Lecture 1 – A Complex Introduction

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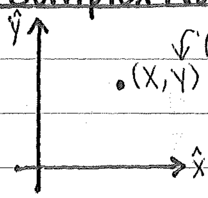
Course Outline:

- Complex analysis
- ODEs – review + new techniques
  - ↳ asymptotic functions
- Sturm–Liouville theorem
  - ↳ special functions (spherical bessel functions, etc.)
- Greene's functions
- Initial value and Boundary value functions
- PDEs (Jackson math, PHYS606)

Complex Numbers

composed of two locations



Addition:

complex number addition is vectorial addition

$$(x_1, y_1) + (x_2, y_2) \equiv (x_1 + x_2,\ y_1 + y_2)$$

Multiplication:

$$(x_1, y_1) * (x_2, y_2) \equiv (x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1)$$

Let $(x, y) \equiv z$

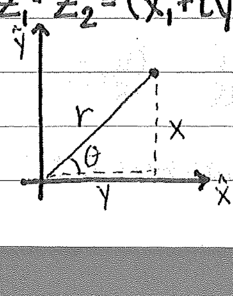
$\Rightarrow z = x + iy$

Where i is defined such that $i^2 = -1$

$$z_1 = z_2 \iff x_1 = x_2,\quad y_1 = y_2$$

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

$$z_1 * z_2 = (x_1 + iy_1)(x_2 + iy_2)$$



Polar representation

$$z = (x, y) = (r, \theta)$$
$$= x + iy = r\cos(\theta) + ir\sin(\theta)$$

Amplitude: $r^2 = x^2 + y^2 \equiv |z|^2$

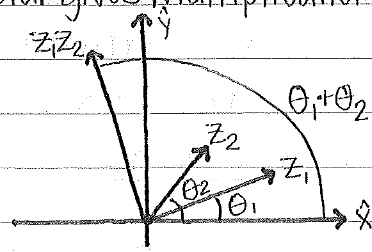
Phase/Argument: $\tan(\theta) \equiv y/x$

Multiplication:

$$z_1 * z_2 = \underbrace{r_1 r_2(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)) + i r_1 r_2(\cos(\theta_1)\sin(\theta_2) + \cos(\theta_2)\sin(\theta_1))}_{\text{"Trig Cheer" expansion}}$$

$$= r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

Polar gives multiplication a nice geometric interpretation



Next, since $z = r\cos(\theta) + ir\sin(\theta)$...

expand over power series

$$z = r\left[1 - \frac{\theta^2}{2!} + \dots\right] + ir\left[\theta - \frac{\theta^3}{3!} + \dots\right]$$

$$= r\left[1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \dots\right] + r\left[(i\theta) + \frac{(i\theta)^3}{3!} + \dots\right]$$

A complete series for all values of $i\theta$ (with convergence pre-proven for cosine and sine

recall definition: $e^p = \sum_{n=0}^{\infty} \frac{p^n}{n!}$

"Why do this? It's completely magical!"

$$z \equiv re^{i\theta} = r\underbrace{\sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}}_{e^{i\theta}}$$

Application: 1-D Schrödinger's Equation

Free particle – no potential

$$-i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}$$

linear PDE → can be solved

(without i, this is just a numerical diffusion eq.)

It is possible to write this as two wholly real equations!

Let $\Psi(x,t) = Re^{i\varphi}$; $R = R(x,t)$, $\varphi = \varphi(x,t)$

$\partial_t \Psi = R_t e^{i\varphi} + Ri\varphi_t e^{i\varphi}$; $\partial_x \Psi = R_x e^{i\varphi} + Ri\varphi_x e^{i\varphi}$

$$\partial_{x^2}\Psi = R_{xx}e^{i\varphi} + R_x i\varphi_x e^{i\varphi} + R_x i\varphi_x e^{i\varphi} - R\varphi_x^2 e^{i\varphi} + Ri\varphi_{xx} e^{i\varphi}$$

$\hookrightarrow -i\partial_t \Psi = -iR_t e^{i\varphi} + R\varphi_t e^{i\varphi}$

$\hbar(-iR_t + R\varphi_t) = \hbar^2/2m\,(R_{xx} + 2R_x i\varphi_x - R\varphi_x^2 + iR\varphi_{xx})$

Real: $\hbar R\varphi_t = \hbar^2/2m\,(R_{xx} - R\varphi_x^2) \Rightarrow \hbar\varphi_t = \hbar^2/2m\,(R_{xx}/R - \varphi_x^2)$

Imag: $-\hbar R_t = \hbar^2/2m\,(2R_x\varphi_x + R\varphi_{xx})$

Free-Particle Schrödinger's equation in two real equations

(-two non-linear PDEs - no way to solve without conversion)

"Those of you who will be real trained physicists in a year or two will look back on this and say 'YUCK!'"

DeMoivre's Theorem

multiplication is a rotation in the complex plane

We showed that $z_1 z_2 = r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

We also showed that $z_1 = r_1 e^{i\theta_1}$

$\therefore z_1 z_2 = r_1 r_2 e^{i\theta_1} e^{i\theta_2}$

$\hookrightarrow z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

So it follows that $e^{i\theta_1} e^{i\theta_2} \equiv e^{i(\theta_1+\theta_2)}$

DeMoivre:

$e^{i\theta} e^{i\theta} = e^{2i\theta}$

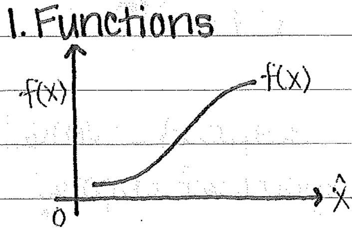
$(e^{i\theta})^n = e^{ni\theta}$

$\therefore (\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$

Now, Analysis

First, review real analysis

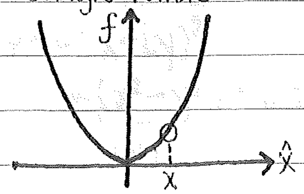
1. Functions



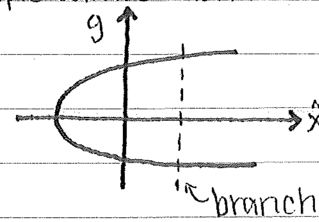
$\rightarrow$

2. Single Valued?

Single value



Multiple values - define branch



3. Define:

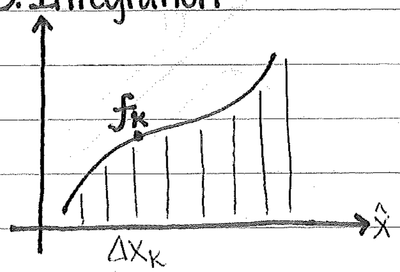
$\frac{df}{dx} \equiv \frac{\Delta f}{\Delta x} \lim_{x \to 0}$ — df/dx exists IFF the limit exists independent of path/direction of approach (i.e., no piece-wise, no cusps)

4. df/dx definition leads to:

$\frac{d}{dx}(x^3) \to 3x^2$ (trick)

└ from → $\frac{(x+\Delta x)^3 - x^3}{\Delta x}$ } Don't forget you once proved this!

5. Integration



$I = \sum_{k=0}^{\infty} \Delta x_k f(x_k) \quad \Delta x \to 0$

└ Only for single-valued functions (must define branch for multi-valued)

6. Taylor series

7. Fundamental Theorem of Calculus

Also, if $f(x) \equiv \int_a^x dx' F(x')$ ← for F is continuous in $[a, x]$ (the domain of a to x)

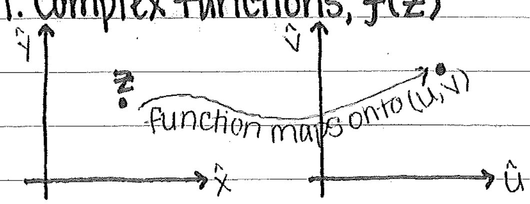
then we can prove:

a) $f(x)$ is differentiable $\forall_x$ (as long as it is continuous in the domain of x)

b) $df/dx = F(x)$

Complex Analysis

1. Complex functions, $f(z)$



A function is a mapping:

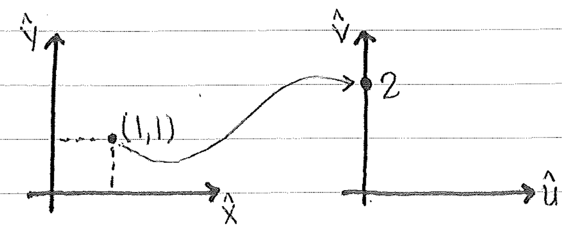
$(x,y) \xrightarrow{f(z)} (u,v)$

ex:

$f(z) = z^2$

$z^2 = (x+iy)^2$

$= \underbrace{(x^2-y^2)}_{u(x,y)} + i\underbrace{(2xy)}_{v(x,y)}$



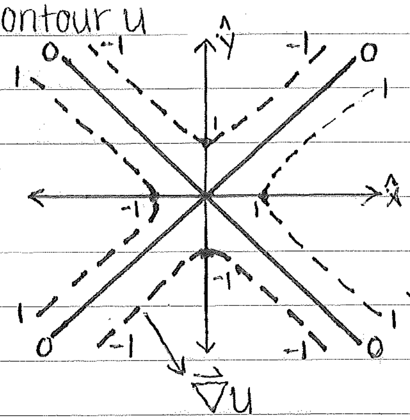
Contour Plots of u and v

$z^2 = u(x,y) + iv(x,y)$

$u = x^2 - y^2$

$v = 2xy$

Contour u



When $u=0$

$x^2 = y^2$

$y = \pm x$

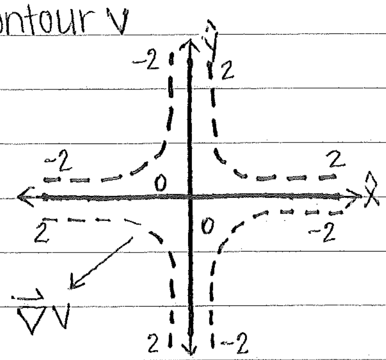
When $u=1$

$y^2 = 1 + x^2$

$y = \pm\sqrt{-1+x^2}$

Saddle point @ origin looking in x-direction

Contour v



When $v=0$

either $x=0$

or $y=0$

When $v=2$

$x = \pm 1$

$y = \pm 1$

When $v=-2$

$x = \pm 1$

$y = \mp 1$

In vector analysis:

gradient

$\vec{\nabla} u = 2x\hat{x} - 2y\hat{y}$

$\vec{\nabla} v = 2x\hat{y} + 2y\hat{x}$

$\vec{\nabla} u \cdot \vec{\nabla} v = 4xy - 4xy$

$= 0$

They're orthogonal!

(as expected)