Euler's Equation for Rigid Body Rotation

Euler's equations with zero torque

→ look at the case $\vec{\Gamma}=0$

$$\dot{\omega}_1\lambda_1 = \omega_2\omega_3(\lambda_2-\lambda_3)$$
$$\dot{\omega}_2\lambda_2 = \omega_1\omega_3(\lambda_3-\lambda_1)$$
$$\dot{\omega}_3\lambda_3 = \omega_1\omega_2(\lambda_1-\lambda_2)$$

Now suppose the object has 3 different principle momenta ($\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$)

1) If the object starts at $t=0$ rotating about one of the principle axes, it remains there

$\omega_3 > 0$ about $\hat{e}_3$, with $\omega_1=\omega_2=0$ at $t=0$

The right-hand-side of all equations are zero here, $\dot{\omega}_i=0$, $i=1,2,3$

Also, $\vec{L}=\lambda_3\vec{\omega}$ will be constant

this will also appear to be constant in the space frame.

It will persist indefinitely.

2) Suppose at $t=0$ $\vec{\omega}$ is not along a principle axis

then $\vec{\omega}$ is not constant

$\vec{\omega}(t=0)$ will have at least two components

say $\omega_1, \omega_2 \Rightarrow \vec{\omega}=\omega_1\hat{e}_1+\omega_2\hat{e}_2$

Then $\dot{\omega}_3\lambda_3=\omega_1\omega_2(\lambda_1-\lambda_2)$ and ω_3 starts to change

→ this leads to a wobbling of the motion (precession)

3) Is the rotation about a principle axis stable?

Start an object rotating about $\hat{e}_3$ with $\omega_1=\omega_2=0$.

Now give it a kick. Will ω_1 and ω_2 grow?

$\omega_1=\omega_2=$ small, such that $\lambda_3\dot{\omega}_3=(\lambda_1-\lambda_2)\omega_1\omega_2 \simeq 0$

$\Rightarrow$ thus $\omega_3 \simeq$ constant

The other equations become:

$$\lambda_1\dot{\omega}_1=[(\lambda_2-\lambda_3)\omega_3]\omega_2$$
$$\lambda_2\dot{\omega}_2=\underbrace{[(\lambda_3-\lambda_1)\omega_3]}_{\text{constant}}\omega_1$$

coupled first-order differential equations

Differentiate the first and substitute the second:

$$\lambda_1\ddot{\omega}_1=\left[(\lambda_2-\lambda_3)\omega_3\right]\dot{\omega}_2=\left[\frac{(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)}{\lambda_2}\omega_3^2\right]\omega_1$$

Yielding:

$$\ddot{\omega}_1=-\left[\frac{(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)}{\lambda_1\lambda_2}\omega_3^2\right]\omega_1$$

If the value in the square brackets is positive, then ω_1 oscillates in time and so does ω_2.

→ This is stable provided that

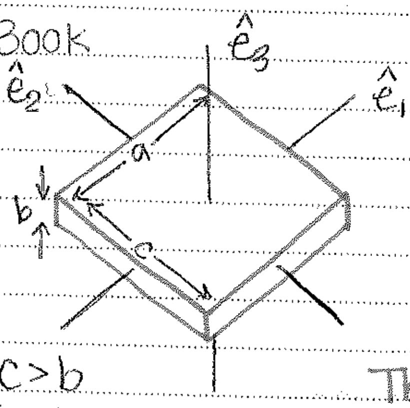
$$\left.\begin{matrix} \lambda_3 > \lambda_1, \lambda_2 \\ \text{or } \lambda_3 < \lambda_1, \lambda_2 \end{matrix}\right\} \text{If } \lambda_3 \text{ is the largest or smallest MOI, then the motion is stable}$$

But not if

$$\left.\begin{matrix} \lambda_1 < \lambda_3 < \lambda_2 \\ \text{or } \lambda_2 < \lambda_3 < \lambda_1 \end{matrix}\right\} \text{If } \lambda_3 \text{ is intermediate, then it is unstable}$$

Thus, if the object rotates around the axis with the <u>largest</u> or the <u>smallest</u> moment of inertia, it is stable.

Ex: Book



$a > c > b$

$\lambda_3 = \frac{m}{12}(a^2 + c^2)$, $\lambda_3 > \lambda_1, \lambda_2$
└ largest moment of inertia

$\lambda_1 = \frac{m}{12}(b^2 + c^2)$, $\lambda_1 < \lambda_2, \lambda_3$
└ smallest moment of inertia

$\lambda_2 = \frac{m}{12}(a^2 + b^2)$, $\lambda_1 < \lambda_2 < \lambda_3$
└ intermediate moment of inerta

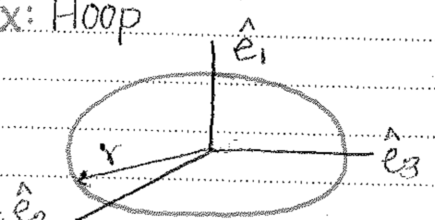
The motion around $\hat{e}_2$ is unstable!

→ For the unstable direction:

$\ddot{\omega}_1 = +k^2\omega_1 \Rightarrow \omega_1(t) = \omega_1 e^{kt}$ $k > 0$

The other directions pick up angular velocity and the object rotates wildly.

Ex: Hoop



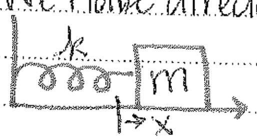
$\lambda_1 = mr^2$
$\lambda_2 = \frac{1}{2}mr^2$
$\lambda_3 = \frac{1}{2}mr^2$
Start along $\hat{e}_3$

$\lambda_3 - \lambda_2 = \frac{1}{2}mr^2 - \frac{1}{2}mr^2 = 0$
$\lambda_3 - \lambda_1 = \frac{1}{2}mr^2 - mr^2 = -\frac{1}{2}mr^2$

→ Can't analyze with this formalism.

Coupled Oscillators

We have already discussed SHM for a mass on a spring



$T = \frac{1}{2}m\dot{x}^2$, $U = \frac{1}{2}kx^2$
$\ddot{x} = -k/m\, x$, $\omega^2 = k/m$
$x(t) = \mathrm{Re}\{z(t)\}$, $z(t) = Ce^{i\omega t} = Ae^{i(\omega t - \delta)}$

Now life gets interesting when you have two or more coupled oscillators.