Lecture 4 - Formations of Lagrangians. 09/10/15

Today
1. Finish up conservation laws
2. Noether's Theorem

1. Conservation

For cyclic coordinate q_j ($\partial\mathcal{L}/\partial q_j = 0$) $\Rightarrow p = \partial\mathcal{L}/\partial\dot{q}$ is conserved
canonical/conjugate momentum

Last lecture: General coordinate q_j chosen such that dq_j is a small uniform translation of the entire closed system:

$$\mathcal{L} = \sum_i \frac{1}{2} m\dot{\vec{r}}^2 - V(|\vec{r}_i - \vec{r}_j|)$$

- p is linear total momentum of all the particles in the closed system

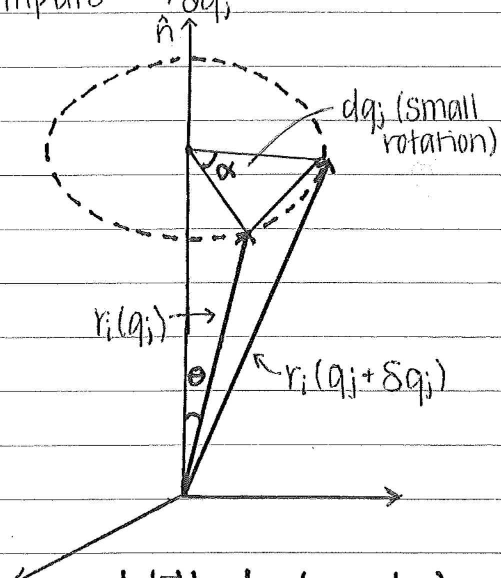
ex: What else is conserved for $\mathcal{L} = \sum_i \frac{1}{2} m\dot{\vec{r}}^2 - V(|\vec{r}_i - \vec{r}_j|)$?

Angular momentum is conserved!

(At least in this case. Whether or not it is conserved for all cases depends on the form of the potential.)

Choose q_j such that dq_j is a small rotation of the entire system about some axis

... compute $\partial\vec{r}_i/\partial q_j$



$\hat{n}$ - unit vector along axis of rotation

α - must be small

θ - can be large

$$|d\vec{r}_i| = |r_i(q_j + dq_j) - r_i(q_j)|$$
$$= r_i \sin(\theta)\, dq_j$$

magnitude of the two position vectors is nonzero

$$\left|\frac{\partial r_i}{\partial q_j}\right| = r_i \sin(\theta)$$ direction is perpendicular to both $\vec{r}_i$ and $\hat{n}$

$$\Rightarrow \frac{\partial \vec{r}_i}{\partial q_j} = \hat{n} \times \vec{r}_i$$

... compute the canonical momentum to this new coordinate, q_j:

① $$P_j = \underbrace{\frac{\partial T}{\partial \dot{q}_j}}_{U \text{ only dependent on } \vec{r}, \text{ not on } \dot{\vec{r}}} = \sum_i (m_i \vec{v}_i) \cdot \left(\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}\right)$$ time derivatives here cancel out (do not need to write)

$$= \sum_i (m_i \vec{v}_i) \cdot (\hat{n} \times \vec{r}_i)$$ angular momentum

$$= \sum_i \hat{n} \cdot [\vec{r}_i \times (m_i \vec{v}_i)] = \hat{n} \cdot \sum_i \vec{L}_i = \hat{n} \cdot \vec{L}$$ ← total orbital angular momentum along axis of rotation (direction $\hat{n}$)

② **Use $\underbrace{V(|\vec{r}_i - \vec{r}_j|)}_{\text{i.e. the fact that it's only dependent on } \vec{r}}$** assumed sum over all values

$\Rightarrow$ **q_j is cyclic**

form of the potential is unchanged (because it takes the magnitude), so that small change is cyclic

→ **Energy is conserved** ←

energy "function", $h(q, \dot{q}, t)$ ← may or may not be explicitly time dependent

for n independent variables

$$h \equiv \sum_j \left(\dot{q}_j \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \mathcal{L}\right)$$

h constant if Lagrangian is independent of time ($\partial \mathcal{L}/\partial t = 0$)

$$\frac{dh}{dt} = \sum \left[\ddot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} + \dot{q}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \dot{q}\frac{\partial \mathcal{L}}{\partial q} - \frac{\partial \mathcal{L}}{\partial \dot{q}}\ddot{q}\right]$$ last term cancels with first

$$= \sum \left\{\cancel{\ddot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}}} + \dot{q}\underbrace{\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q}\right]}_{0 \text{ by Lagrange's equation}} - \cancel{\frac{\partial \mathcal{L}}{\partial \dot{q}}\ddot{q}}\right\}$$

$dh/dt = 0$ ← constant h

Aside:

$\mathcal{H}(q, p = \partial \mathcal{L}/\partial \dot{q})$ → **a function of 2n independent variables**

Hamiltonian

For a closed system:

$$h = \sum(\dot{\vec{r}}_i \cdot m_i\dot{\vec{r}}_i - \tfrac{1}{2}m_i\dot{\vec{r}}_i^{\,2} + V)$$
$$= \sum(+\tfrac{1}{2}m_i\dot{\vec{r}}_i^{\,2} + V)$$
$$\mathcal{L} = \sum \tfrac{1}{2}m\dot{\vec{r}}_i^{\,2} - V(|\vec{r}_i - \vec{r}_j|)$$

$h = \mathcal{L}$; energy is conserved

2. Noether's Theorem

The succinctly uniform process of showing that energy and momentum are conserved.

→ Can also be used for electric charge, Baryon number, etc.

General transformation: $q_i(t) \rightarrow \underbrace{q_i'(t)}_{\text{new coordinates}} = \underbrace{q_i(t) + \overset{\text{small}}{\alpha}\underbrace{\Delta q_i(t)}_{\text{shift}}}_{\text{continuous transformation}}$

C.f. Change of Coordinates:

- $\Delta q_i(t)$ were functions of q_i & t ONLY
  - → We will not be making this assumption for generalized Noether's Theorem
- Spatial translation – Δq_i = constant
  - → Also not assuming here that $\Delta q_i(t)$ "vanishes at the endpoints (t_0 & t_f)"; c.f. variational principle where δ (vs. Δ) denoted that the change = 0 at the endpoints
- Such a transformation is symmetric ("a symmetry") if it leaves the final equations of motion unchanged
  - → i.e. the physics is unchanged. But not just the form of the EOMs, their values must be unchanged, too!
  - – as opposed to our previous change to a rotating system...

Old EOM: $m\ddot{\vec{r}} = 0$

new EOM: $\underbrace{m\ddot{\vec{r}}' = [\text{centrifugal + coriolis forces}]}_{\text{this is still Lagrange's equation}}$

final EOM changed

One possibility to satisfy these transformation conditions is to require that $\mathcal{L}$ is unchanged.

– This is possible if $\mathcal{L}$ shifts by a total time derivative

$$\underbrace{\mathcal{L} \to \mathcal{L} + \alpha \frac{dK}{dt}}_{\alpha \Delta \mathcal{L}} \;;\; \underset{\overset{\parallel}{S,\text{ action}}}{I} \to I + \alpha \int_{t_0}^{t_f} dt \frac{dK}{dt} = \alpha \underbrace{\left[K(t_f) - K(t_0)\right]}$$

some value

if change is a total time derivative, then this term = 0 and the EOM is left unaltered

Compare this $\alpha\, dK/dt$ to a shift in $\mathcal{L}(q, \dot{q}, t)$ from transformation on q's:

$$\alpha \frac{dK}{dt} = \left(\frac{\partial \mathcal{L}}{\partial q}\right)(\alpha \Delta q) + \left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)(\alpha \Delta \dot{q}) \Big\}$$

$\partial\mathcal{L}/\partial t = 0$

Change in $\mathcal{L}$ due to those in $q, \dot{q}$

indices neglected for simplicity – still a sum!

Compare this to proving linear & angular momentum conservation where we only changed q_i's!

→ Use Lagrange's equation $\partial\mathcal{L}/\partial q = \frac{d}{dt}(\partial\mathcal{L}/\partial\dot{q})$

$$= \left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)(\alpha \Delta q) + \left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)(\alpha \Delta \dot{q})$$

$$= \alpha \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \Delta q\right)$$

$$\Rightarrow \frac{d}{dt}\underbrace{\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\Delta q_i - K\right]}_{\text{constant of motion}} = 0 \Big\}$$ Noether's Theorem – simplest form

Example: **Uniform translation**

$\Delta \vec{r}_i = \hat{n}$

unit vector along the direction of translation

Lagrangian invariant → $K = 0$

- **Homogeneity of Space** (linear momentum is conserved)

$$\text{conserved quantity} = \sum_i \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}_i} \Delta \vec{r}_i = \underbrace{\sum_i m\dot{\vec{r}}_i \cdot \hat{n}}_{}$$

component of linear momentum in the direction of translation

• **Isotropy of Space** (angular momentum is conserved)

$\Delta \bar{r}_i = -(\hat{n} \times \bar{r}_i)$

• **Homogeneity of Time** (energy is conserved)

$\underbrace{t'}_{\text{new-time}} = t - \alpha \rightarrow q'(t') = q(t)$

↑ Because we are only shifting time, the [new] position at the new-time is the same as it was at the old time (translation or not)

$\Rightarrow q'(t-\alpha) = q(t)$

$\underbrace{q'(t) = q(t+\alpha)}_{\text{Changing time results in a change in coordinates}} \approx q(t) + \alpha \overset{\Delta q}{\dot{q}}$

***Conservation of energy is a change in time that results in a change in space rather than a direct change solely in space as with conservation of momentum**

→ Time translation results in a change in the Lagrangian as well

$\mathcal{L} \rightarrow \mathcal{L} + \alpha \frac{d\mathcal{L}}{dt}$ ← K, now non-zero

$\text{conserved quantity} = \sum_i \left(\frac{\partial \mathcal{L}}{\partial \dot{r}_i}\right)(\dot{\bar{r}}_i) - \mathcal{L} = K$ ← extra term here compared to isotropy & homogeneity of space

↑ conservation of energy