ENEE 324

Sections 0101, 0102, FR01 Spring 2011

Homework #11

*Remember, the goal here is to understand **how** to do the calculations, not to produce the correct answer using pure instinct. Please explain **how** you have arrive at your answers.*

1) Random variables X and Y have the joint PDF $f_{XY}(x,y) = c\,e^{-(x^2+4xy+8y^2)}$.
   a) What are $E[X]$ and $E[Y]$.
   b) Find ρ, the correlation coefficient of X and Y.
   c) What are $VAR[X]$ and $VAR[Y]$?
   d) What is the constant c?
   e) Are X and Y independent? Why or why not?

2) Suppose X and Y are random variables with $E[X|Y]= (Y+1)/2$ and $E[Y|X]= (X+1)/2$.
   a) Calculate $E[X]$ and $E[Y]$.

   Now suppose you are also told that X and Y are jointly Gaussian random variables with $COV[X,Y] = 2$.
   b) Determine the correlation coefficient ρ.
   c) Determine $VAR[X]$ and $VAR[Y]$.

3) Let X be a standard normal random variable, $X \sim N(0,1)$.
   a) Consider $V = -X$. Show that V is also a standard normal random variable.
   b) Consider the discrete random variable W, independent of X, which can be equal to either +1 or -1 with equal probability (i.e. $P_W(w) = 1/2$ for $w = \pm 1$ and zero otherwise). Let $Y = WX$. Are X and Y independent? Why or why not.
   c) Calculate $E[XY \mid W = 1]$ and $E[XY \mid W = -1]$.
   d) Use the result of (c) and the law of iterated expectations to show that X and Y are uncorrelated. Is that consistent with your answer to (b)?
   e) Show that Y has the same probability distribution as X. Hint: Show that $P(Y \le x) = P(X \le x)$ using probabilities conditioned on W and the result of (a).

   You have now proven that uncorrelated Gaussian random variables need not be independent (though uncorrelated *jointly* Gaussian variables must be independent).

4) Suppose the position of a particle within a cube is described by a random vector $\mathbf{X} = [X_1\ X_2\ X_3]^{\mathrm{T}}$. The joint PDF of $\mathbf{X}$ is given by:

$$f_{\mathbf{X}}(x_1, x_2, x_3) = \begin{cases} c(x_1 + x_2 + x_3) & 0 \le x_1, x_2, x_3 \le 2 \\ 0 & \text{otherwise} \end{cases}$$

a) Determine the value of the constant c.

b) Calculate the marginal PDF of X_1.

c) Find $f_{X_1X_2}(x_1, x_2)$

d) Calculate the mean $\mu_{\mathbf{X}} = E[\mathbf{X}]$

e) Calculate the correlation matrix $\mathbf{R}_{\mathbf{X}}$.

f) Calculate the covariance matrix $\mathbf{C}_{\mathbf{X}}$.

5) In class, we considered the random vector $\mathbf{V} = [X\ Y\ Z]^{\mathrm{T}}$ described by the following joint PDF:

$$f_{XYZ}(x, y, z) = \begin{cases} 6 & 0 \le x, y, z, \quad x + y + z \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a) Calculate the mean $\mu_{\mathbf{V}} = E[\mathbf{V}]$.

b) Calculate the correlation matrix $\mathbf{R}_{\mathbf{V}}$.

c) Calculate the covariance matrix $\mathbf{C}_{\mathbf{V}}$.

d) Suppose $W = X - 3Y + 2Z + 4$. Use your result from (c) to find μ_W and σ_W^2.

e) Consider a new random vector $\mathbf{U}$ defined by: $\mathbf{U} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \mathbf{V}$. Calculate $\mu_{\mathbf{U}}$ and $\mathbf{C}_{\mathbf{U}}$.

6) Suppose X_1 and X_2 are independent, identically distributed Gaussian random variables with mean 0 and variance 1. Consider a random vector $\mathbf{Y} = [Y_1\ Y_2]^{\mathrm{T}}$ whose components are related to X_1 and X_2 by:

$$Y_1 = X_2 - X_1, \qquad Y_2 = 2X_1 + 3X_2$$

a) Calculate the mean $\mu_{\mathbf{Y}} = E[\mathbf{Y}]$.

b) Calculate the covariance matrix $\mathbf{C}_{\mathbf{Y}}$.

c) What is $E[Y_2|Y_1]$?