Lecture 5 – Galilean Relativity & Hamiltonians 09/15/15

Reminder: Useful Concepts

① $S = \int \mathcal{L}(q, \dot{q}, t)\, dt$

$\delta S = 0$ with $\delta q(t_0, t_f) = 0$

$$\Rightarrow \frac{d}{dt}\underbrace{\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)} = \frac{\partial \mathcal{L}}{\partial q}$$

② $p \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}}$

Today

1. Galilean relativity and the form of the Lagrangian
2. Introduction to Hamiltonians

1. Galilean Relativity

Why choose $\mathcal{L}_{free} = \frac{1}{2}mv^2$

What is the advantage of this?

- inertial reference frames – frames move with constant velocity relative to each other

$$\vec{r}' = \vec{r} + \underbrace{\vec{V}}_{\text{relative velocity between frames}} t\ ;\ t' = t;\ \vec{v}' = \underbrace{\vec{v}}_{\text{velocity of one particle in the old frame}} + \vec{V}$$

→ Laws of mechanics are the same in all reference frames

Assumptions:

- homogeneity/isotropy of space (x)
- homogeneity of time (t)
  - a particular time is not special

} $\mathcal{L}_{free}$ a function of $|\vec{v}|$ or v^2
- no explicit dependence on x & t

Now, go to a new reference frame (+ infinitesimal relative velocity)

$\vec{v}' = \vec{v} + \vec{\alpha}$

$\vec{\alpha}$: $\vec{V}$, small

α^2 term very small → negligible

$$\mathcal{L}(v^2) \rightarrow \underbrace{\mathcal{L}(v'^2)}_{\text{Lagrangian in the new frame}} \approx \mathcal{L}(v^2 + 2\vec{v}\cdot\vec{\alpha})$$

→

$$\simeq \mathcal{L}(v^2) + \underbrace{\frac{\partial \mathcal{L}}{\partial v^2} 2\vec{v}\cdot\vec{\alpha}}$$

$\alpha \cdot \Delta\mathcal{L}$ – this shift must be by a total time derivative

$$\Delta\mathcal{L} = \underbrace{\frac{\partial \mathcal{L}}{\partial v^2}} 2\vec{\alpha}\cdot\frac{d\vec{x}}{dt}$$

If $\Delta\mathcal{L}$ must be a total time derivative, this prefactor must be a constant (no v ← depends on t)

$\Rightarrow \mathcal{L} \propto v^2$

$\mathcal{L} = \frac{1}{2}mv^2$

└ coefficient must be positive otherwise variational principle does not go through

<u>Closed system of particles</u>

$$\mathcal{L} = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2} - \underbrace{V(\vec{r}_i)}$$

embodies an "action-at-a-distance" potential (instantaneous propagation of the effects of change within the system)

Action-at-a-distance, this instantaneous spread of the effects of potentials, is necessary in Galilean relativity

→ if propagation speed was finite ($<c$), then that speed would be different in different frames of reference

The effects of potentials cannot (and do not) change with frame of reference, this would imply that the laws of motion are different for different frames (which is absolutely not true)

2. Hamiltonian Formalism

$\mathcal{L}$-formalism: n degrees of freedom (q_i)

└ $\dot{q}$ not independent here

- **produces n 2nd-order differential equations (in t)**

⇒ Must specify $2n$ initial conditions to solve

e.g. $q_i(t_0)$, $\dot{q}_i(t_0)$

Idea of $\mathcal{H}$-formalism:

- We want to replace the 2nd-order differential equations with first-order differential equations
  - ⇒ Need $2n$ 1st-order equations
    - └ $2n$ independent variables to work with

Choose:

- n variables to be q_i
- other n variables are $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ conjugate momentum
  - ↳ not a required choice, but it is the obvious one.
  - ↳ this choice makes the final equation(s) of motion symmetric between q and p

Motivation to use $\mathcal{H}$-formalism

- reveals more structure of underlying physics (e.g. symmetries, chaos)
- connection to fields beyond classical mechanics (quantum mechanics, electrodynamics, etc)
  - e.g. Poisson brackets in $\mathcal{H}$-formalism → commutators in Quantum Mechanics

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Math detour:

Legendre transform (mathematical trick)

$f(x,y)$ $\quad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

think of f as the Lagrangian

define: $g(x,y,u) = ux - f(x,y)$

$$dg = \cancel{u\,dx} + x\,du - \cancel{\frac{\partial f}{\partial x}dx} - \frac{\partial f}{\partial y}dy$$

- choose $u = \partial f / \partial x$ (causes cancellations)

g is only a function of u & y (dg only dependent on du & dy)

$$dg = \underbrace{x(u,y)}_{\text{Solve for } x(u,y) \text{ by "inverting" } u = \frac{\partial f}{\partial x}}du - \frac{\partial f}{\partial y}dy$$

$\Rightarrow g(u,y) = ux(u,y) - f[x(u,y), y]$ old function

Once you've solved for g this way, "forget" f & $u = \partial f / \partial x$

defining relation

Does the new function 'g' contain the same information as the old function 'f'?

like g — like f — like x

* $\mathcal{H}$ is the Legendre Transform of $\mathcal{L}$ with respect to $\dot{q}$

like y — time plays no role in this transform

$$\mathcal{H}(q_i, p_i, t) = \Sigma p_i \dot{q}_i - \mathcal{L}(q, \dot{q}, t)$$

independent variable in $\mathcal{H}$-form

Solve for $\dot{q}$ as a function of q, p, t using $\underbrace{p_i = \partial \mathcal{L} / \partial \dot{q}_i}_{\text{analogous to definition of } u}$

→ Physics | Equations of Motion will connect all these variables p, q, and "their dots"

On to Physics | EOM:

$\mathcal{L}(q, \dot{q}, t)$

NOT an independent variable (variation $\delta \dot{q}$ not independent of q in variational principle)

Changing to $\mathcal{H}$

$$d\mathcal{H} = \frac{\partial \mathcal{H}}{\partial p} dp + \frac{\partial \mathcal{H}}{\partial q} dq + \frac{\partial \mathcal{H}}{\partial t} dt$$

We can't discuss the physics contained in $\mathcal{H}$ without bringing back the defining relation → the Lagrangian

"bring back" $\mathcal{L}$: $\mathcal{H} = p\dot{q} - \mathcal{L}$

$$d\mathcal{H} = dp\dot{q} + \cancel{pd\dot{q}} - \frac{\partial \mathcal{L}}{\partial q} dq - \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}}}_{p} \cancel{d\dot{q}} - \frac{\partial \mathcal{L}}{\partial t} dt$$

equate the differentials:

$$\left.\begin{aligned} &① \; dp \rightarrow \dot{q} = \frac{\partial \mathcal{H}}{\partial p} \\ &② \; dq \rightarrow -\frac{\partial \mathcal{L}}{\partial q} = \frac{\partial \mathcal{H}}{\partial q} \end{aligned}\right\}$$

2n 1^{st}-order differential equations

— Hamilton's Equations —

$$-\frac{\partial \mathcal{L}}{\partial q} = \frac{-d}{dt}\underbrace{\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)}_{p} = \frac{\partial \mathcal{H}}{\partial q} \Rightarrow \dot{p} = -\frac{\partial \mathcal{H}}{\partial q}$$

③ $dt \longrightarrow -\frac{\partial \mathcal{L}}{\partial t} = \frac{\partial \mathcal{H}}{\partial t}$

- equation ① seems to be the "inverse" of the defining relation $p = \partial\mathcal{L}/\partial\dot{q}$

The Hamiltonian gives the evolution of both position (q) and conjugate momentum (p)

Now ask: Can we get Hamilton's equations directly from the variational principle instead of going through the Lagrangian?
- We suspect that we can as all laws of physics can supposedly be obtained through the variational principle (a.k.a. the principle of least action)

<u>Hamilton's Equations</u>: direct from variational principle

$S = \int dt\,(p\dot{q} - \mathcal{H})$

solve $p = \partial\mathcal{L}/\partial\dot{q}$ for $\dot{q}(q,p,t)$

Vary p and q independently (because we've "forgotten" $p = \partial\mathcal{L}/\partial\dot{q}$)
- this is <u>unlike</u> how $\delta\dot{q}$ followed from δq in the Lagrangian

$$\delta S = \int dt\left(\delta p\,\dot{q} + p\,\delta\dot{q} - \frac{\partial \mathcal{H}}{\partial q}\delta q - \frac{\partial \mathcal{H}}{\partial p}\delta p\right)$$

integration by parts, no motion at endpoints

$$= \int dt\left[\delta p\left(\dot{q} - \frac{\partial \mathcal{H}}{\partial p}\right) + \delta q\left(-\dot{p} - \frac{\partial \mathcal{H}}{\partial q}\right)\right] + p\,d\dot{q}\Big|_{t_0}^{t_f}$$

Requirements: $p\,d\dot{q}\Big|_{t_0}^{t_f} = 0$

(only) δq @ $t_0, t_f = 0$