ENEE 324

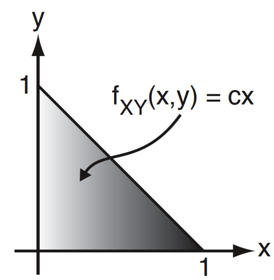
Sections 0101, 0102, FR01 Spring 2011

Homework #10

Remember, the goal here is to understand ***how*** *to do the calculations, not to produce the correct answer using pure instinct. Please explain* ***how*** *you have arrive at your answers.*

1) Random variables X and Y have the following joint PDF: $f_{X,Y}(x,y) = \begin{cases} cx & x \geq 0, y \geq 0, x+y \leq 1 \\ 0 & \text{otherwise} \end{cases}$

   a) Find the value of the constant c.
   b) Calculate and sketch the marginal PDFs $f_X(x)$ and $f_Y(y)$.
   c) Are X and Y independent?
   d) Calculate $E[X]$, $E[Y]$, and $COV[X, Y]$.
   e) Calculate and make a sketch of $f_{X|Y}(x \mid y)$.
   f) Find an expression for $E[X|Y]$.
   g) Use the law of iterated expectations to calculate $E[X]$, and verify that your result agrees with (d).
   h) Calculate and make a sketch of $f_{Y|X}(y \mid x)$.
   i) Find an expression for $E[Y|X]$.
   j) Use the law of iterated expectations to calculate $E[Y]$, and verify that your result agrees with (d).
   k) Let $W = X+Y$. Calculate the PDF $f_W(w)$. Make a labeled sketch of your answer.



2) Random variables X and Y are distributed according to the joint PDF

$$f_{XY}(x,y) = \begin{cases} ax & 3 \leq x \leq y \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

   a) Sketch the region in the x-y plane where $f_{XY}(x,y)$ is nonzero.
   b) Evaluate the constant a.
   c) Determine the marginal PDF $f_Y(y)$.
   d) Determine the marginal PDF $f_X(x)$.
   e) Are X and Y independent? Why or why not?
   f) Calculate $COV[X,Y]$.
   g) Calculate the conditional PDF of X given Y, $f_{X|Y}(x \mid y)$. Make a fully labeled sketch of $f_{X|Y}(x \mid 4)$.
   h) Find $E[X|Y]$. Your answer should be a new random variable that is derived from Y.
   i) Calculate the conditional PDF of Y given X, $f_{Y|X}(y \mid x)$. Make a fully labeled sketch of $f_{Y|X}(y \mid 4)$.
   j) Find $E[Y|X]$. Your answer should be a new random variable that is derived from X.

k) Find the expected value of $1/X$, given that $Y = 4$.

3)

a) Prove that if X and Y are independent, then $E[X|Y] = \mathrm{E}[X]$.

b) Prove that if $E[X|Y] = \mathrm{E}[X]$, then X and Y are uncorrelated.

c) Think of an example of two random variables that are uncorrelated for which $E[X|Y] \neq \mathrm{E}[X]$. You must specify the joint PMF or PDF of X and Y. (There are many correct answers to this question.)

4) Suppose X is an positive continuous random variable described by the following exponential PDF: $f_X(x) = \mathrm{e}^{-x}\, u(x)$. Now let Y be a second continuous random variable, related to X, where the conditional PDF for Y given X is also exponential: $f_{Y|X}(y\,|\,x) = \frac{1}{x}\mathrm{e}^{-y/x}\, u(y)$.

a) Find $E[Y]$.

b) Find $VAR[Y]$.

c) Find $COV[X,Y]$.

d) Does Y have an exponential distribution?

Hint: You should be able to solve this entire problem without ever calculating $f_{XY}(x,y)$ or $f_Y(y)$, and without calculating a single integral, provided you know where to look up the mean and variance of an exponential distribution. You may find it helpful to use the law of iterated expectations and the relationship $E[U^2] = VAR[U] + E[U]^2$.

5) For each of the cases shown in (a) . . . (d) below, determine whether the random variables X and Y are independent. If they are independent, then calculate and sketch the marginal PDFs, $f_X(x)$ and $f_Y(y)$. In all cases, you may assume that the joint PDF is zero at all points outside of the unit square.