Lecture 3 - Structuring Lagrangians

09/08/15

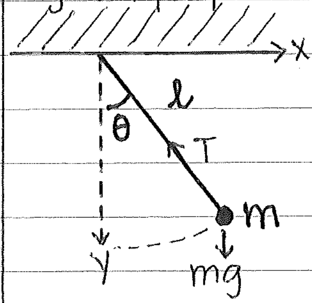
* HW 1&2 deadline extended to this Friday, 09/11, at 1700

Today

1. Finish up constraints
2. Velocity-dependent forces
3. Conservation Theorems

1. Constraints

e.g. simple pendulum



constraining "force" - constant value

$$f_\alpha = x^2 + y^2 - \ell^2$$

à la Newton:

$$m\ddot{x} = -T\,x/\ell$$

$$m\ddot{y} = mg - T\,y/\ell$$

vs. Lagrange:

Lagrange Multipliers; treat as extra variables

① $\mathcal{L}'(\text{"new"}) = \mathcal{L}(x_A, \dot{x}_A) + \lambda_\alpha f_\alpha(x_A, t)$

└ constraint

$$\Rightarrow x_A = x_A(\overbrace{\underbrace{q_1 \dots q_n}_{\text{generalized coordinates}}}^{\text{degrees of freedom}}, t)$$

indicates 3-D

$A = 1 \dots 3N$ - # of particles

$\alpha = 1 \dots (3N - n)$ - # of constraints

→ For a simple pendulum:

$$\left.\begin{aligned} x &= \ell\sin(\theta) \\ y &= \ell\cos(\theta) \end{aligned}\right\} x_{A_1} = x_{A_N}$$

Lagrange's equation for λ_α gives $\underline{f_\alpha = 0}$

consequence of constraint

Lagrange's equation for x_A gives:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_A}\right) - \frac{\partial \mathcal{L}}{\partial x_A} = \underbrace{\lambda_\alpha \frac{\partial f_\alpha}{\partial x_A}}_{\text{new term (normally nonzero)}}$$

here we can now treat x as being unconstrained — the constraints are encoded in Lagrange's equation for λ_α

$$\mathcal{L}' = \mathcal{L}(\text{free particle in the plane}) - (-mgy) + \underline{\lambda(x^2 + y^2 - \ell^2)}$$

└ new (modified) Lagrangian

this $= 0$ is the constraint on the system

Equations of motion:

$m\ddot{x} = 2\lambda x$

$m\ddot{y} = mg + 2\lambda y$

Same as Newton: $2\lambda = -T/\ell$

② "Get rid of λ"

Claim: There are equations of motion for $q_{i=1\ldots n}$ (e.g. θ) ← degrees of freedom

→ You can get this directly from the original Lagrangian with the x's written in terms of the degrees of freedom, q.

$$\mathcal{L}(q,\dot{q},t) = \mathcal{L}[x^A(q,t);\ \dot{x}^A(q,\dot{q},t)]$$

Proof:

$\mathcal{L}' = \mathcal{L} + \lambda_\alpha f_\alpha$ with a change of coordinates

$$x_A \rightarrow \begin{cases} q_i,\ i = 1\ldots n & \text{- degrees of freedom} \\ f_\alpha,\ \alpha = 1\ldots 3N-n & \text{- constraints} \end{cases}$$

└ f_α orthogonal to q_i

Lagrange's equation for q_i (only): ← θ in the case of the simple pendulum

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = \lambda_\alpha \underbrace{\frac{\partial f_\alpha}{\partial q_i}}_{=0 \text{ (result of orthogonality)}} \quad \leftarrow \text{same form} \rightarrow \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_A}\right) - \frac{\partial \mathcal{L}}{\partial x_A} = \lambda_\alpha \frac{\partial f_\alpha}{\partial x_A}$$

$$\underbrace{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0}_{\text{standard form!}}$$

For the pendulum:

$\mathcal{L}(x,y) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy$

$x = \ell\sin(\theta),\ y = \ell\cos(\theta)$

$\Rightarrow \mathcal{L}(\theta) = \frac{1}{2}\ell^2 m\dot{\theta}^2 + mg\ell\cos(\theta)$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \Rightarrow m\ell^2\ddot{\theta} + mg\ell\sin(\theta) = 0$$

You never need to introduce constraint explicitly!

2. Velocity-Dependent Forces/Potentials

V no longer just a function of x

e.g. friction $= -k\dot{x}$

opposes the direction of motion of the particle

*Lagrangian-formulation does not usually work in general for non-conservative forces

Non-conservative force: $F \neq -\frac{\partial V(x)}{\partial x}$

However, it does work for the Lorentz force

→ Re-write fields in terms of potential

$$\underbrace{\overline{B} = \overline{\nabla} \times \overline{A}}_{\text{magnetic field is the curl of the vector potential}} \; ; \; \overline{E} = \overbrace{-\overline{\nabla}\varphi}^{\text{gradient of potential}} - \partial\overline{A}/\partial t$$

Claim:

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 - \overset{\text{charge}}{e}(\varphi - \dot{\overline{r}} \cdot \overline{A}(\overline{r}, t))$$

gives the Lorentz force law as a Lagrangian equation of motion

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \frac{\partial \mathcal{L}}{\partial r}$$

$$\frac{d}{dt}(m\dot{r} + e\overline{A}) = -e\overline{\nabla}\varphi + e\overline{\nabla}(\dot{\overline{r}} \cdot \dot{\overline{A}})$$

→ generalize to x-component →

derivative only acts on A's

$$m\ddot{x} + \frac{e\,dA_x}{dt} = -e\frac{\partial \varphi}{\partial x} + e\frac{\partial}{\partial x}(v_x A_x + v_y A_y + v_z A_z)$$

velocity only a function of t (not x)

$$= -e\frac{\partial \varphi}{\partial x} + e\left(v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_y}{\partial x} + v_z \frac{\partial A_z}{\partial x}\right) \quad ②$$

velocities in x, y, and z

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \left(\underbrace{\frac{\partial A_x}{\partial x}\frac{\partial x}{\partial t}}_{} + \frac{\partial A_x}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial A_x}{\partial z}\frac{\partial z}{\partial t}\right) \quad ③$$

extra dependence of $\overline{A}$ on time because of $\overline{r}$'s dependence on time

→

$$= \frac{\partial A_x}{\partial t} + v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_x}{\partial y} + v_z \frac{\partial A_x}{\partial z}$$

└ be careful of different forms than in ②

$m\ddot{x} + e③ = ②$

$$m\ddot{x} + e\left(\frac{\partial A_x}{\partial t} + \cancel{v_x \frac{\partial A_x}{\partial x}} + v_y \frac{\partial A_x}{\partial y} + v_z \frac{\partial A_x}{\partial z}\right) = -e\frac{\partial \varphi}{\partial x} + e\left(\cancel{v_x \frac{\partial A_x}{\partial x}} + v_y \frac{\partial A_y}{\partial x} + v_z \frac{\partial A_z}{\partial x}\right)$$

$$m\ddot{x} = \underbrace{-e\frac{\partial \varphi}{\partial x} - e\frac{\partial A_x}{\partial t}}_{\text{x-component of the electric field, } +eE_x} + e\left[v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + v_z\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\right]$$

$$m\ddot{x} = +eE_x + e[v_y(\bar{\nabla}\times\bar{A})_z - v_z(\bar{\nabla}\times\bar{A})_y]$$

$$= +eE_x + e(\underbrace{v_y B_z - v_z B_y}_{(\bar{v}\times\bar{B})_x})$$

$$= +eE_x + e(\bar{v}\times\bar{B})_x$$

Generalized for $\bar{r}$:

$$m\ddot{\bar{r}} = e\bar{E} + e(\bar{v}\times\bar{B})$$

└ solve for how coordinates behave with time

3. __Conservation Theorems / Laws__

From symmetry principles — not manifest in Newton's laws

For a system of point particles, with forces from $V(q_i)$:

└ all interacting

$$\mathcal{L} = T - V$$

$$= \sum_i \frac{1}{2}m(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) - V(q_i)$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = m\dot{x}_i$$

$$= p_{ix}$$

└ linear momentum in the $\hat{x}$-direction of the i^{th} particle

Generalized momentum (canonical/conjugate):

$$p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j}$$

the dimensions here are not the same as for standard linear momentum

→

In the velocity-dependent case:

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 - e(\varphi - \dot{\bar{r}}\cdot\bar{A})$$

$$\bar{p} \equiv \partial\mathcal{L}/\partial\dot{r} = m\dot{r} + \underbrace{e\bar{A}}$$

extra component added to linear momentum because of force dependence

Back to $V(q_i)$:

Coordinate q_j is called cyclic/ignorable if $\partial\mathcal{L}/\partial q_j = 0$

($\partial\mathcal{L}/\partial\dot{q}_j$ need not $= 0$ though)

→ For cyclic coordinates, p_j is conserved

momentum is a constant of motion; $\frac{d}{dt}p_j = 0$

$$\frac{dp_i}{dt} = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_j}\right) = \frac{\partial\mathcal{L}}{\partial q_j}$$

— assuming no constraints here

$= 0$

the relationship established by Lagrange's equations holds only for the path followed by the particle

ex 1: 1 Free Particle (moving in 3D)

$$\mathcal{L} = \frac{1}{2}mv^2$$

x, y, and z are cyclic $\Rightarrow \bar{p}$ conserved

Note: for interacting particles, the $\bar{p}$ of one particle is not conserved because of the interaction potential

ex 2: Closed System of Interacting Particles

$$\mathcal{L} = \frac{1}{2}\sum m_i \dot{\bar{r}}_i^2 - V(|\bar{r}_i - \bar{r}_j|)$$

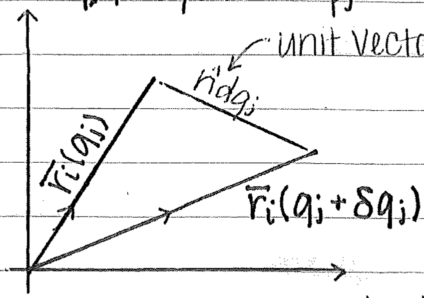
→ We know $\bar{p}_i$ will not be conserved, but we do expect $\bar{p}_{Total}$ to be conserved

Use a generalized coordinate system q_j such that δq_j corresponds to a translation of the whole system in some [small] given direction

e.g. the x-coordinate of the Center of Mass of the system

- Is q_j cyclic?
  - q_j does not appear in T
  - δq_j moves whole system, so V doesn't depend on q_j either

$\rightarrow q_j$ is cyclic $\Rightarrow p_j = \partial\mathcal{L}/\partial\dot{q}_j$ is conserved



Need to ask: How does the position of the i^{th} particle, $\bar{r}_i$, change with the system shift δq_j?

V not velocity-dependent

$$\frac{\partial r_i}{\partial q_j} = \hat{n} \; ; \quad p_j = \frac{\partial T}{\partial \dot{q}_j} = \sum_i \underbrace{(m_i \dot{\bar{r}}_i)}_{\partial T/\partial \dot{r}_i} \underbrace{\frac{\partial \dot{r}_i}{\partial \dot{q}_j}}_{\text{time derivatives cancel}}$$

$$= \sum_i (m_i \bar{v}_i) \underbrace{\frac{\partial r_i}{\partial q_j}}_{n}$$

$$= \sum_i (m_i \bar{v}_i) \cdot \bar{n}$$