# ENEE 324

Sections 0101, 0102, FR01 Spring 2011

**Homework #12**

*Remember, the goal here is to understand* ***how*** *to do the calculations, not to produce the correct answer using pure instinct. Please explain* ***how*** *you have arrive at your answers.*

1) $\mathbf{Z} = [Z_1\ Z_2]^{\mathrm{T}}$ is a random vector of independent, identically distributed Gaussian random variables with mean of 0 and variance of 1. (You could generate instances of this random vector using the Matlab `randn` function.)
   a) Find a linear transformation $\mathbf{X} = \mathbf{AZ} + \mathbf{b}$ so that the new random vector $\mathbf{X}$ is Gaussian with

$$\mathbf{C_X} = \begin{bmatrix} 73 & -36 \\ -36 & 52 \end{bmatrix}, \qquad \mu_{\mathbf{X}} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

   b) Derive equations for $Z_1$ and $Z_2$ in terms of $X_1$ and $X_2$.

2) Consider a random vector $\mathbf{X} = [X_1\ X_2\ X_3]^{\mathrm{T}}$ with covariance matrix

$$\mathbf{C_X} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

   a) Using eigenvector analysis, find a matrix $\mathbf{A}$ such that for $\mathbf{Y} = \mathbf{AX}$ the covariance matrix of $\mathbf{Y}$, $\mathbf{C_Y}$, is diagonal. This transform guarantees that $\mathbf{Y}$'s component random variables are uncorrelated.
   b) Similarly, find a matrix $\mathbf{B}$ such that for $\mathbf{W} = \mathbf{BX}$ the covariance matrix of $\mathbf{W}$, $\mathbf{C_W}$, is not only diagonal but also proportional to the identity matrix. This transform guarantees that $\mathbf{W}$'s component random variables are not only uncorrelated, but also have equal variance (this process is called "whitening"). [If your answer to (a) already has this property, then you need not find another matrix, just state that this is the case.]

3) Suppose $X$ is a continuous random variable with the Laplacian distribution,

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda|x|}, \qquad -\infty < x < \infty$$

   a) Calculate $\Phi_X(s)$, the moment generating function of $X$.

b) Use a Taylor series expansion of that result to find a formula for $E[X^n]$. Hint: You shouldn't need to compute any derivatives if you remember the following standard formula for a geometric sum:

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \ldots$$

c) Now suppose that $W = X^3$. Use the results from (b) to find $VAR[W]$.

d) Let $Y$ be a continuous random variable with an exponential distribution and a mean of $1/\lambda$, where $X$ and $Y$ are independent. Find the moment generating function of $Z = X + Y$.

4) Let $X$ be a Poisson random variable with mean $\alpha$.

a) Find the moment generating function $\Phi_X(s)$.

b) Use that result to calculate the $E[X]$ and $E[X^2]$.

c) Suppose $X_1$ and $X_2$ are independent Poisson random variables with means $\alpha_1$ and $\alpha_2$ respectively. Show that the sum $W = X_1 + X_2$ is also a Poisson random variable, with mean $\alpha_1 + \alpha_2$.

5) The total number of fish caught by a fisherman in one day ($X$) can be modeled as a Poisson random variable with mean $\alpha$:

$$X \sim \text{Poisson}(\alpha)$$

a) In terms of $\alpha$, find an expression for $E[b^X]$, where $b$ is a constant. You may find it helpful to use:

$$e^u = \sum_{x=0}^{\infty} \frac{u^x}{x!}$$

Of the $X$ fish that he has caught, on average 1/3 of them will be red-snapper. Suppose $Y$ is a random variable that represents the total number of red-snapper that he catches within a single day. Given $X$, it therefore seems reasonable to model $Y$ as a Binomial random variable with $N = X$ and $p = 1/3$.

b) What is $E[e^{sY} | X]$? (You may refer to the textbook's table of moment-generating functions, Table 6.1.)

c) Using your results from (a) and (b), calculate the moment generating function $\Phi_Y(s) = E[e^{sY}]$.

d) What is the PDF of $Y$? (You may use the table of moment generating functions.)