ENEE 324

Sections 0101, 0102, FR01 Spring 2011

Homework #13

*Remember, the goal here is to understand **how** to do the calculations, not to produce the correct answer using pure instinct. Please explain **how** you have arrive at your answers.*

1) Let the set of random samples $X_1, X_2, \ldots$ consist of i.i.d. versions of a random variable X with variance VAR[X]. Define a derived set of random samples $Y_1, Y_2, \ldots$ by $Y_n = X_{2n-1} - X_{2n}$. That is, $Y_1 = X_1 - X_2$, $Y_2 = X_3 - X_4$, etc..
   a) What is $E[Y_n]$?
   b) What is VAR[Y_n]?
   c) What are the mean and variance of $M_n(Y)$?

2) Let X and Y be a pair of random variables.
   a) Consider the case when their means are known, μ_X and μ_Y, and we construct this estimator of the covariance of X and Y: $\hat{C}_{X,Y} = \frac{1}{n}\sum_{j=1}^{n}(X_j - \mu_X)(Y_j - \mu_Y)$. Compute its expected value in terms of the true (unknown) covariance COV[X,Y].
   b) Consider the case when their means are unknown, and we construct this estimator of the covariance of X and Y: $\hat{K}_{X,Y} = \frac{1}{n-1}\sum_{j=1}^{n}(X_j - M_n(X))(Y_j - M_n(Y))$. Compute its expected value in terms of the true (unknown) covariance COV[X,Y].

3) Let the sample $X_1, X_2, \ldots, X_n$ consist of n i.i.d. versions of a random variable X whose cumulative distribution function (CDF) is $F_X(x)$ and probability density function (PDF) is $f_X(x)$. Consider the derived random variables that are the maximum and minimum values of the sample:

$$Y = \max(X_1, X_2, \ldots, X_n) \qquad W = \min(X_1, X_2, \ldots, X_n).$$

   a) Show that the CDF of Y is $F_Y(x) = \left(F_X(x)\right)^n$. What is its PDF?
   b) Show that the CDF of W is $F_W(x) = 1 - \left(1 - F_X(x)\right)^n$. What is its PDF?

4) When estimating an unknown θ, the mean square error e of the estimator $\hat{\theta}$ is defined by

$$e = E[(\hat{\theta} - \theta)^2].$$

a) Show that $e = \mathrm{VAR}[\hat{\theta}] + \left(\mathrm{Bias}[\hat{\theta}]\right)^2$. Hint: Try subtracting and adding $E[\hat{\theta}]$ from within the squared term in the definition of e.

b) Let the output of a communication system be $Y = \theta + N$ where θ is the input signal and N is a noise signal with $N \sim \mathrm{Uniform}(0,2)$. Suppose the signal is transmitted n times and the noise contributions are iid. Find the bias and mean square error of the estimator $\hat{\theta} = M_n(Y)$.

5) Prove for a Gaussian random variable X, that the sample mean M_n of n i.i.d. random samples X_i is within one standard error, $\mathrm{se} = \sqrt{\mathrm{VAR}[M_n]}$, of the expected value $E[X]$ with probability $2\Phi(1) - 1 \approx 0.68$. Or, equivalently, that $P\left[\mu_X - \mathrm{se} \le M_n \le \mu_X + \mathrm{se}\right] = 2\Phi(1) - 1 \approx 0.68$.