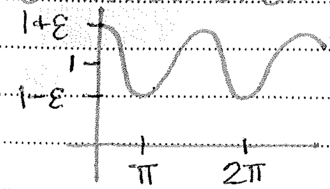
Orbits & Collisions

Bounded Orbit ($E<0$)

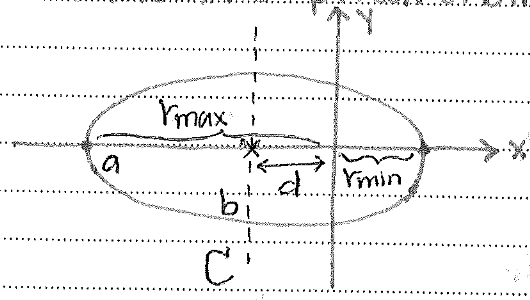


$r_{min} = \frac{c}{1+\varepsilon}$ $\varphi = 0$, $r_{max} = \frac{c}{1-\varepsilon}$ $\varphi = \pi$

$\ell = \mu r^2 \dot{\varphi}$

Periodic = closed orbit, $r(\varphi + 2\pi) = r(\varphi)$

→ Forms an elliptical orbit



Ellipse equation

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1$$

$d = a\varepsilon$ (offset)

$a = c/_{1-\varepsilon^2}$ (semimajor axis)

$b = c/\sqrt{1-\varepsilon^2}$ (semiminor axis)

Ellipticity of an ellipse: (ε)

$$\frac{b}{a} = \frac{c}{\sqrt{1-\varepsilon^2}} \times \frac{1-\varepsilon^2}{c} = \sqrt{1-\varepsilon^2}$$

$\varepsilon = 0$, $b/a \to 1$ ⇒ circle, $d=0$

$\varepsilon = 1$, $b/a \to 0$ ⇒ elongated, highly elliptical orbit (comets)

Kepler's Law

$\frac{dA}{dt} = \frac{\ell}{2\mu}$ (A constant, conservation of orbital angular momentum)

Orbital Period:

$$T = \frac{A}{dA/dt} = \frac{\pi ab}{\ell/2\mu}$$

→ yields Kepler's 3rd law of motion

$T^2 = \frac{4\pi^2}{GM_\odot} a^3$ • relating period to semi-major axis
• assumed $m_{planet} \ll M_\odot$

Energy in the Orbit

E @ r_{min} (radial kinetic energy = 0)

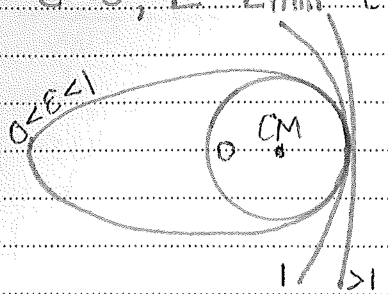
$E = \frac{-\gamma}{r_{min}} + \frac{\ell^2}{\mu r_{min}^2}$ ← $r_{min} = c/(1+\varepsilon)$

→ $E = \frac{\gamma^2 \mu}{2\ell^2}(\varepsilon^2 - 1)$ (applicable for all values of ε)

where $F(r) = -\gamma/r^2$

$\varepsilon > 1$, $E > 0$ unbounded orbits

$0<\varepsilon<1$, $E<0$ bound orbits (ellipse)
$\varepsilon=0$, $E=E_{min}$ circular orbits

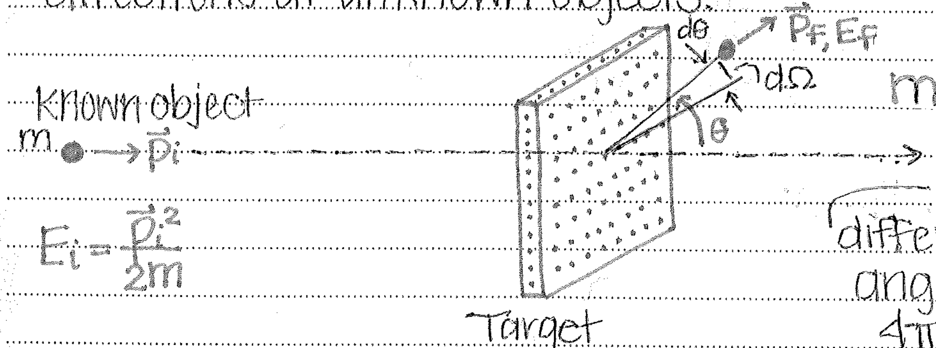


$\varepsilon=1 \rightarrow$ parabola
$\varepsilon>1 \rightarrow$ hyperbola

Collisions → Scattering Theory

"If you don't know what it is, always try to destroy it first."

- Find the structure and properties of small things by means of throwing known stuff at known energies in known directions at unknown objects.



measure $\vec{p}_F, E_F, \theta$

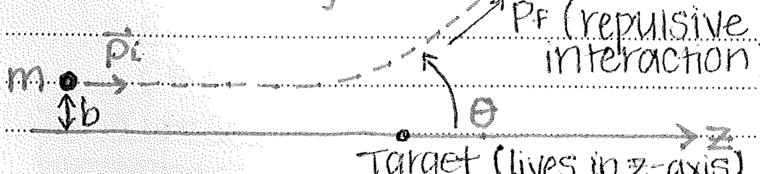
$d\Omega = \sin(\theta)\,d\theta\,d\varphi$
differential solid angle (steradians)
4π sr = sphere

→ the exact incident particle reemerging is very rare and very boring

Example of note: Rutherford Scattering Experiment

- atoms are mostly empty space

How scattering works:



* assume spherically symmetric scattering potential (uniform in φ)

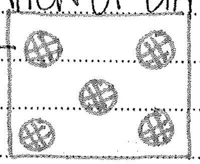
b – impact parameter, distance from z-axis of incident particle (not controlled or measured)
$\theta=0 \rightarrow$ no scattering

Must resort to
Statistical Description of Scattering
because we don't know the impact parameter
* this ignores quantum statistics for scattering!

-- point of view of an alpha particle as it approaches a gold film

radius R —

$R \approx 10^{-14}$ m



→ See atoms as classically hard spheres

Projected 2D density of atoms

$n_{target} = \frac{\#\ atoms}{area}$

area of the foil = A

atoms to interact with = $n_{target} A$

Target atoms area $\sigma = \pi R^2$

units of $\sigma = 10^{-28}\ m^2$ (barns)

total scattering area = $n_{target} \sigma A$

Probability of scattering = $\frac{n_{target} \sigma A}{A} = n_{target} \sigma$

┌ # of scattered particles ┌ probability of scattering

$N_{scattered} = N_{inc} \cdot n_{target} \cdot \sigma$

└ # of incident particles

The concept of scattering is really general:

- Elastic scattering $\sigma_{elastic} = \pi R^2$
- Inelastic scattering $\sigma_{inelastic}$
- Capture scattering $\sigma_{capture}$
  └ target goes OM NOM NOM to incident particle
  $N_{e^- abs} = N_{inc} \cdot n_{target} \cdot \sigma_{capture}$
  └ # electrons absorbed by target
- Ionization scattering $\sigma_{ionization}$
  └ highly energetic electrons in ⇒ multiple electrons out
- Fission [scattering] $\sigma_{fission}$
  └ Neutrons in, hit Uranium/Plutonium, lots of stuff out

Total scattering cross section:

$\sigma_T = \sigma_{elas} + \sigma_{ion} + \sigma_{fis} + \ldots$

Given $\vec{F}$ ⇒ predict how many particles come out at $\theta, E_f, \vec{p}_f$

→ Want $N_{scat}(d\Omega$ at $\theta, \varphi)$

(generalize scattering cross section to be an angle-dependent quantity)

→

$N_{scat}(d\Omega, \theta, \varphi) = N_{inc} \cdot n_{target} \cdot d\sigma(d\Omega, \theta, \varphi)$
└ partial cross-section for a specific $d\Omega$

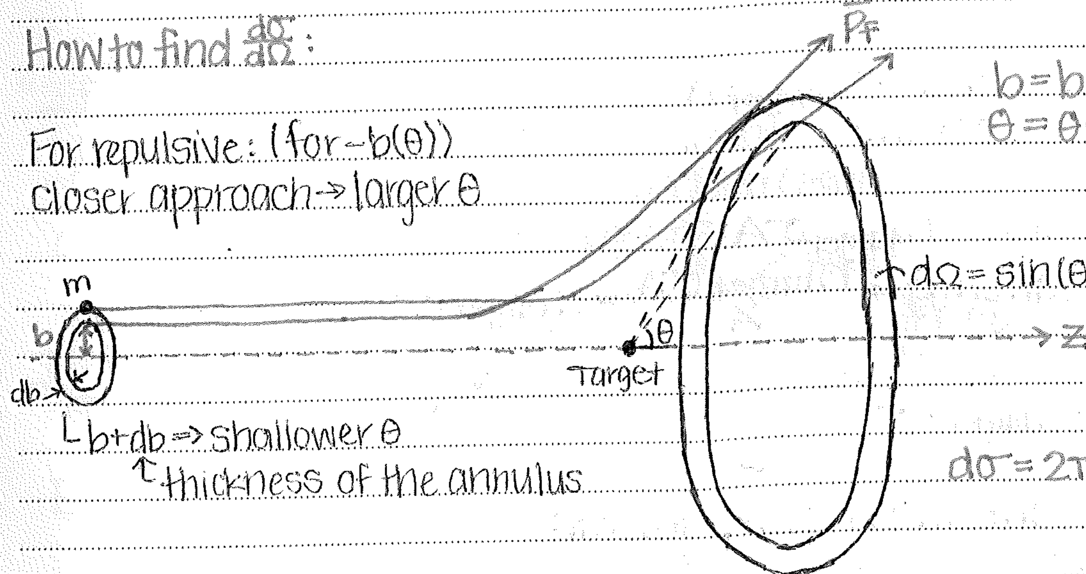
$d\sigma(\theta,\varphi) = \underbrace{\frac{d\sigma}{d\Omega}} d\Omega$
differential scattering cross section – what fraction of incoming particles are deflected at a certain angle

$$\sigma = \int_0^{2\pi}\int_0^{\pi} \frac{d\sigma(\theta,\varphi)}{d\Omega} \sin(\theta)\, d\theta\, d\varphi$$

How to find $\frac{d\sigma}{d\Omega}$:

For repulsive: (for $b(\theta)$)
closer approach → larger θ

$b = b(\theta)$
$\theta = \theta(b)$



$d\Omega = \sin(\theta)\, d\theta\, d\Omega$

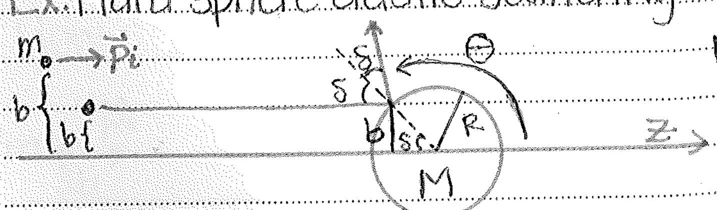
└ $b + db \Rightarrow$ shallower θ
↑ thickness of the annulus

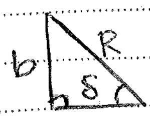
$d\sigma = 2\pi b\, db$

Angle subtended by input beam → angle subtended by output beam

$$\frac{d\sigma}{d\Omega} = \frac{2\pi b\, db}{2\pi \sin(\theta)\, d\theta} = \frac{b}{\sin(\theta)} \underbrace{\left|\frac{db}{d\theta}\right|}_{\text{find this value}}$$
← physics determines $b(\theta)$, $db/d\theta$

Ex. Hard sphere elastic scattering





$b = R\sin(\delta)$
$\theta + 2\delta = \pi$
$\delta = \frac{\pi - \theta}{2}$
$b = R\sin(\frac{1}{2}(\pi - \theta))$
$= R\cos(\theta/2)$

$b > R$ no scattering $\theta = 0$, $b \leq R$ scattering interaction

$\frac{db}{d\theta} = -\frac{R}{2}\sin(\theta/2)$ ┌ trig identity: $\cos(\theta/2)\sin(\theta/2) = \frac{1}{2}\sin(\theta)$

$\frac{d\sigma}{d\Omega} = \frac{R\cos(\theta/2)}{\sin(\theta)} \frac{R}{2}\sin(\frac{\theta}{2}) = \frac{R^2}{4}$ ← θ independent! same amount of stuff in every direction

$$\sigma = \int_0^{2\pi}\int_0^{\pi} \frac{R^2}{4} \sin(\theta)\, d\theta\, d\varphi$$

$= \pi R^2$ (as expected)