S 9.1

Using 2*cos(theta) = exp(j*theta) + exp(-j*theta), we see that s(t) can be expressed as the (unweighted) sum of eight complex sinusoids of the form

```
    exp(2*pi*f*t) ,
```

where f (Hz) takes the four positive values (in Hz)

```
    23+25+30 = 78
    23+25-30 = 18
   -23+25+30 = 32
    23-25+30 = 28 ,
```

as well as the negatives of these values. The largest value of fo such that all four frequencies shown are integer multiples of fo equals fo = 2 Hz. Thus the fundamental cyclic frequency of s(t) is fo = 2Hz and the fundamental period is To = 1/fo = 0.5 sec.

S 9.2

The sinusoids in s(t) have frequencies

```
    15, 18.75, and 26.25  Hz
```

The largest value of fo such that all four frequencies shown are integer multiples of fo is fo = 3.75 Hz. This is the fundamental cyclic frequency, and the fundamental period is To = 1/fo = 4/15 sec.

The nonzero complex Fourier series coefficients correspond to

```
    k = 0, (+/-)4 , (+/-)5 and (+/-)7
```

Using

```
    2*cos(theta) = exp(j*theta) + exp(-j*theta)
    2*sin(theta) = -j*(exp(j*theta) - exp(-j*theta))
```

we have

```
    S_0     =  11
    S_4     =  0.5
```

```
    S_(-4)  =  0.5
    S_5     =       j*2.5
    S_(-5)  =      -j*2.5
    S_7     = -4.5 - j
    S_(-7)  = -4.5 + j
```

s(t) is neither odd nor even about t=0. In the time domain, this can be seen from the fact that s(t) is a sum of both cosines (even) and sines (odd). In the frequency domain, S_k is neither purely real (corresponding to even s(t)) nor purely imaginary (corresponding to odd s(t)).

S 9.3

When s(t) is sampled at a rate of N samples per period (or N*fo samples/second), the kth sinusoid in the Fourier series for s(t) becomes a discrete-time sinusoid of frequency

```
    k*(2*pi*fo)/(N*fo) = k*(2*pi/N)
```

i.e, the kth Fourier sinusoid for a vector of size N. If the Fourier series is finite, i.e,

```
    S_k = 0 for |k|>K,
```

and if

```
    N > 2*K
```

then the N discrete-time Fourier sinusoids obtained in this manner will be distinct. The Fourier series for s(t) will then serve as a synthesis equation for the sample vector

```
    s = [s[0] s[1] ... s[N-1]].'
```

(which consists of N uniform samples of s(t) over [0,To)).

```
    S_k = kth Fourier series coefficient for s(t)

        = (1/N)*(kth (or (N-k)th) entry in the DFT of s)
```

In this particular case, K = 8. We start with

```
    N = 17 = 2*8+1
```

uniform samples contained in the vector s. We obtain the Fourier series coefficients using

```
    S = fft(s)/17
```

We then generate 340 uniform samples over [0,To) using

```
    340*ifft( [ S(1:9) ; zeros(323,1) ; S(10:17) ] )
```

S **9.4.**

$$S_k = \begin{cases} 1, & |k| \le M; \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$s(t) = \sum_{k=-M}^{M} e^{jk\Omega_0 t} = 1 + 2 \cdot \sum_{k=1}^{M} \cos(k\Omega_0 t)$$

Following the hint, we set $z = e^{j\Omega_0 t}$ and use the geometric sum formula in (*) below:

$$\begin{aligned} s(t) &= \sum_{k=-M}^{M} z^k \\ &\overset{(*)}{=} z^{-M} \cdot \frac{1 - z^{2M+1}}{1 - z} \\ &= \frac{z^{-M-1/2}}{z^{-1/2}} \cdot \frac{1 - z^{2M+1}}{1 - z} \\ &= \frac{z^{M+1/2} - z^{-M-1/2}}{z^{1/2} - z^{-1/2}} \end{aligned}$$

Since

$$\begin{aligned} z^{M+1/2} - z^{-M-1/2} &= 2j \sin((M + 1/2)\Omega_0 t) \\ z^{1/2} - z^{-1/2} &= 2j \sin(\Omega_0 t/2) \end{aligned}$$

we obtain

$$s(t) = \frac{\sin((M + 1/2)\Omega_0 t)}{\sin(\Omega_0 t/2)}$$

This is the Dirichlet kernel introduced in the detection of sinusoids using the DFT, i.e., $s(t) = \mathcal{D}_{2M+1}(\Omega_0 t)$.

S **9.5.** Since $S_k = 2^{-|k|}$ for all k, we have

$$\begin{aligned} s(t) &= \sum_{k=-\infty}^{\infty} 2^{-|k|} e^{jk\Omega_0 t} \\ &= 1 + \sum_{k=1}^{\infty} 2^{-k} e^{jk\Omega_0 t} + \sum_{k=1}^{\infty} 2^{-k} e^{-jk\Omega_0 t} \end{aligned}$$

(Using real-valued sinusoids, $s(t) = 1 + \sum_{k\geq 1} 2^{-k+1}\cos(k\Omega_0 t)$.)

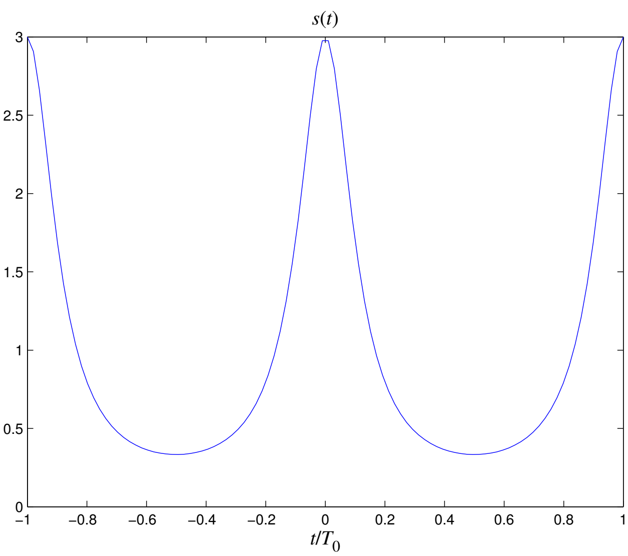
Both infinite geometric sums converge, since their common ratios $e^{j\Omega_0 t}/2$ and $e^{-j\Omega_0 t}/2$ have modulus less than unity. Using the summation formula

$$\sum_{k=1}^{\infty} z^k \;=\; z\cdot\sum_{k=0}^{\infty} z^k \;=\; \frac{z}{z-1}$$

(valid for $|z| < 1$),we obtain

$$\begin{aligned} s(t) &= 1 + \frac{1}{2}\cdot\frac{e^{j\Omega_0 t}}{1-\frac{1}{2}e^{j\Omega_0 t}} + \frac{1}{2}\cdot\frac{e^{-j\Omega_0 t}}{1-\frac{1}{2}e^{-j\Omega_0 t}} \\ &= 1 + \frac{2(e^{j\Omega_0 t}+e^{-j\Omega_0 t})-2}{5-2(e^{j\Omega_0 t}+e^{-j\Omega_0 t})} \\ &= 1 + \frac{4\cos(\Omega_0 t)-2}{5-4\cos(\Omega_0 t)} \\ &= \frac{3}{5-4\cos(\Omega_0 t)} \end{aligned}$$

The signal is plotted below (over two periods).

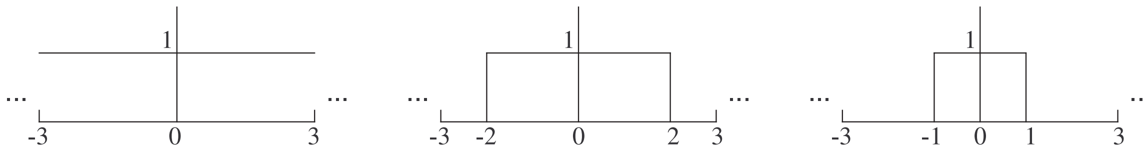


S **9.5. (i)** $\Omega_0 = \pi/3$, hence $T = 6$.

$s(t)$ is a linear combination of a constant and two rectangular pulse trains whose Fourier series coefficients are given by

$$\frac{\sin(k\pi/3)}{k\pi} \qquad \text{and} \qquad \frac{\sin(2k\pi/3)}{k\pi}$$

The corresponding duty factors are 1/3 and 2/3.



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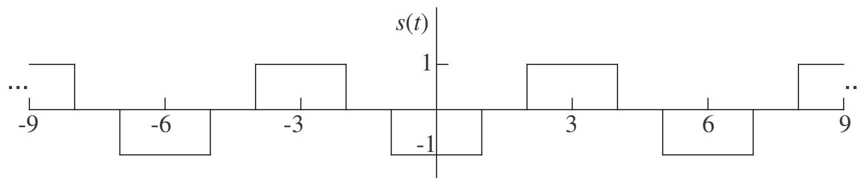
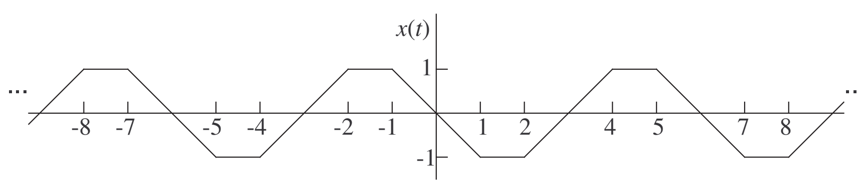


(ii) We have, for $-3 < t \leq 3$,

$$x(t) \;=\; \int_{-3}^{t} s(\tau)\, d\tau$$

Thus $x(t)$ is the area under the $s(\cdot)$ graph over the interval $(-3, t]$. This function

- increases linearly (in t) for $-3 \leq t \leq 2$;
- remains constant for $-2 < t \leq -1$;
- decreases linearly for $-1 < t \leq 1$;
- remains constant for $1 < t \leq 2$;
- increases linearly for $2 < t \leq 3$.



(iii) The signal $x(t)$ is odd-symmetric, i.e., $x(-t) = -x(t)$.
There are no discontinuities in $x(t)$. This is because

$$x(3) \;=\; \int_{-3}^{3} s(\tau)\, d\tau \;=\; 6S_0 \;=\; 0$$

(That the mean value of $s(t)$ equals zero is easily seen from its graph.) Thus $x(3)$ equals the limit of $x(t)$ as $t \downarrow -3$, and the periodic extension will have no discontinuities at the points where the graph segments adjoin, i.e., at $t = -3 + 6n$.

Clearly,

$$s(t) \;=\; \frac{dx(t)}{dt}$$