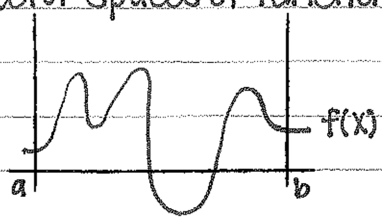
Lecture 21 – Sturm-Liouville

Sturm-Liouville System

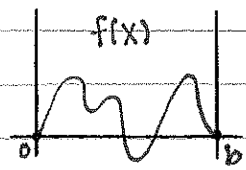
Vector spaces of functions include boundary conditions



* If you define a Hermitian operator in the vector space, you can find a complete set of eigenfunctions

Last time:

Consider
$$A = \frac{d^2}{dx^2}$$



If A is Hermitian, $(f, Ag) = (Af, g)$

↳ equivalent to checking ↴

$$(f, Ag) = \int_0^b dx\, f^*(x)\frac{d^2 g}{dx^2}$$

⇒ found H

$$\frac{d^2}{dx^2}\varphi = -k^2\varphi \quad \text{(oscillatory)}$$

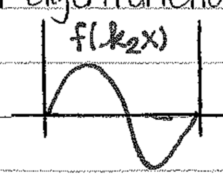
sign chosen for convenience

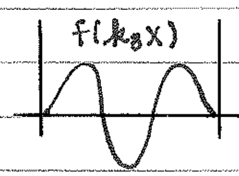
$\rightarrow \varphi \sim \{\sin(kx), \cancel{\cos(kx)}\}$

must satisfy boundary conditions

$\sin(kb) = 0;\ k_n b = n\pi,\ n = 1, 2, \ldots$

$\underbrace{\varphi_n = \sin(k_n x)}_{\text{a complete set of eigenfunctions}}$





$f(x) = \sum_{n=1}^{\infty} a_n\varphi_n$, find a_n by inversion

→ Utilize orthonormality →

$(\varphi_m, \varphi_n) = 0,\ m \neq n$

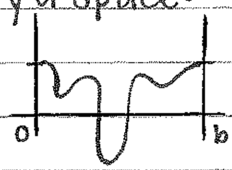
$$\int_0^b dx\, \sin(k_n x)\sin(k_m x) = \delta_{mn}(b/2)$$

→

$$(\varphi_m, f) = \sum_{n=1}^{\infty} a_n (\varphi_m, \varphi_n)$$
$$= a_m\left(\frac{b}{2}\right)$$

$(\varphi_m, \varphi_m) > 0$ (they're real!)

Try a space:



$f'(0) = 0, \; f'(b) = 0$

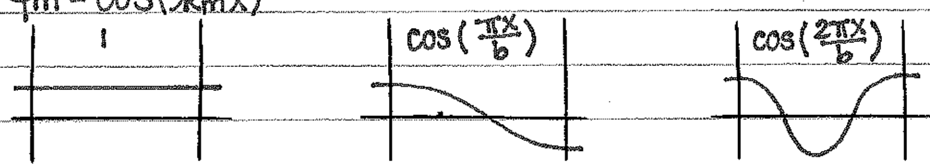
Is A Hermitian in this space?

→ Yes, but you must now use the $\cos(kx)$ solution

$\cos(kx)'\big|_b = 0$

$k_m b = m\pi, \; m = 0, 1, 2, \ldots$

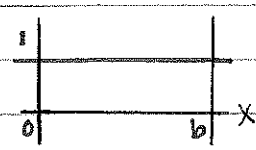
$\Rightarrow \varphi_m = \cos(k_m x)$



$f = \sum \varphi_m$

Suppose

$f(x) =$



→ expand in a complete set (choose cosine solution)

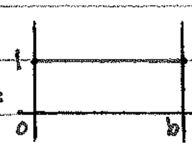
$f(x) = \sum_m A_m \cos(m\pi \underbrace{x(b)}_{\text{function of } b})$

$f(x) = A_0 \cos\left((m=0)\frac{\pi x}{b}\right)$

$\quad \llcorner A_0 = 1$

Now suppose

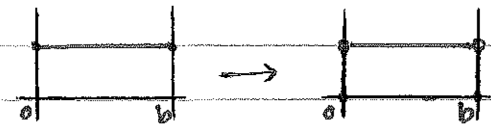
$f(x) =$



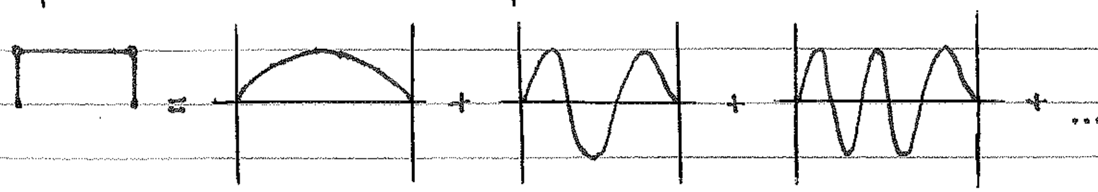
$= \underbrace{\sum_m A_m \sin(k_m x)}$

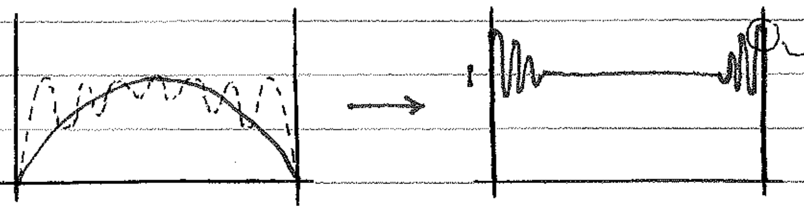
This solution does not satisfy the boundary conditions!

→ We can approximate this as a step function to achieve boundary requirements



Expand sine solution in a complete set:





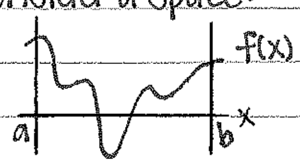
$\Rightarrow$ No absolute convergence! This approaches the area of absolute convergence, but the fringe terms never fully disappear.

S-L Systems for General 2nd Order Linear Operators

e.g. $\mathcal{L} = p\frac{d^2}{dx^2} + p_1\frac{d}{dx} + q$

Where $\mathcal{L}$ is a linear operator and $p(x)$, $p_1(x)$, and $q(x)$ are known

Consider a space:



* Under appropriate boundary conditions on $f(x)$ we can make $\mathcal{L}$ Hermitian provided we redefine the inner product with a weight function $W(x)$

namely, let

$$(f,g) = \int_a^b dW \underbrace{W(x)}_{W(x)>0 \text{ in the domain (except possibly @ } a,b)} f^*(x)\, g(x)$$

Claim:

If you can rewrite $\mathcal{L}$ as

$$\mathcal{L}_W = \frac{1}{W}\frac{d}{dx}\left(Wp\frac{d}{dx}\right) + q,$$

then $\mathcal{L}$ is Hermitian for $\underbrace{W(x)>0}_{\text{and in the appropriate domain}}$ and with appropriate boundary conditions

$\longrightarrow$

first - can rewrite

$$\mathcal{L} = p\frac{d^2}{dx^2} + \frac{(pW)'}{W}\frac{d}{dx} + q$$

equations for $\mathcal{L}$ equivalent if:

$$\frac{(pW)'}{W} = p_1$$ ← only works if $p(x) \neq 0$ in the domain!

⇒ yields appropriate weight function

Proof:

$\mathcal{L}$ is Hermitian ⟺ $(f, \mathcal{L}g) = (\mathcal{L}f, g)$

$$(f, \mathcal{L}g) = \underbrace{{}_a\int^b dx\, \cancel{W(x)} f^*(x) \frac{1}{\cancel{W}} \frac{d}{dx}\left(pW\frac{dg}{dx}\right)}_{\text{integration by parts}} + {}_a\int^b dx\, W(x) f^*(x) q(x) g(x)$$

$$= -{}_a\int^b dx \underbrace{\frac{df^*}{dx}}_{\int \text{down}} \underbrace{(pW)\frac{dg}{dx}}_{\int \text{up}} + \left[pWf^*g'\right]_a^b + {}_a\int^b dx\, Wf^*qg$$

$$= {}_a\int^b dx\, W \overbrace{\left(\underbrace{\frac{1}{W}}_{\text{reinserted}}\frac{d}{dx}\left[pW\frac{df^*}{dx}\right]\right)}^{\mathcal{L}_W} g + \left[pWf^*g' - pWf^{*\prime}g\right]_a^b + {}_a\int^b dx\, Wf^*qg$$

$$= (\mathcal{L}f, g) + \underbrace{\left[pW(f^*g' - f^{*\prime}g)\right]_a^b}$$

$\mathcal{L}$ is Hermitian if this = 0!

(must define inner product appropriately)

Summary: The Completeness Theorem for Hilbert Spaces

Given a space and a domain $[a,b]$, we can make any operator $\mathcal{L} = \frac{1}{W}\frac{d}{dx}\left[pW\frac{d}{dx}\right] + q$ Hermitian provided $(f,g) \equiv {}_a\int^b dx\, Wf^*g$, $W(x) > 0$ and $p(x) \neq 0$ in the domain, and we satisfy the boundary condition $\left[pW(f^*g' - f^{*\prime}g)\right]_a^b = 0 \;\forall \{f, g\} \in$ space

-- You can use Dirichlet or Neumann boundary conditions or force the work function to $= 0$ @ a and/or b to constrain the functions, etc.

Examples of Operators

- Harmonic

$\frac{d^2}{dx^2}$ $\quad W = 1,\ p = 1,\ q = 0$

• (Bessel)$_0$

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{d}{dx}\right) \quad W=x,\ p=1,\ q=0$$

• (Bessel)$_\nu$ (where ν is a given parameter)

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{d}{dx}\right)-\frac{\nu^2}{x^2} \quad W=x,\ p=1,\ q=\frac{-\nu^2}{x^2}$$

• Airy

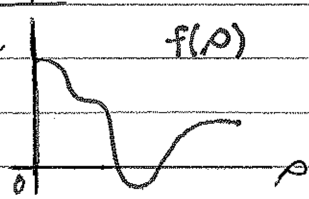
$$\frac{d^2}{dx^2}-1 \quad W=1,\ p=1,\ q=-1$$

• Legendre

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right] \quad W=1,\ \underbrace{p=1-x^2}_{\text{we will need to be careful with this!}},\ q=0$$

(Bessel)$_0$ ex:

Suppose



Consider

$$\mathcal{L}=\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right)$$

→ find some complete sets

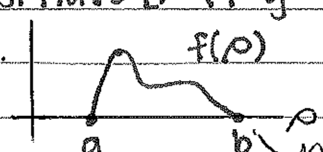
$(f,g)={}_a\int^b d\rho\ \rho\ f^*g$

└ can only do this for $\underbrace{\rho>0}$ ┌ or ≥ 0 at the endpoints

must satisfy $W(\rho)>0$ in the domain

must have $[\rho(f^*g'-f^{*\prime}g)]_0^b=0\ \forall\ f,g$ in the space

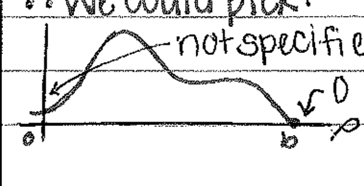
e.g.



would work

note that $W(0)=0$ (allowed)

∴ We could pick:



Want φ_n for this space

$\mathcal{L}\varphi = -k^2\varphi$

$\varphi(b) = 0$

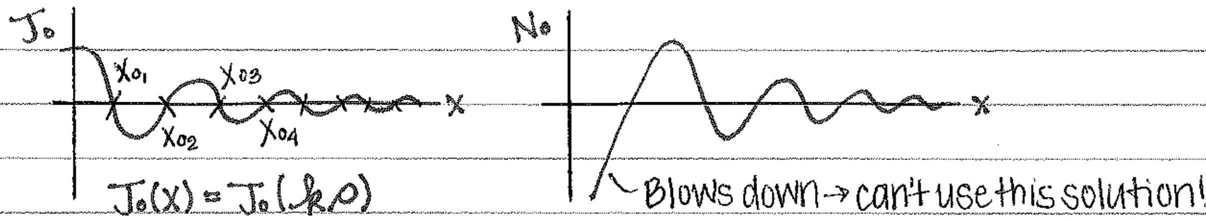
$\varphi(a)$ well-behaved (satisfies $W(\rho)$ requirement)

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\varphi}{d\rho}\right) = -k^2\varphi$$

can define $x \equiv k\rho$; $k>0$, real

$$\rightarrow \underbrace{\frac{1}{x}\frac{d}{dx}\left(x\frac{d\varphi}{dx}\right) = -\varphi}_{\text{Just the Bessel function!}}$$

→ known solutions are $J_0(x)$ and $N_0(x)$



$\therefore \varphi \rightarrow J_0(x)$

$\varphi \sim J_0(k\rho)$

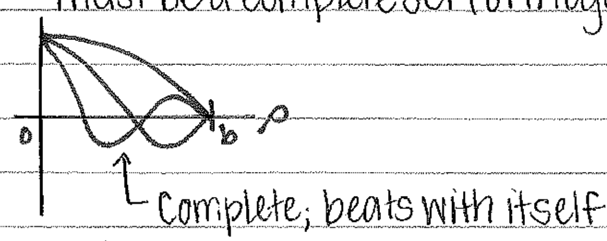
must have:

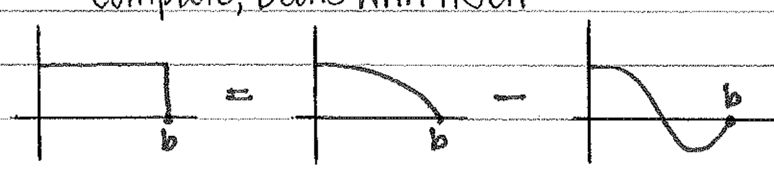
$J_0(kb) = 0 \rightarrow J_0(x_{0n}) = 0$

$\quad \llcorner k_n b = x_{0n},\ n = 1,2,3,\ldots$

$$\Rightarrow \underbrace{\varphi_n = J_0(k_n\rho)}_{}$$

must be a complete set (orthogonal) by the completeness theorem





Orthogonality: Must be that $(\varphi_m, \varphi_n) = 0$, $m \neq n$

– for $m \neq n$

$${}_a\int^b d\rho\, \rho\, J_0(k_m\rho) J_0(k_n\rho) = 0$$

- for $m = n$

$$\int_a^b d\rho\, \rho\, J_0(k_m\rho) J_0(k_m\rho) = \underbrace{\# C_{m,b}}_{J_1(k_m b)}$$ a constant

$$f(\rho) = \sum_{m=1}^{\infty} a_m J_0(k_m\rho)$$

$$\int_0^b d\rho\, \rho\, J_0(k_m\rho) f(\rho) = a_m \int_0^b d\rho\, \rho\, J_0^2(k_m\rho)$$

$\Longrightarrow$ yields a_m values