# Lecture 14 - Asymptotic Behavior of Special Functions

**So Far:** ODE's

- **Separation of Variables**
  $1^{st}$-order equations; both linear & non-linear
- **Integrating Factor**
  $1^{st}$-order equations; linear only
- **Constant Coefficients**
  General solution of the form: $x = Ae^{\alpha t}$, $\alpha$ = constant
  $n^{th}$-order homogeneous equations; linear only
- **Equi-dimensional**
  General solution of the form: $y = x^{\alpha}$
  $n^{th}$-order homogeneous equations; linear only
- **Energy Method**
  Find $y(x)$ for $\frac{d^2y}{dx^2} = F(y) = \frac{-dV}{dy}$ ← potential
- **Integral Transform Method**
  General solution of the form: $\int_c dk\, e^{kx} f(x) = y(x)$
  $n^{th}$-order equations; linear only

## Integral Transform Method

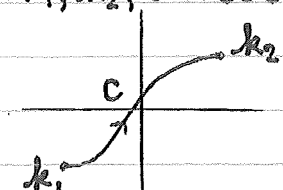
(can reduce equation order)

e.g. The Airy equation

$$y'' = xy$$

Try: $y(x) = k_1 \int_c^{k_2} dk\, f(k) e^{kx}$
← standard solution form for this method

$k_1, k_2$, C to be specified later




$y' \rightarrow \int k$

$y'' \rightarrow \int k^2$ ← LHS

$xy \rightarrow \int x \rightarrow \int \underbrace{xe^{kx}}_{\text{rewrite as } \frac{d}{dk}(e^{kx}) \text{ to consolidate x-term}}$

$$xy = \left[f(k)e^{kx}\right]_{k_1}^{k_2} - \int_c dk \frac{df}{dk} e^{kx} \leftarrow \text{RHS}$$

→

LHS = RHS

$$\int_C dk\, e^{kx} \underbrace{\left[k^2 f + \frac{df}{dk}\right]}_{\text{integrand}} = \left[f(k)\, e^{kx}\right]_{k_1}^{k_2}$$

**Regular Method - Solution:**

① **Find $k_1, k_2$ such that $[f(k)e^{kx}] = 0$**

② **Set the integrand = 0; $\overbrace{k^2 f + df/dk = 0}^{\text{first-order in } k!}$** – solve by using separation of variables

└ must maintain equality with RHS

$$\int \frac{df}{f} = -\int dk\, k^2$$

$\ln(f) = -k^3/3 + C$ solution of the form $f = Ae^{-k^3/3}$

③ **Plug in for $k_1, k_2$ requirements**

Want $k_1, k_2 \rightarrow \left[e^{-k^3/3} e^{kx}\right]_{k_1}^{k_2} = 0$

* Note that for any $k$, $\mathrm{Re}\{k^3\} > 0$

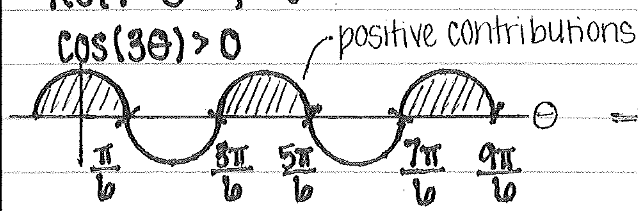
$\Rightarrow e^{-k^3/3} \rightarrow 0$ as $|k| \rightarrow \infty$

**Now, demand $\mathrm{Re}\{k^3\} > 0$**

let $k = re^{i\theta}$

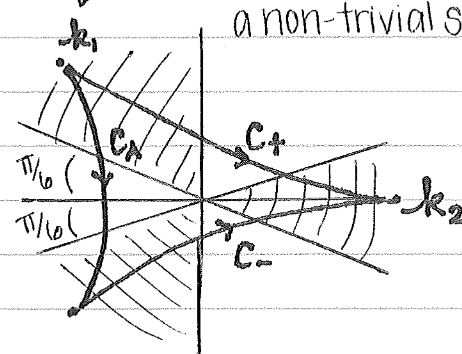
$\mathrm{Re}\{r^3 e^{3i\theta}\} > 0$

$\cos(3\theta) > 0$




$\Longrightarrow$

endpoints must be in separate cones to produce a non-trivial solution




**3 possible contours:**

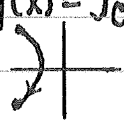
$Y(x) = \int_C dk\, e^{kx} e^{-k^3/3}$

$y = \int_{C_1}$ $y = \int_{C_2}$ $y = \int_{C_3}$

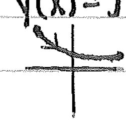
BUT - by Cauchy's Theorem, to integrate over all three (closed loop, entirely analytic region) would give you 0!

**Instead, we choose two independent solutions**

$\Rightarrow$ **Solution 1:** $y(x) = \int_{C_A}$



$\Rightarrow$ **Solution 2:** $y(x) = \int_{C_+}$ **OR** $y(x) = \int_{C_-}$



## If none of the previous solutions work...

i.e., linear, but non-constant coefficients, not equi-dimensional, and you can't reduce the order

Examples:

$-(\text{Bessel})_\nu \quad y'' + \frac{y'}{x} + \frac{(\nu^2 - x^2)}{x^2} y = 0$

$-(\text{Bessel})_0 \quad y'' + \frac{y'}{x} + y = 0$ $\quad \nu = 0$

$\quad$ (or) $\underbrace{x^2 y''}_{\text{can't reduce order}} + xy' + x^2 y = 0$

$-(\text{Legendre}) \quad \frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + k^2 y = 0$

$\quad \to$ standard form: $\nu(\nu+1)y$

$$(1-x^2)y'' - 2xy' + \nu(\nu+1)y = 0$$

Bessel, Legendre, and Airy: Main functions for this class

**Standard method is to try:**

**Frobenius' Method**

$$y = x^\lambda \sum_{n=0}^{\infty} a_n x^n$$

But before we're ready to apply Frobenius, we need to check some asymptotic behavior.

$(\text{Bessel})_\nu \quad y'' + \frac{y'}{x} + \left(1 - \frac{\nu^2}{x^2}\right) y = 0$ (with $\nu$ $O \sim 1$)

**Find the asymptotic behavior as $x \to 0$ and $x \to \infty$** (separately)

First, study the equation as $x \to 0$

**① Scale the equation**

$y' = \frac{dy}{dx} \sim \frac{y}{x} \quad ; \quad y'' \sim \frac{y}{x^2}$

$\quad \sim$ means scaling (compare with respect to each other)

$\frac{\cancel{y}}{\cancel{x^2}} : \frac{\cancel{y}}{\cancel{x^2}} : x^2 \cancel{y} : \frac{\nu^2}{\cancel{x^2}} \cancel{y}$

$\to 1 : 1 : x^2 : \nu^2$

since $x \to 0$ ↗ is a small term

$\therefore$ let $y = y_0 + y_1 + y_2 + \dots$

$$(y_0+y_1+\dots)'' + \frac{(y_0+y_1+\dots)'}{x} + \underset{\uparrow}{(y_0+y_1+\dots)} - \frac{\nu^2}{x^2}(y_0+y_1+\dots) = 0$$

Small compared to other $y_0$ terms (can throw out for lowest order)

**Lowest Order:**

$$y_0'' + \frac{y_0'}{x} - \frac{\nu^2 y_0}{x^2} = 0$$ ← equi-dimensional solution here

**First Order:**

$$y_1'' + \frac{y_1'}{x} + \underbrace{y_0} - \frac{\nu^2 y_1}{x^2} = 0$$

bring this term back in – large compared to $y_1$ terms

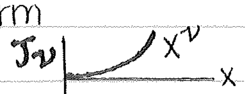
Equi-dimensional solution:

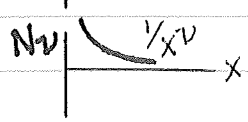
Try $y_0 = x^\alpha$ ← standard form

$\alpha(\alpha-\cancel{1}) + \cancel{\alpha} - \nu^2 = 0$

$\alpha^2 = \nu^2 \Longrightarrow \alpha = \pm\nu$

$y_0 \sim \begin{Bmatrix} x^\nu \\ x^{-\nu} \end{Bmatrix} \nu \neq 0$




Special case: $\nu = 0$

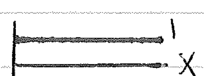
$$y_0'' + \frac{y_0'}{x} = 0 \Longrightarrow \frac{x(xy_0')'}{x^2}$$

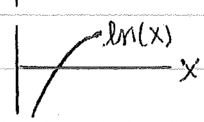
$xy_0' = C$

$y_0' = C/x$

$y_0 = C\ln(x) + D$

$y_0 \sim \begin{Bmatrix} 1 \\ \ln(x) \end{Bmatrix} \nu = 0$




**Now, find $y_1$:** (but do it only for $\nu = 0$)

$\nu = 0, \; y_0 = \{1, \ln(x)\}$

$$y_1'' + \frac{y_1'}{x} = -y_0 \Longrightarrow (xy_1')' = -xy_0$$

1st Sol'n: $(xy_1')' = -x$

$(y_0 = 1)$

→

$$\Rightarrow y_1' = \frac{-x}{2} \;, \quad y = \frac{-x^2}{4}$$

$$\to y^{(1)} \sim 1 - x^2/4 + \ldots$$

$2^{nd}$ sol'n: $(xy_1') = -\underbrace{x\ln(x)}$

$(y_0 = \ln(x))$ : $\quad \int x\ln(x)\,dx = \frac{x^2}{2}\left(\ln(x) - \frac{1}{2}\right)$

$$y_1' = \frac{-x}{2}\left(\ln(x) - \frac{1}{2}\right)$$

$$y_1 = -\frac{1}{2}\frac{x^2}{2}\left(\ln(x) - \frac{1}{2}\right) + \frac{1}{4}\frac{x^2}{2}$$

$$= \frac{-x^2}{4}\left(\ln(x) - 1\right)$$

$$\to y^{(2)} \sim \ln(x) - x^2/4\left(\ln(x) - 1\right) + \ldots$$

**This is the generation of an asymptotic solution!**

- We can now place constraints on $x$

**For self-consistency, the only constraint we've put on the system/ solution is that the terms must get successively smaller**

$$1, \ln(x) \to |x| \ll 1$$

## Airy equation asymptotics

as $x \to 0$

$$y'' = xy$$

Scale: $\frac{y}{x^2} : xy \sim 1 : x^3$

let $y = y_0 + y_1 + y_2 + \ldots$

$y_0'' = 0 \leftarrow$ there's only one other large term possible and it's smaller, so we can throw it out

$y_1'' = xy_0$

$y_2'' = xy_1$

$$\to y_0 = \{1, x\}$$

$$y_1'' = x\begin{Bmatrix} 1 \\ x \end{Bmatrix}, \quad y_1' = \begin{Bmatrix} \frac{1}{2}x^2 \\ \frac{1}{3}x^3 \end{Bmatrix}, \quad y_1 = \begin{Bmatrix} \frac{1}{6}x^3 \\ \frac{1}{12}x^4 \end{Bmatrix}$$

$$\Rightarrow y^{(1)} \sim 1 + x^3/3\cdot 2 + \ldots = A_i(x)$$

$$y^{(2)} \sim x + x^4/4\cdot 3 + \ldots = B_i(x)$$

Self-consistent so long as $|x| \ll 1$