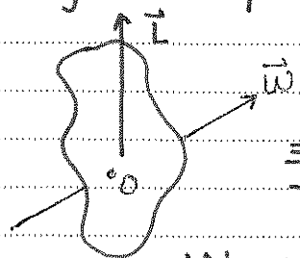
04/15/13

Rigid Body Rotation, con't



$\vec{L} = \overline{\overline{I}}\,\vec{\omega}$

↑ Inertia tensor

$$\overline{\overline{I}} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$ ← a symmetric matrix $I_{ij} = I_{ji}$

Want to find $\hat{e}_1, \hat{e}_2, \hat{e}_3$ such that

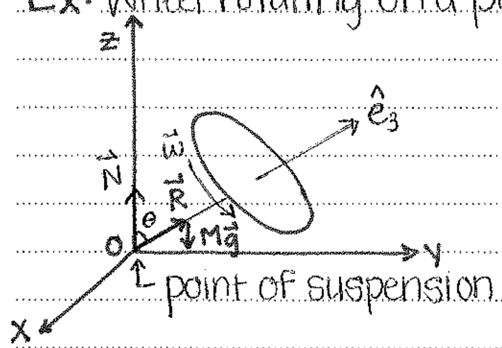
$$\overline{\overline{I}} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$ — a diagonalized matrix

└ here demands $\vec{L} \parallel \vec{\omega}$

λ – principle moments of inertia

$\hat{e}$ – principle axes of rotation

Ex: Wheel rotating on a point of suspension (Gyroscope/Top)



$\vec{L} = \lambda_3 \omega \hat{e}_3$

$\dot{\vec{L}} = \vec{\Gamma}^{ext}_{net} = \vec{R} \times M\vec{g}$

↱ change in $\hat{e}_3$ → precession

$\dot{\vec{L}} = \lambda_3 \omega \dot{\hat{e}}_3$

└ $|\vec{L}|$ is fixed

$= R\hat{e}_3 \times M(-g\hat{z})$

$= MgR\,\hat{z} \times \hat{e}_3$

Forces on the top:

1) $M\vec{g}$

2) $\vec{N}$ ($\vec{\Gamma}_N = 0$)

$\vec{v} = \vec{\omega} \times \vec{r}$

$\dot{\vec{r}} = \vec{\omega} \times \vec{r}$

$$\Rightarrow \dot{\hat{e}}_3 = \left(\frac{MgR}{\lambda_3 \omega}\hat{z}\right) \times \hat{e}_3 = \vec{\Omega} \times \hat{e}_3$$

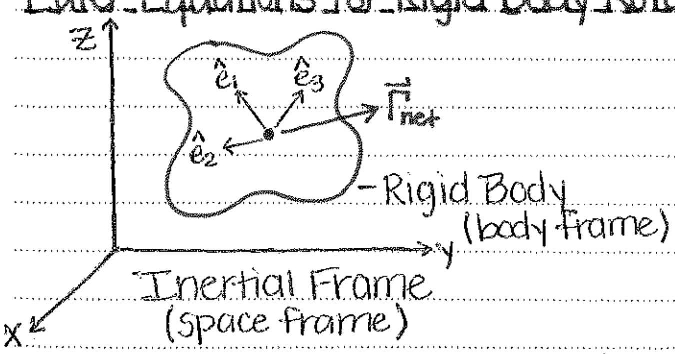
$$\vec{\Omega} = \frac{MgR}{\lambda_3 \omega}\hat{z}$$

$\hat{e}_3$ rotates around $\hat{z}$ at an angular velocity of $MRg/\lambda_3\omega$.

→ Why don't we see motion around the other two principle axes?

- The net torque does not excite the other two

Euler Equations for Rigid Body Rotation



A rotating body becomes a non-inertial reference frame (must compensate)

→ The Euler equations transfer motion from body frame ⟺ space frame.

$\vec{L} = \overline{\overline{I}}\omega$ — not $\omega_x, \omega_y, \omega_z$

$$= \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \leftarrow \text{equation in the body frame}$$

$$= (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3)$$

From a perspective inside the space reference frame:

$$\left(\frac{d\vec{L}}{dt}\right)_{space} = \vec{\Gamma}_{net}$$

$$\left(\frac{d\vec{L}}{dt}\right)_{space} = \left(\frac{d\vec{L}}{dt}\right)_{body} + \underbrace{\vec{\omega} \times \vec{L}}_{\text{angular momentum moving around at some speed}}$$

state of motion of the object

axis of interest

$$\vec{\Gamma}_{net} = (\dot{\vec{L}})_{body} + \underbrace{\vec{\omega} \times \vec{L}}_{\text{intermediate between the two frames}}$$

$$= (\lambda_1\dot{\omega}_1, \lambda_2\dot{\omega}_2, \lambda_3\dot{\omega}_3) + \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ \lambda_1\omega_1 & \lambda_2\omega_2 & \lambda_3\omega_3 \end{vmatrix}$$

$$= (\lambda_1\dot{\omega}_1, \lambda_2\dot{\omega}_2, \lambda_3\dot{\omega}_3) + \hat{e}_1\omega_2\omega_3(\lambda_3 - \lambda_2) - \hat{e}_2\omega_1\omega_3(\lambda_3 - \lambda_1) + \hat{e}_3\omega_1\omega_2(\lambda_2 - \lambda_1)$$

Euler's Equations

$$\begin{cases} \Gamma_1 = \lambda_1\dot{\omega}_1 - \omega_2\omega_3(\lambda_2 - \lambda_3) \\ \Gamma_2 = \lambda_2\dot{\omega}_2 - \omega_1\omega_3(\lambda_3 - \lambda_1) \\ \Gamma_3 = \lambda_3\dot{\omega}_3 - \omega_1\omega_2(\lambda_1 - \lambda_2) \end{cases}$$

Top Precession:

Initial Conditions: $t=0$, $\omega_3 \neq 0$, $\omega_1 = \omega_2 = 0$

λ_3 – principle moment of inertia

$\lambda_1 = \lambda_2$, by symmetry

Using equation 3:

$$0 = \lambda_3\dot{\omega}_3 - \omega_1\omega_2\cancel{(\lambda_1 - \lambda_2)}^{\,0}$$

$\hookrightarrow$ ω_3 is a constant (all rotational energy in $\hat{e}_3$ axis)

$\Rightarrow$ ω_1 & ω_2 never excited!