Solved Example 10.1

We have $x(t) = s(t) + s(-t)$, and thus

$$X_k = S_k + S_{-k}$$

Since $s(t)$ is real-valued, $S_{-k} = S_k^*$ and thus also

$$X_k = 2\Re e\{S_k\}$$

The spectrum is real and even, as is $x(t)$.

For $y(t)$, we have

$$y(t) = x(t) + x(t - T_0/2)$$

and thus the two sets of Fourier series coefficients—both evaluated with respect to the fundamental period T_0 of $x(t)$—are related by

$$\begin{aligned} Y_k &= S_k + e^{-jk\Omega_0 T_0/2} \cdot S_k \\ &= (1 + e^{-jk\pi})S_k \\ &= (1 + (-1)^k)S_k \\ &= \begin{cases} 2S_k, & k \text{ even;} \\ 0, & k \text{ odd.} \end{cases} \end{aligned}$$

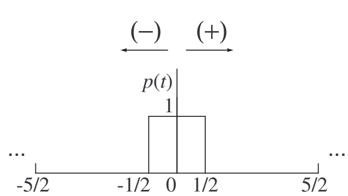
The fact that $Y_k = 0$ for all odd k is not surprising, since the fundamental period of $y(t)$ is $T_0/2$. Thus $y(t)$ is a sum of sinusoids of frequency $k(2\Omega_0) = (2k)\Omega_0$.

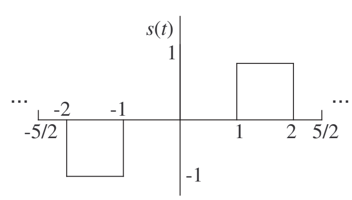
If we were to express $y(t)$ as a Fourier series with respect to its true fundamental period $T_0/2$, then the k^{th} coefficient of the series would be given by $2S_{2k}$ (note the doubling in the subscript).

Note: No change in time scale is involved between $s(t)$ and $y(t)$, i.e., $y(t) \neq s(2t)$.

Solved Example 10.2

Here $T_0 = 5$ and $\Omega_0 = 2\pi/5$. The signal $s(t)$ is obtained from an even rectangular pulse train using two time shifts:



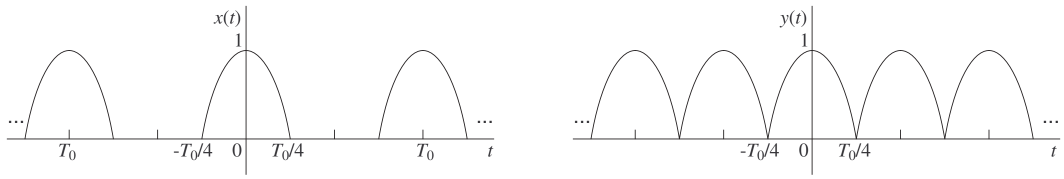


$$s(t) = p\left(t - \frac{3}{2}\right) - p\left(t + \frac{3}{2}\right)$$

Therefore, using the time-shift property,

$$\begin{aligned} S_k &= \left(e^{-jk(2\pi/5)(3/2)} - e^{jk(2\pi/5)(3/2)}\right) \cdot P_k \\ &= -\frac{2j}{k\pi} \cdot \sin\left(\frac{k\pi}{5}\right) \cdot \sin\left(\frac{k3\pi}{5}\right) \end{aligned}$$

Solved Example 10.3



We can write

$$y(t) \;=\; x(t) + x(t - T_0/2) \;=\; x(t) + x(t + T_0/2)$$

and equivalently,

$$y(t) \;=\; 2x(t) - \cos \Omega_0 t \;=\; 2x(t) - \frac{e^{j\Omega_0 t}}{2} - \frac{e^{-j\Omega_0 t}}{2}$$

If $\{X_k\}$ and $\{Y_k\}$ are the Fourier series coefficients with respect to fundamental period T_0, when we also have two equivalent relationships between the two sets of coefficients:

$$\begin{aligned} Y_k \;&=\; X_k + e^{-jk\Omega_0 T_0/2} \cdot X_k \\ &=\; \left(1 + (-1)^k\right) X_k \\ &=\; \begin{cases} 2X_k, & k \text{ even;} \\ 0, & k \text{ odd.} \end{cases} \end{aligned}$$

and also

$$Y_k \;=\; \begin{cases} 2X_k, & k \neq \pm 1; \\ 2X_k - 1/2, & k = \pm 1. \end{cases}$$

The two relationships together reveal two facts about $\{X_k\}$:

- $X_1 = X_{-1} = 1/4$;
- $X_k = 0$ for odd values of k other than ± 1,

both of which are consistent with the formula for X_k derived in the lecture notes.

Finally, we note that $y(t)$ has fundamental period $T_0/2$, which explains why the odd harmonic coefficients Y_k obtained above for fundamental period T_0 were all zero. Clearly, the Fourier series expansion of $y(t)$ with respect to its fundamental period $T_0/2$ has k^{th} coefficient equal to $2X_{2k}$.

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S 10.4 (P 4.5)
--------------

i) MATLAB code:

    b = [1 -3 1 1 -3 1].';
    H = fft(b,256);
    A = abs(H);
    q = angle(H);
```

```
ii) H(exp(j*w)) = 1 - 3*exp(-j*w) + exp(-j*2*w) + exp(-j*3*w)
            - 3exp(-j*4*w) + exp(-j*5*w)

            = exp(-j*5*w/2)*(exp(j*5*w/2) - 3*exp(j*3*w/2) + exp(j*w/2)
              + exp(-j*w/2) - 3*exp(-j*3*w/2) + exp(-j*5*w/2))

    = exp(-j*5*w/2)*(2*cos(w/2) = 6*cos(3*w/2) + 2*cos(5*w/2))

iii) y[n] = H(1/2)*x[n] = H(1/2)*(1/2)^n

        H(1/2) = 1 - 3*2 + 2^2 + 2^3 -3*2^4 + 2^5 = -9
So
        y[n] = -9*(1/2)^n
```

S 10.5 (P 4.6)

The vector H (in MATLAB) is obtained by sampling

```
        sum(a[n]*exp(-j*w*n)), where n ranges from 0 to 4
```

at 500 equally spaced frequencies. This means that the
coefficient vector for the filter is simply = a, i.e.,

```
        [1 -3 5 -3 1].'
```

```
(i) y[n] = x[n] -3*x[n-1]+ 5*x[n-2] -3*x[n-3] + x[n-4]

(ii) H(z) = 1-3/z + 5/(z^2) - 3/(z^3) + 1/z^4
```

Thus

```
        H(1/3) = 1 - 3*3 + 5*9 - 3*27 + 81 = 37
```

and if the input sequence is x[n] = (1/3)^n (all n), the output sequence is

```
        y[n] = H(1/3)*(1/3)^n = 37*(1/3)^n, all n
```

(iii)

```
    H(exp(j*w))= 1-3*exp(-j*w)+5*exp(-j*2*w)-3*exp(-j*3*w)+exp(-j*4*w)

        = exp(-j*2*w)*(exp(j*2*w)-3*exp(j*w)+5-3*exp(-j*w)+exp(-j*2w))
```

```
= exp(-j*2*w)*(5-6*cos(w)+2*cos(2*w))
```