11 / 05 / 15

Lecture 19 - Time Evolution of Orbits

1. Kepler's 3rd Law
2. r, θ as Functions of t
3. Constants of Motion
4. Scattering

1. Kepler's 3rd Law of Motion

Areal Velocity, $\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{\ell}{2m}$ constant

$\Rightarrow$ area of orbit, $A = \frac{\ell \tau}{2m} = \pi a b$ semimajor axis, semiminor axis

$= \pi a^2 \sqrt{1-e^2}$

for the Kepler case:

$e^2 = 1 + \frac{2E\ell^2}{mk^2}$, $a = \frac{-k}{2E}$

$\rightarrow$ τ = orbital period $= 2\pi a^{3/2} \sqrt{\frac{m}{k}}$

3rd Law:

- The square of the orbital period is proportional to the cube of the semimajor axis of the orbit.

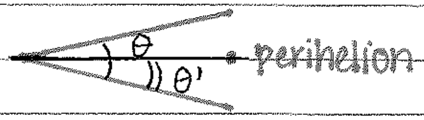
Reference: Orbit Equation $r(\theta)$

- derived from equation of motion

$$\frac{1}{r} = \frac{mk}{\ell^2}\left(1 + \underbrace{\sqrt{1 + \frac{2E\ell^2}{mk^2}}}_{e \text{ - orbital eccentricity}} \cos(\theta - \theta')\right)$$

using $a = -k/2E$

$$r(\theta) = \frac{a(1-e^2)}{1 + e\cos(\theta - \theta')}$$



2. r, θ as Functions of t

Formal solution:

$$(1)\ t = \sqrt{\frac{m}{2}} \int_{r_0}^{r} dr \left[\frac{k}{r} - \frac{\ell^2}{2mr^2} + E\right]^{-1/2}$$

↳ invert to get r(t)

(2) $mr^2\dot{\theta} = l$

$$\Rightarrow dt = \frac{mr^2}{l}\,d\theta + r(\theta)$$ just the orbit equation

$$t = \frac{l^3}{mk^2}\int_{\theta_0}^{\theta}\frac{d\theta}{[1+e\cos(\theta-\theta')]^2}$$

↳ invert to get θ(t)

Orbital cases:

① e=1, parabola

θ'=0 here

$$1+(e=1)\cos(\theta) = 2\cos^2(\theta/2)$$

→ plugging back into t(θ) →

$$t = \frac{l^3}{4mk^2}\int_0^{\theta}\sec^4\left(\frac{\theta}{2}\right)d\theta$$

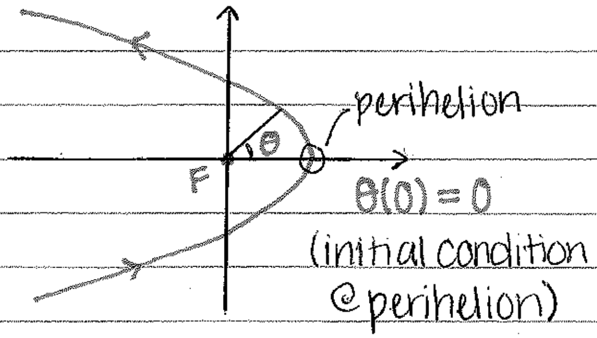
change of variables:

let $x = \tan(\theta/2)$

$$\rightarrow t = \frac{l^3}{2mk^2}\int_0^{\tan(\theta/2)}(1+x^2)\,dx$$

$$= \frac{l^3}{2mk^2}\left[\tan\left(\frac{\theta}{2}\right) + \underbrace{\frac{1}{3}\tan^3\left(\frac{\theta}{2}\right)}\right]$$

requires us to solve a cubic equation for θ(t)



Choose θ=0 to be the perihelion
→θ'=0 (to maximize r)

$-\pi < \theta < \pi$ ($\theta\to\pi$: $t\to+\infty$; $\theta\to-\pi$: $t\to-\infty$)

② 0<e<1, ellipse

∫dr will be a little bit easier than ∫dθ

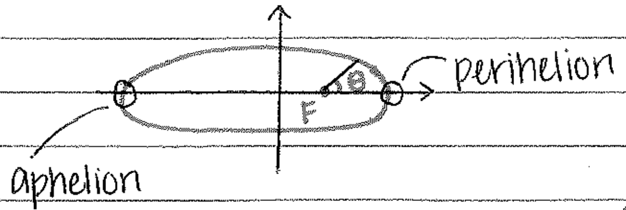
$$r = a(1-e\cos(\psi)) = \frac{a(1-e^2)}{1+e\cos(\theta)}$$

↳ ψ ≡ eccentric anomaly

* One complete cycle in θ = one complete cycle in Ψ

$\theta = 0 \rightarrow r = a(1-e) \rightarrow$ perihelion

$\theta = \pi \rightarrow r = a(1+e) \rightarrow$ aphelion



$$t = \sqrt{\frac{m}{2}} \int_{r_0}^{r} dr \left[\frac{k}{r} - \frac{\ell^2}{2mr^2} + E\right]^{-1/2}$$

$\rightarrow$ Replace ℓ and E with expressions for the semimajor axis $\rightarrow$

$E = -k/2a$, $\ell^2 = mka(1-e^2)$

$$t = \sqrt{\frac{m}{2k}} \int_{\boxed{r_0}}^{r} r\,dr \left[r - \frac{r^2}{2a} - \frac{a}{2}(1-e^2)\right]^{-1/2}$$

└ initial position @ perihelion (assumed)

$\rightarrow$ Use eccentric anomaly expression $r(\Psi) \rightarrow$

$$t = \sqrt{\frac{ma^3}{k}} \int_0^{\Psi} (1 - e\cos(\Psi))\,d\Psi$$

$$\tau = \int_0^{\boxed{2\pi}} \ldots d\Psi$$

special case for one full period/ one complete orbit

└ orbital period

$$= 2\pi a^{3/2} \sqrt{\frac{m}{k}}$$, Kepler's 3rd law recovered

- Want to find $\Psi(t)$ for any Ψ (not just the full period case)

$\rightarrow$ plug into $r(\Psi)$ to yield $r(t)$

Introduce orbital frequency:

$$\omega = \frac{2\pi}{\tau} = \sqrt{\frac{k}{ma^3}}$$

$\omega t = \Psi - e\sin(\Psi) \Rightarrow$ Kepler's Equation

(a transcendental equation)

By comparing equations for $r(\theta)$ and $r(\Psi)$:

$$\theta(\Psi) \rightarrow \tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+e}{1-e}} \tan\left(\frac{\Psi}{2}\right)$$

3. Constants of Motion

- $\ell, \theta', E, r, \theta(t_0)$, 4 total constants of motion
- 3 degrees of freedom (x,y,z) or (r,θ,φ), etc.

$\Rightarrow$ We expect 6 constants of motion
(formed out of initial position and velocity)

Laplace-Runge-Lenz vector is also a constant of motion!

$$\bar{A} = \bar{p} \times \bar{L} - mk\overbrace{\frac{\bar{r}}{r}}^{\hat{r}}$$

-Recall that previously we showed $\bar{A}$ to be a constant of motion using poisson brackets

$\{\mathcal{H}, A\} = 0$

↳ Show it now using equations of motion (i.e. $d\bar{A}/dt = 0$)

For a general central force, $f(r)$:

$$\dot{\bar{p}} = f(r)\frac{\bar{r}}{r} \leftarrow \text{Newton's 2nd law}$$

$$\dot{\bar{p}} \times \bar{L} = \frac{mf(r)}{r}\left[\bar{r} \times (\bar{r} \times \dot{\bar{r}})\right]$$

$$= \frac{mf(r)}{r}\left[\bar{r}\underbrace{(\bar{r} \cdot \dot{\bar{r}})}_{\frac{1}{2}\frac{d}{dt}(\bar{r} \cdot \bar{r}) = r\dot{r}} - r^2\dot{\bar{r}}\right]$$

→ because $\vec{L}$ is a constant, we can write →

$$\frac{d}{dt}(\bar{p} \times \bar{L}) = -mf(r)r^2\left(\frac{\dot{\bar{r}}}{r} - \frac{\bar{r}\dot{r}}{r^2}\right)$$

$$= -mf(r)r^2\frac{d}{dt}\left(\frac{\bar{r}}{r}\right)$$

specify $f(r) = -k/r^2$; $V = -k/r$

$$\frac{d}{dt}(\bar{p} \times \bar{L}) = \frac{d}{dt}\left(\frac{mk\bar{r}}{r}\right)$$

$\Rightarrow$ For the Kepler problem there is a conserved vector $\bar{A}$

$$\bar{A} = \bar{p} \times \bar{L} - mk\frac{\bar{r}}{r}$$

↳ 3 conserved vectors at different positions in the orbit

- Now have $\bar{L}, \bar{E}, \bar{A}$... 7 constants total

↳ One too many!

In this case, it must be that not all of the constants are independent

relation:

$\bar{A}\cdot\bar{L}=0$, $\bar{L}=\bar{r}\times\bar{p}$ } By the orthogonality of $\bar{A}$ to $\bar{L}$, $\bar{A}$ must be a fixed vector in the plane of the orbit

($\bar{L}$ is perpendicular to the orbital plane)

1 relation → reduced to [the expected] 6 constants of motion

BUT! All of these constants are "global", that is they all provide information on the overall orbital properties

(none encode information about the initial position of the particle on the orbit -- $r(t_0)$ or $\theta(t_0)$)

→ One constant of motion must encode information about the initial position - brings us back to 7 constants

Need one more relation!

$\bar{A}$ = constant ⇒ the orbit is a conic section

→ The conic equation as a function of $\bar{A}$ removes the extra constant and encodes information about the initial position

define:

$\theta \equiv$ the angle between $\bar{r}$ & $\bar{A}$ ← Where $\bar{A}$ is fixed/constant in both direction and magnitude

$$\bar{A}\cdot\bar{r} = Ar\cos(\theta) = \underbrace{\bar{r}\cdot(\bar{p}\times\bar{L})} - mkr$$

permutation of the terms of the triple dot product:

$$\bar{r}\cdot(\bar{p}\times\bar{L}) = \bar{L}\cdot(\bar{r}\times\bar{p}) = \ell^2$$

$$Ar\cos(\theta) = \ell^2 - mkr$$

$$\Rightarrow \frac{1}{r} = \frac{mk}{\ell^2}\left(1+\frac{A}{mk}\cos(\theta)\right) \quad (1)$$

← confirmed: orbit is a conic section

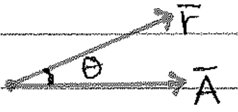
Where the earlier $r(\theta)$ found using EOM was:

$$\frac{1}{r} = \frac{mk}{\ell^2}\left(1+\sqrt{1+\frac{2E\ell^2}{mk^2}}\cos(\theta-\theta')\right) \quad (2)$$

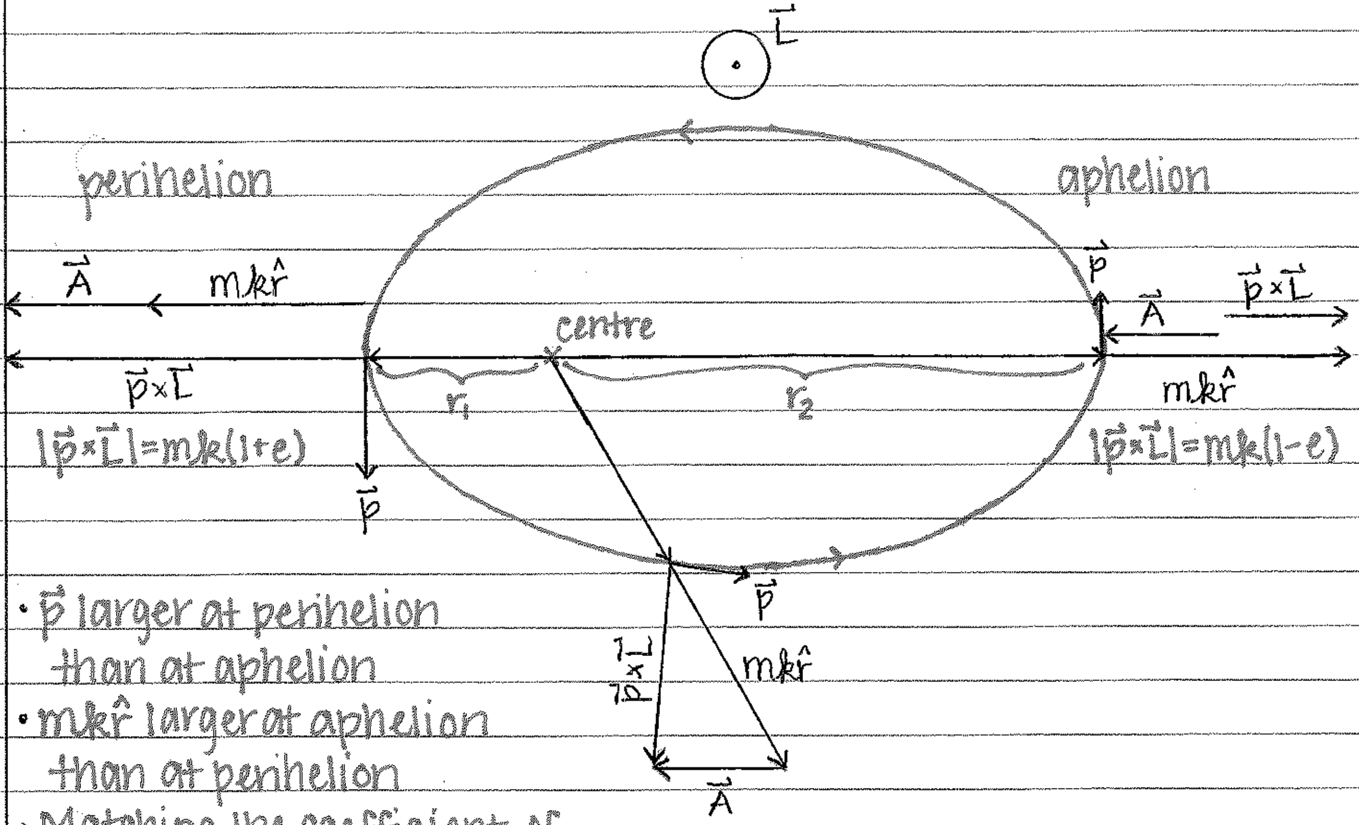
where θ here is with reference to the perihelion

→ Match (1) & (2) →

$\bar{A}$ must point from centre to perihelion for this match to be true!



Visualizing the vectors $\vec{p}$, $\vec{L}$, and $\vec{A}$ in a Keplerian orbit:



- $\vec{p}$ larger at perihelion than at aphelion
- $mk\hat{r}$ larger at aphelion than at perihelion
- Matching the coefficient of $\cos(\theta)$, $\vec{A}$ always points in the same direction with magnitude

  $A = mke$

→ expressing the eccentricity in terms of E and ℓ →

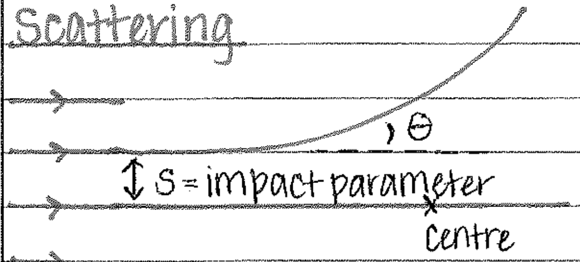
$A^2 = m^2k^2 + 2mE\ell^2$

└ of the seven constants we had before, only five are independent

We now have:

- 5 global constants
- 1 constant encoded with initial conditions (e.g. time @ perihelion, etc.)

4. Scattering



Repulsive force ($f \to 0$ as $r \to \infty$)

→ Only focus on θ (ignore φ) due to azimuthal symmetry