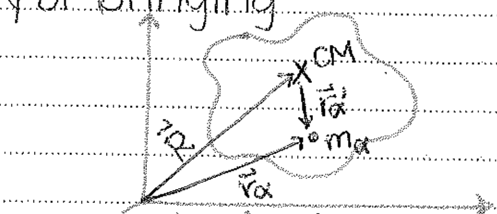
Rotation of Rigid Bodies

Angular momentum – difficulty of bringing a moving body to rest



$\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha'$

$\dot{\vec{r}}_\alpha = \dot{\vec{R}} + \dot{\vec{r}}_\alpha'$

$$\vec{L} = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times m_\alpha \vec{v}_\alpha$$

$$= \vec{L}_{CM} + \underbrace{\vec{L}_{\text{relative to CM}}}_{\text{can only be rotation}}$$

Rotation of Rigid Bodies

Kinetic energy of a system of particles

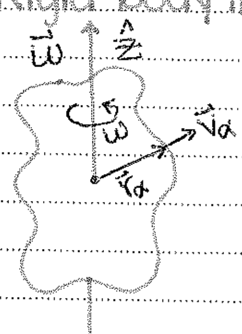
$$T = \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha \dot{\vec{r}}_\alpha^{\,2} = \sum_\alpha \frac{1}{2} m_\alpha \dot{\vec{R}}^2 + \underbrace{\cancel{\sum_\alpha \frac{1}{2} m_\alpha 2\dot{\vec{R}} \cdot \dot{\vec{r}}_\alpha'}}_{\sum m_\alpha \dot{\vec{r}}_\alpha' = 0} + \sum_\alpha \frac{1}{2} m_\alpha (\dot{\vec{r}}_\alpha')^2$$

$$= \frac{1}{2} M \dot{\vec{R}}^2 + \sum_\alpha \frac{1}{2} m_\alpha (\dot{\vec{r}}_\alpha')^2 + \sum_\alpha \frac{1}{2} m_\alpha (\dot{\vec{r}}_\alpha')^2$$

$$= T_{CM} + T_{Rel}$$

Rigid bodies: All "relative motion" is rotation

→ Rigid body in rotation about an arbitrary fixed axis



$\vec{L}$ Angular Momentum

Naively, $\vec{L} = \overline{\overline{I}}\vec{\omega}$

$\vec{p} = m\vec{v}$

$$\vec{L} = \sum_\alpha \underbrace{\vec{r}_\alpha \times m_\alpha \vec{v}_\alpha}_{\vec{l}_\alpha} \leftarrow \vec{v}_\alpha = \vec{\omega} \times \vec{r}_\alpha$$

$$= \sum_\alpha \vec{r}_\alpha \times m_\alpha (\vec{\omega} \times \vec{r}_\alpha)$$

$\vec{\omega} = (0, 0, \omega)$, $\vec{r}_\alpha = (x_\alpha, y_\alpha, z_\alpha)$

$$\vec{\omega} \times \vec{r}_\alpha = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 0 \\ x_\alpha & y_\alpha & z_\alpha \end{vmatrix} = (-y_\alpha \omega, x_\alpha \omega, 0)$$

$$\vec{l}_\alpha = \vec{r}_\alpha \times m_\alpha (\vec{\omega} \times \vec{r}_\alpha) = m_\alpha \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_\alpha & y_\alpha & z_\alpha \\ -y_\alpha \omega & x_\alpha \omega & 0 \end{vmatrix}$$

$$= m_\alpha (-x_\alpha z_\alpha \omega, -y_\alpha z_\alpha \omega, (x_\alpha^2 + y_\alpha^2)\omega)$$

$$\vec{L}_z = \underbrace{\sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2)}_{I_{zz} = \sum_\alpha m_\alpha \rho_\alpha^2} \omega$$

I_{zz} – Moment of inertia

$L_z = I_{zz}\omega$

→ however, we cannot extrapolate and say $\vec{L} = I_{zz}\vec{\omega}$ (not true)

$L_x = \underbrace{-\sum_\alpha m_\alpha x_\alpha z_\alpha}_{I_{xz}}\omega$

product of inertia (includes negative)

$L_y = \underbrace{-\sum_\alpha m_\alpha y_\alpha z_\alpha}_{I_{yz}}\omega$

$\vec{L} = (I_{xz}\omega, I_{yz}\omega, I_{zz}\omega)$

$\vec{L}$ not parallel to $\vec{\omega}$ in general

→ $\vec{L} \parallel \vec{\omega}$ in special cases (such as tires)

$\vec{L} = (I_{xz}\hat{i} + I_{yz}\hat{j} + I_{zz}\hat{k})\omega$

In general: No fixed axis

$\vec{L} = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times m_\alpha \vec{v}_\alpha$

$= \sum_\alpha \vec{r}_\alpha \times m_\alpha(\vec{\omega} \times \vec{r}_\alpha)$

└ $\vec{\omega}$ is arbitrary

$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

$\vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha) = \vec{\omega}(\vec{r}_\alpha \cdot \vec{r}_\alpha) - \vec{r}_\alpha(\vec{r}_\alpha \cdot \vec{\omega})$

$\vec{L} = (\omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k})(x_\alpha^2 + y_\alpha^2 + z_\alpha^2) - (x_\alpha\hat{i} + y_\alpha\hat{j} + z_\alpha\hat{k})(x_\alpha\omega_x + y_\alpha\omega_y + z_\alpha\omega_z)$

$= [(y_\alpha^2 + z_\alpha^2)\omega_x - x_\alpha y_\alpha \omega_y - x_\alpha z_\alpha \omega_z$

$- x_\alpha y_\alpha \omega_x + (x_\alpha^2 + z_\alpha^2)\omega_y - y_\alpha z_\alpha \omega_z$

$- x_\alpha z_\alpha \omega_x - y_\alpha z_\alpha \omega_y + (x_\alpha^2 + y_\alpha^2)\omega_z]$

$L_x = \underbrace{\sum_\alpha m_\alpha[(y_\alpha^2 + z_\alpha^2)}\omega_x - x_\alpha y_\alpha \omega_y - x_\alpha z_\alpha \omega_z$

$= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$

└ component of the angular velocity vector

└ component of the angular momentum vector

$I_{xx} = \sum_\alpha m_\alpha(y_\alpha^2 + z_\alpha^2)$, $I_{xy} = -\sum_\alpha m_\alpha x_\alpha y_\alpha$, $I_{xz} = -\sum_\alpha m_\alpha x_\alpha z_\alpha$

└ off-diagonal terms in the inertia tensor

$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$

$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$

- diagonal elements → moments of inertia
- off-diagonal elements → products of inertia

$L_i = \sum_{j=1}^{3} I_{ij}\omega_j$ where $x=1$, $y=2$, and $z=3$

$\vec{L} = \overset{\leftrightarrow}{I}\vec{\omega}$ ← General relationship between axis of rotation $\vec{\omega}$ and the resulting angular momentum, $\vec{L}$

└ inertia tensor

→

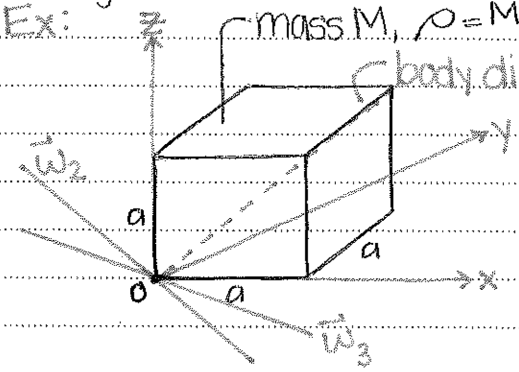
$$\overleftrightarrow{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$ "Inertia Tensor" → a real matrix

tensor is symmetric $I_{ji} = I_{ij}$

Principle axes of rotation:
$\vec{L} \parallel \vec{\omega}$ is satisfied

- there are always three → allows you to diagonalize the inertia tensor
- angular momentum doesn't change as a function of time

Ex:



Choice of Origin
→ assume that one corner is fixed during rotation
$\vec{\omega}$ has to go through the origin

$I_{xx} = \sum_\alpha m_\alpha (y_\alpha^2 + z_\alpha^2)$

$\sum_\alpha m_\alpha \vec{r}_\alpha \to \int \vec{r}\, dm$

$\Rightarrow I_{xx} = \int_0^a dx \int_0^a dy \int_0^a dz\, \rho (y^2 + z^2)$

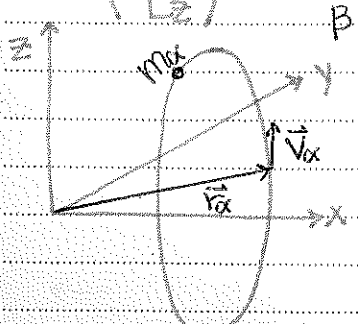
$$\overleftrightarrow{I} = \frac{Ma^2}{12} \begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}$$

We've chosen $\vec{\omega} = (\omega, 0, 0)$

Solve $\vec{L} = \overleftrightarrow{I} \vec{\omega}$

$$\vec{L} = \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \frac{Ma^2}{12} \begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix} \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \underbrace{\boxed{\frac{Ma^2 \omega}{12}}}_{B} \begin{pmatrix} 8 \\ -3 \\ -3 \end{pmatrix}$$



$\vec{l}_\alpha = \vec{r}_\alpha \times m_\alpha \vec{v}_\alpha$
└ angular momentum wobbles around here

$\vec{l}_\alpha$ is not parallel to $\vec{x}$
(a symmetric object rotated in a non-symmetric way)

Want: 3 axes of rotation through a constraining point that make $\vec{L} \parallel \vec{\omega}$, such as the body diagonal

different real numbers

$$\vec{L} = \lambda_1 \vec{\omega}_1 + \lambda_2 \vec{\omega}_2 + \lambda_3 \vec{\omega}_3$$

3 principle axes of rotation

$\overline{\overline{I}}$ can be diagonalized

$$\vec{L} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \vec{\omega}$$

$\rightarrow$ for each fixed point, there are 3 principle axes of rotation

Propose that the following exists:

$\vec{L} = \lambda \vec{\omega} \leftarrow$ Special choice(s) of $\vec{\omega}$ direction

$\quad \uparrow$ real scalar number

Then this is true:

$\vec{L} = \overline{\overline{I}}\vec{\omega} \Rightarrow \lambda \vec{\omega} = \overline{\overline{I}}\vec{\omega}$

$\quad$ └ matrix · vector = # · same vector (Eigenvalues!)

$\lambda \vec{\omega} = \lambda \overline{\overline{\mathbb{1}}} \vec{\omega}$

$(\overline{\overline{I}} - \lambda \overline{\overline{\mathbb{1}}})\vec{\omega} = 0$

$\quad$ └ unit matrix with λ's on the diagonal

For a non-trivial solution: $\det(\overline{\overline{I}} - \lambda \overline{\overline{\mathbb{1}}}) = 0$

$$\det \begin{pmatrix} 8\beta - \lambda & -3\beta & -3\beta \\ -3\beta & 8\beta - \lambda & -3\beta \\ -3\beta & -3\beta & 8\beta - \lambda \end{pmatrix} = 0$$

$$= (2\beta - \lambda)(11\beta - \lambda)^2 = 0$$

$$\Rightarrow \left.\begin{aligned} \lambda_1 &= 2\beta = Ma^2/6 \\ \lambda_2 &= \lambda_3 = \tfrac{11}{12} Ma^2 \end{aligned}\right\} \text{principle moments of inertia}$$

For $\lambda_1 = 2\beta$

$$\begin{pmatrix} 6\beta & -3\beta & -3\beta \\ -3\beta & 6\beta & -3\beta \\ -3\beta & -3\beta & 6\beta \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

$\omega_x = \omega_y = \omega_z$

$\vec{\omega} = 1/\sqrt{3}\,(1,1,1) \leftarrow$ Body diagonal direction

$\lambda_1 = Ma^2/6 \leftarrow$ Moment of inertia for this ω

For λ_2 & λ_3

$$\begin{pmatrix} -3\beta & -3\beta & -3\beta \\ -3\beta & -3\beta & -3\beta \\ -3\beta & -3\beta & -3\beta \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

$\omega_x + \omega_y + \omega_z = 0$

$\rightarrow$

$\vec{\omega}_2 \cdot \vec{\omega}_1 = 0$
$\vec{\omega}_3 \cdot \vec{\omega}_1 = 0$
$\vec{\omega}_2 \cdot \vec{\omega}_3 = 0$
} mutually perpendicular to $\vec{\omega}_1$, the body diagonal for the second and third principle axes

$I = \frac{11}{12} Ma^2$ ← Moment of Inertia for both principle axes here

Ex. of vector that satisfy this requirement:

$\vec{\omega}_2 = 1/\sqrt{6}\,(2,-1,-1)$
$\vec{\omega}_3 = 1/\sqrt{2}\,(0,1,-1)$

$$\Rightarrow \overline{\overline{I}} = \begin{pmatrix} \frac{1}{6}Ma^2 & 0 & 0 \\ 0 & \frac{11}{12}Ma^2 & 0 \\ 0 & 0 & \frac{11}{12}Ma^2 \end{pmatrix}$$ for the principle axes of rotation