Lecture 2 – Advantages of Lagrangians 09/03/15

Advantages of $\mathcal{L}$-Formulation

within mechanics:
- change of coordinates
- constraints
- conserved quantities / symmetries
- velocity-dependent forces

- "structural analogy" between fields

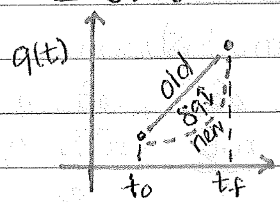
Stationary Action

$$\delta I (=0) = \int \underbrace{\left[\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\right]}_{\text{must be } 0} dt\, \delta q$$

$I = \int dt\, \mathcal{L}$

$\delta q = \text{new } q(t) - \text{old } q(t)$ (small change)

$\delta \dot{q} = (\dot{q + \delta q}) - \dot{q}$
$= (\dot{\delta q}) = \frac{d}{dt}(\delta q)$



Euler-Lagrange Equation

$$\frac{\partial \mathcal{L}}{\partial q} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}$$

– Gives the path of the particle

ex:

$\mathcal{L}(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - V(x)$

L's eq. gives: $\frac{\partial \mathcal{L}}{\partial x} = \underbrace{\frac{-dV}{dt}}_{\text{force}} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)$

$= \frac{d}{dt}(m\dot{x}) = m\ddot{x}$

$= ma$, recreating Newton's 2^{nd} Law of Motion

*In Quantum Mechanics, the particle takes many paths, which are represented by the probability [amplitude] of the particle taking each path option

Change of Coordinates

$q_i = q_i(\underbrace{S_1 \ldots S_N}_{\text{new coordinates}}, t)$ $\iff$ $S_I = S_I(q_1 \ldots q_n, t)$

old coordinates

a good coordinate change is invertible

function of S_I

① $\dot{q}_i = \frac{d}{dt} q_i = \frac{\partial q_i}{\partial S_I} \dot{S}_I + \frac{\partial q_i}{\partial t} \Rightarrow \dot{S}_I = \frac{\partial S_I}{\partial q_i} \dot{q}_i + \frac{\partial S_I}{\partial t}$

$\mathcal{L}(q,\dot{q})$ re-expressed in terms of s and $\dot{s}$:

$$\frac{\partial \mathcal{L}}{\partial s_I} = \overbrace{\frac{\partial \mathcal{L}}{\partial q_i}}^{f(s)}\,\overbrace{\frac{\partial q_i}{\partial s_I}}^{f(q)} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial s_I} \leftarrow \text{use ①}$$

$$= \left(\frac{\partial \mathcal{L}}{\partial q_i}\right)\left(\frac{\partial q_i}{\partial s_I}\right) + \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\left[\frac{\partial^2 q_i}{\partial s_I \partial s_J}\dot{s}_J + \frac{\partial^2 q_i}{\partial t\,\partial s_I}\right] \text{ ②}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{s}_I}\right) = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \dot{s}_I}\right) = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\underbrace{\left(\frac{\partial q_i}{\partial s_I}\right)}_{③}$$

$$= \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\right]\frac{\partial q_i}{\partial s_I} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\left[\frac{\partial^2 q_i}{\partial s_I \partial s_J}\frac{\partial s_J}{\partial t} + \frac{\partial^2 q_i}{\partial s_I \partial t}\right] \text{ ④}$$

Subtract ② from ④

$$\underbrace{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{s}_I}\right)}_{④} - \underbrace{\frac{\partial \mathcal{L}}{\partial s_I}}_{②} \underset{\text{cancel out}}{\overset{\text{second derivatives}}{=}} \underbrace{\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i}\right]}_{=0\text{; just Lagrange's equation in old coordinate system}} \overbrace{\frac{\partial q_i}{\partial s_I}}^{\text{change of coordinates factor}}$$

ex:

$\mathcal{L} = \frac{1}{2}m\dot{r}^2 \rightarrow$ Go to a new rotating frame about the z-axis with an angular frequency ω (non-inertial)

*The form of Lagrange's equation will not change in the new non-inertial reference frame

$$\text{new system}\begin{cases} z' = z \\ x' = x\cos(\omega t) + y\sin(\omega t) \\ y' = y\cos(\omega t) - x\sin(\omega t) \end{cases}$$

$\mathcal{L}$ in terms of $\vec{r}\,' = \frac{1}{2}m[(\dot{x}' - \omega y')^2 + (\dot{y} + \omega x')^2 + \dot{z}^2]$

$$= \frac{1}{2}m[\dot{\vec{r}}\,' + (\vec{\omega} \times \vec{r}\,')]^2$$

$$\frac{\partial \mathcal{L}}{\partial r'} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}'}$$

$$m[\dot{r}' \times \omega - \omega \times (\omega \times r')] = m(\ddot{r}' + \omega \times \dot{r}')$$

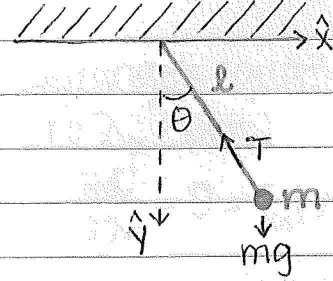
In old coordinates: $m\ddot{r} = 0$ -- free particle

In new coordinates: $m\ddot{r}' = \underbrace{-m\cdot\omega \times (\omega \times r')}_{\text{centrifugal force}} - \underbrace{2m\omega \times \dot{r}'}_{\text{Coriolis force}}$

"Fictitious" forces – the modifiers to Newton's Law – arise from reference frame change.

Constraints

ex: (simple) Pendulum



X and y are not independent, so there is only one degree of freedom in this system – θ

$x^2 + y^2 = \ell^2$

$x = \ell \sin(\theta)$

$y = \ell \cos(\theta)$

Newton's Equations:

$m\ddot{x} = -(T)(x/\ell)$

$m\ddot{y} = mg - T(y/\ell)$

$\ddot{\theta} = -(g/\ell)\sin(\theta)$

$T = m\ell\dot{\theta}^2 + mg\cos(\theta)$

We want to treat the constraints on the same footing as the coordinates (the constraint is the relationship between coordinates)

holonomic constraint:

(holonomic – when controllable degrees of freedom are equal to the total degrees of freedom)

$$f_\alpha(\underset{1 \ldots 3N}{x_A}, t) = 0, \quad \overset{\text{number of constraints}}{\alpha} = 1 \ldots (\underset{\text{3-D}}{3}\underset{\text{number of coordinates}}{N} - \underset{\text{\# degrees of freedom}}{n})$$

α = # of relations between x and t $\Rightarrow$ # of constraints

#coords.		#constr.		#deg. of freedom
2	−	1	=	1 ← θ, for the example above
3N		α		n

$$X_A = X_A(\underbrace{q_1 \ldots q_n}_{\#q\text{s} = \#\text{d.o.f.}})$$

→ need new variable(s) →

Introduce λ_α, Lagrange Multipliers

└ introduce as many new variables as you have constraints

$\mathcal{L}' = \mathcal{L}(x^A, \dot{x}^A) + \lambda_\alpha f_\alpha(x^A, t)$

└ new └ treat λ_α just like additional coordinates

With these λ_α, x^A taken as <u>unconstrained</u>

Lagrange's equation for λ_α:

(no $\dot{\lambda}_\alpha$ exists)

$$\frac{\partial \mathcal{L}'}{\partial \lambda_\alpha} = \underbrace{0 = f_\alpha(x^A, t)}_{\text{enforces constraint}}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^A}\right) - \frac{\partial \mathcal{L}}{\partial x^A} = \lambda_\alpha \frac{\partial f_\alpha}{\partial x^A}$$