11/03/15

Lecture 18 – Keplerian Motion

1. Finishing Central Force
2. Kepler

1. Finalizing Central Force

Formal Solution:

$$t = \int_{r_0}^{r} dr' \left[\frac{2}{m}\left(E - V - \frac{\ell^2}{2mr^2}\right)\right]^{-1} \Rightarrow t(r)$$

$t=0$

$$mr^2\dot{\theta} = \ell \Rightarrow \theta = \theta_0 + \frac{\ell}{m}\int_0^t \frac{dt'}{r^2(t')}$$ angular momentum is conserved

· Equivalent 1D potential

$$V' = V + \frac{\ell^2}{2mr^2}$$

for $V = -k/r$

(1) & (2) $E \geq 0 \Rightarrow$ unbounded

(3) $E < 0$ (but $> V'_{min}$) $\Rightarrow$ bounded

(4) $E = V'_{min} = \frac{-mk^2}{2\ell^2} \Rightarrow$ circle

for the circular orbit (fixed radius)

$$T = -\frac{1}{2}V \Rightarrow E = \frac{V}{2}$$

E: total energy

→ generalize (for bounded)

- usually T & V change as the particle goes around the ellipse
- for a circle, they are unvarying

(T+V) is constant at all points in the orbit ⇒ want the time-averaged T-related-V

Virial Theorem

(proof mostly follows GPS)

m_i, r_i, F_i for a many-particle system

$\hookrightarrow \dot{p}_i = F_i$

define $G \equiv \sum \bar{p}_i \cdot \bar{r}_i$

$$\underbrace{\frac{dG}{dt}}_{\text{units of Energy}} = \sum(\dot{\vec{r}}_i \cdot \vec{p} + \dot{\vec{p}} \cdot \vec{r})$$

$$= \sum(\underbrace{m\dot{r}}_{p}\dot{r} + \underbrace{F \cdot r}_{\dot{p}})$$

$$= 2T + \sum F \cdot r$$

↑ total KE of all particles in the system

Taking the time-average

$$\frac{1}{\tau}\int_0^{\tau} \frac{dG}{dt}\,dt = \frac{G(\tau) - G(0)}{\tau}$$

initial condition

time-interval, τ

$$= 2\overline{T} + \overline{\sum F \cdot r}$$

← bar indicates time-averaged value

If the motion is periodic:

τ = period $\Rightarrow G(\tau) = G(0)$

↳ or p, r are finite/bounded

$\Rightarrow G < C$ (i.e. G also has some upper bound for periodic motion)

- Choose $\tau \to \infty$

$$\frac{G(\tau) - G(0)}{\tau} \to 0$$

For both periodic & non-periodic (but still bounded) motion:

$$\overline{T} = -\frac{1}{2}\sum_{i=1}^{n} \overline{\vec{F}_i \cdot \vec{r}_i}$$

$$= +\frac{1}{2}\sum_{i=1}^{n} \overline{\vec{\nabla} V \cdot \vec{r}_i}$$

General form of the Virial Theorem

→ Effective 1-body problem

$V = ar^{n+1}$ (force $\propto r^n$)

$$\Rightarrow \overline{T} = \frac{1}{2}\overline{\frac{\partial V}{\partial t} r} = \underbrace{\frac{1}{2}(n+1)}_{\text{special case: } n=-2}\overline{V}$$

$n=-2$ is the Kepler case!

Orbit Equation: $r(\theta)$

Yields the shape of the orbit

$$mr^2\dot{\theta} = l \Rightarrow l\,dt = mr^2 d\theta$$

$$t(r) \rightarrow \frac{dr}{dt} = \sqrt{\frac{2}{m}\left(E - V - \frac{\ell^2}{2mr^2}\right)}$$

Want to eliminate dt

$$d\theta = \ell\, dr \left[mr^2 \sqrt{\frac{2}{m}\left(E - V - \frac{\ell^2}{2mr^2}\right)} \right]^{-1}$$

$$\Longrightarrow \theta = \theta_0 + \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\ldots}}$$ easier equation to solve than $r(t)$

after integral, invert to get $r(\theta)$

2. Details of Keplerian Motion ($V = -k/r$)

$$\theta = \theta' - \int du \left[\frac{2mE}{\ell^2} + \frac{2mku}{\ell^2} - u^2 \right]^{-1/2}$$

θ': constant of integration; $u \equiv 1/r$

Standard form for this indefinite integral:

$$\int \frac{dx}{\sqrt{\alpha + \beta x + \gamma x^2}} = \frac{1}{\sqrt{-\gamma}} \arccos\left(-\frac{\beta + 2\gamma x}{\sqrt{q}} \right)$$

$q = \beta^2 - 4\alpha\gamma$, the discriminant

for our case:

$$\alpha = \frac{2mE}{\ell^2}, \quad \beta = \frac{2mk}{\ell^2}, \quad \gamma = -1$$

$$\longrightarrow q = \left(\frac{2mk}{\ell^2}\right)^2 \left(1 + \frac{2E\ell^2}{mk^2}\right)$$

→ Plugging into our equation for θ →

$$\theta = \theta' - \arccos\left[\left(\frac{\ell^2 u}{mk} - 1\right) \left(1 + \frac{2E\ell^2}{mk^2}\right)^{-1/2} \right]$$

→ invert for $r(\theta)$ →

$$\frac{1}{r} = \frac{mk}{\ell^2} \left[1 + \cos(\theta - \theta') \sqrt{1 + \frac{2E\ell^2}{mk^2}} \right]$$

for bounded orbits, θ' corresponds to the turning angle (min. r @ $\theta = \theta'$)

3 $\int$ constants: ℓ, θ', E

But we expected 4! (two 2nd-order ODEs)

→

- 4th is the initial position ($@\,t=0$) of the particle on the orbit

but there is no reference to time in our equation

$\rightarrow$ Only need 4th constant to calculate $r(t)$, $\theta(t)$

The orbit is a conic section with its focus @ origin (center of force)

$$\frac{1}{r} = C\left[1 + \underbrace{e}_{\text{eccentricity }(=0\text{ for a circle})}\cos(\theta-\theta')\right] \text{, general conic}$$

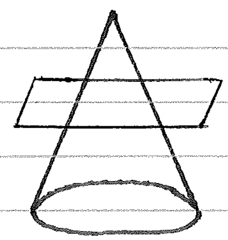
$$e = \sqrt{1+\frac{2E\ell^2}{mk^2}}$$ (found by comparing with Kepler case)

Orbital cases:

① $e=0$, $\frac{1}{r}=$ constant

$$\Rightarrow E_0 = \frac{-mk^2}{2\ell^2}$$

matches result found from using the effective 1D potential

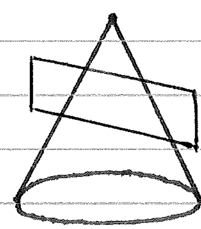


cut parallel to the conic base

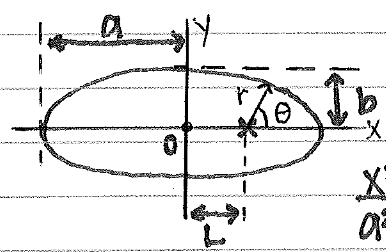
② $E<0$ (but $>E_0$), $e<1$

$\rightarrow$ ellipse

Does our above equation match "usual" ellipses?



cut at an angle to the conic base



a – semimajor axis

b – semiminor axis

$$e = \sqrt{1-\left(\frac{b}{a}\right)^2} = \frac{L}{a}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ (centre @ origin)}$$

distance from centre to focus (x): the center of force

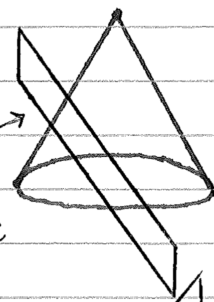
$L = \sqrt{a^2+b^2}$

aka linear eccentricity

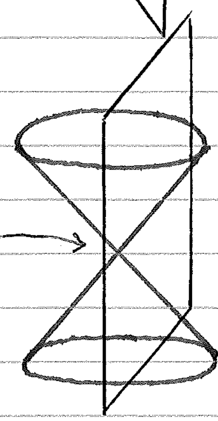
- Shift centre to focus $\rightarrow$ you will recover our new equation

$\rightarrow$

③ e=1, parabola
→ E=0



④ e>1, hyperbola
→ E>0



<u>Elliptical Orbits: Kepler's 3rd Law</u>

Kepler's Laws:

1st – **Planets move in elliptical orbits**

2nd – **Areal velocity is conserved** (the planet sweeps out equal areas in its orbit in equal times)

3rd – **The square of the orbital period is proportional to the cube of the semimajor axis of the orbit**

$$\frac{dA}{dt} \equiv \frac{1}{2}r^2\dot{\theta} = \frac{l}{2m}$$ – constant; conservation of angular momentum

definition of areal velocity

orbital period

semi-major and -minor axes

$$A = \int_0^{\tau} \frac{dA}{dt}\,dt = \frac{l\tau}{2m} = \pi \overbrace{ab}$$

→ Orbit is an ellipse (1^{st} law or $V = -\frac{1}{r}$, central)

$$= \pi a^2 \sqrt{1-e^2}$$

$$\tau = 2\pi a^{3/2}\sqrt{\frac{m}{k}}$$ ⇒ **3rd law: square of the period ∝ cube of the semimajor axis**

k also dependent on mass of planet

$$k = G m_p m_s$$

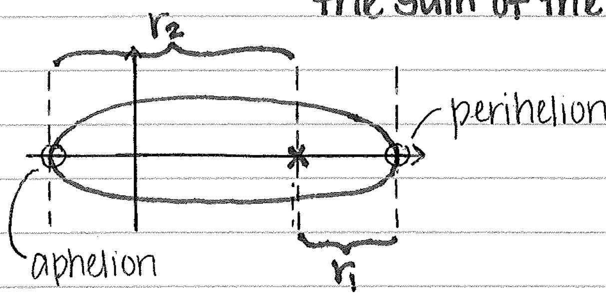
where $m = \mu = \underbrace{\frac{m_p m_s}{m_p + m_s}}_{\text{reduced mass, planet-sun system}} \approx m_p \quad (m_p \ll m_s)$

(I) $$\tau = \frac{2\pi a^{3/2}}{\sqrt{G(m_p + m_s)}}$$

$$\approx \frac{2\pi a^{3/2}}{\sqrt{G m_s}}$$ Independent of planet! Only dependent on orbital axis and solar mass

Claim:

(II) $a = \frac{1}{2}(r_1 + r_2) \rightarrow$ The semimajor axis is proportional to the sum of the apsidal distances



The apsidal distances are the two turning points with respect to the focus (perihelion & aphelion)

At these turning points, by definition

$\dot{r} = 0$ (the only Kinetic Energy is angular)

$$E_{TP} = \frac{k}{r} + \frac{l^2}{2mr^2} \quad (= V')$$

$$r^2 + \underbrace{\frac{k}{E}}_{-(\text{sum of the roots})} r - \frac{l^2}{2mE} = 0$$

$$a = \frac{1}{2}(r_1 + r_2) = \frac{-k}{2E}$$ a is only dependent on energy, not orbital angular momentum, etc.

Rewrite equation for $\frac{1}{r}$:

(using $a = -k/2E$)

$$r = \frac{a(1-e^2)}{1 + e\cos(\theta - \theta')}$$

turning points:

$(\theta - \theta') = 0$ or π

$\rightarrow r = a(1 \mp e)$

Velocity Vector

$$\vec{v} = \underset{\dot{r}}{v_r}\,\hat{r} + \underset{r\dot{\theta} = l/mr}{v_\theta}\,\hat{\theta}$$

$$v_r = \frac{e v_\theta \sin(\theta - \theta')}{(1-e^2)} = 0 \text{ @ turning points}$$