

$$\frac{d^2 f}{dx^2} = 0 \Rightarrow f(x) = ax + b$$

part of the $\cos(kx)$ solution

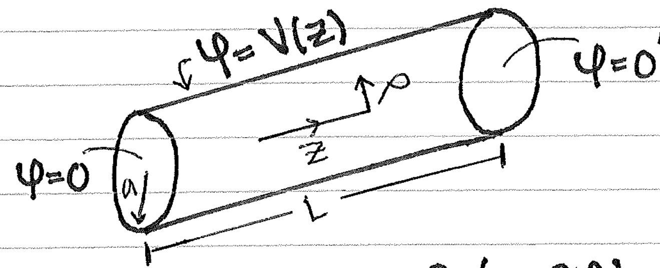
NOT included in $\{\sin(kx), \cos(kx)\}$

$\longrightarrow$

Laplace in Cylindrical

(azimuthal symmetry)

Case ①: $\varphi(\rho, z)$



Potential disappears @ end caps (no matter what the radius)

$$\nabla^2\varphi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\varphi}{\partial\rho}\right) + \frac{\partial^2\varphi}{\partial z^2} = 0$$

Ansatz: $\varphi = R(\rho)\,Z(z)$

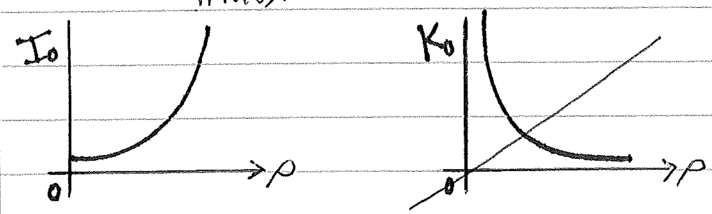
$$\frac{1}{Z}\frac{d^2}{dz^2}Z = \mp k^2 \quad , \quad \frac{1}{R}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) = \pm k^2$$

We want the oscillatory solution in z to achieve $\varphi(z) = 0$ at both end caps

→ pick $-k^2$ for oscillatory solution in z

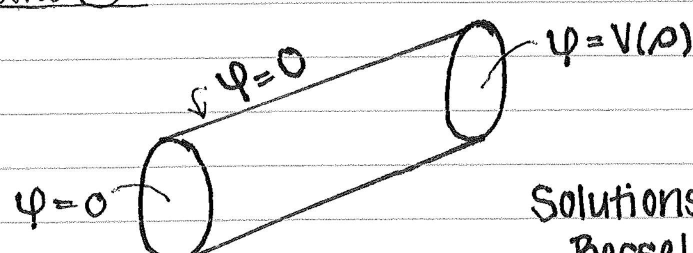
→ R will be [hyperbolic] bessel functions $I_0(k\rho)$ and $K_0(k\rho)$

index



Choose $I(k\rho)$ to satisfy the boundary conditions.

Case ②:



Solutions are just the standard Bessel functions!

$J_0(k\rho)$ & $N_0(k\rho)$

choose J such that $\varphi = 0$ @ $\rho = a$