ENEE 222 0201/2 HOMEWORK ASSIGNMENT 4 Due Tue 03/04/14

Problem 4A

Review the concept of a rotation matrix (e.g., p. 60 in the textbook). Show that any matrix of the form

$$\begin{bmatrix} r & -s \\ s & r \end{bmatrix} \qquad (r, s \in \mathbf{R})$$

represents a counterclockwise rotation on the plane, preceded or followed by scaling (the scaling factor being nonnegative). Express the angle of rotation and the scaling factor in terms of r and s.

In what follows, consider two linear transformations $\mathbf{R}^2 \to \mathbf{R}^2$ with matrices given by $\mathbf{A}$ and $\mathbf{B}$, such that

$$\mathbf{A}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}; \qquad \mathbf{A}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

and

$$\mathbf{B}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 17 \end{bmatrix}; \qquad \mathbf{B}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 19 \\ 22 \end{bmatrix}$$

(ii) (5 pts.) Determine $\mathbf{A}$ and $\mathbf{B}$. Verify that both matrices are of the form given in part (i).

(iii) (5 pts.) Evaluate the products $\mathbf{E} = \mathbf{AB}$ and $\mathbf{F} = \mathbf{BA}$. Are they equal? Are $\mathbf{E}$ and $\mathbf{F}$ of the form given in part (i), and if so, what are the corresponding rotation angles and scaling factors?

(iv) (5 pts.) Without any further algebra, display a matrix $\mathbf{G}$ such that

$$\mathbf{GE} = \mathbf{GF} = \mathbf{I}$$

(where $\mathbf{I}$ is the 2×2 identity matrix).

Problem 4B

Let

$$\mathbf{A} = \begin{bmatrix} 0 & a & 1 & b \\ c & 1 & d & 0 \\ 1 & e & 0 & f \\ g & 0 & h & 1 \end{bmatrix}$$

In each of the following cases, find matrices $\mathbf{P}$ and $\mathbf{Q}$ such that $\mathbf{PAQ}$ equals the matrix shown. (Note that if either $\mathbf{P}$ or $\mathbf{Q}$ is not needed, it can be set equal to $\mathbf{I}$.)

(i) $\begin{bmatrix} h & 1 & g & 0 \\ 0 & f & 1 & e \\ d & 0 & c & 1 \\ 1 & b & 0 & a \end{bmatrix}$

(ii) $\begin{bmatrix} 0 & h & 1 & g \\ a & 1 & b & 0 \\ 1 & d & 0 & c \\ e & 0 & f & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} g & 1 & g & 1 \\ 0 & b & 0 & b \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & a+e & 1 & b+f \end{bmatrix}$

(v) $\begin{bmatrix} d-e+1 \end{bmatrix}$ $(1 \times 1$, i.e., a scalar$)$

Problem 4C

Solve by hand without using your calculator. Show all intermediate steps.

Let

$$\mathbf{U} = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 9 & -6 & -5 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

(i) (6 pts.) Solve $\mathbf{U}\mathbf{x} = \mathbf{b}$ for an arbitrary vector $\mathbf{b}$. Display $\mathbf{U}^{-1}$.

(ii) (4 pts.) Express the matrix

$$\mathbf{G} = \begin{bmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 3 & 12 \\ -2 & 2 & -2 & -2 \\ 0 & 9 & -6 & -5 \end{bmatrix}$$

in the form $\mathbf{G} = \mathbf{DPU}$, where $\mathbf{D}$ is a diagonal matrix, $\mathbf{P}$ is a permutation matrix and $\mathbf{U}$ is as given above. Determine $\mathbf{G}^{-1}$.

(iii) (6 pts.) If

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 3 & 6 & -3 & -2 \\ 3 & -3 & 4 & 7 \\ 6 & 3 & 1 & 3 \end{bmatrix}$$

and $\mathbf{b} = \begin{bmatrix} 1+3c & 2 & 4 & 5 \end{bmatrix}^T$, solve the equation $\mathbf{Ax} = \mathbf{b}$ using Gaussian elimination (without row interchanges).

(iv) (4 pts.) Show that $\mathbf{A} = \mathbf{LU}$, where $\mathbf{L}$ is a lower triangular matrix and $\mathbf{U}$ is as above. Also, determine $\mathbf{L}$. (*Hint*: You know $\mathbf{U}^{-1}$ from part (i).)

Solved Examples

S **4.1**. (*P* **2.6** in textbook). Compute by hand the matrix product $\mathbf{AB}$ in the following cases:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 1 & 5 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} -2 & -3 & 3 \\ 4 & 0 & 7 \end{bmatrix};$$

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 4 & 0 \\ 3 & -5 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 2 & -2 & 3 \\ 0 & 4 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

S **4.2** (*P* **2.7** in textbook). If

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

find vectors $\mathbf{u}$ and $\mathbf{v}$ such that

$$\mathbf{u}^T\mathbf{A}\mathbf{v} = b$$

S **4.3**. (*P* **2.10** in textbook). Let

$$\mathbf{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

If $\mathbf{P}$ and $\mathbf{Q}$ are 4×4 matrices such that

$$\mathbf{P}\mathbf{x} = \begin{bmatrix} c \\ a \\ b \\ d \end{bmatrix} \quad \text{and} \quad \mathbf{Q}\mathbf{x} = \begin{bmatrix} a \\ c \\ d \\ b \end{bmatrix}$$

determine the product $\mathbf{PQ}$.

S **4.4**. (*P* **2.11** in textbook). Matrices $\mathbf{A}$ and $\mathbf{B}$ are generated in MATLAB using the commands

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A = [1:3; 4:6; 7:9] ;
B = [9:-1:7; 6:-1:4; 3:-1:1] ;
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Find matrices $\mathbf{P}$ and $\mathbf{Q}$ such that $\mathbf{B} = \mathbf{PAQ}$.

S **4.5**. (*P* **2.13** in textbook). Express each of the matrices

$$\mathbf{A} = \begin{bmatrix} a & b & c & d \\ 2a & 2b & 2c & 2d \\ -a & -b & -c & -d \\ -2a & -2b & -2c & -2d \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & t & t^2 & t^3 \\ t^{-1} & 1 & t & t^2 \\ t^{-2} & t^{-1} & 1 & t \\ t^{-3} & t^{-2} & t^{-1} & 1 \end{bmatrix}$$

as a product of a row vector and a column vector. Using your result, show that the product $\mathbf{AB}$ can be also expressed as a product of a row vector and a column vector.

The following examples make mention of the LU factorization of a matrix, obtained by Gaussian elimination. The lower triangular matrix is formed used the multipliers obtained in the forward elimination phase, by placing them below the leading diagonal of an identity matrix after reversing their signs. The solution of Ly=b can be incorporated into the forward elimination phase.

S **4.6**. The matrix $\mathbf{A}$ can be factored as $\mathbf{A} = \mathbf{LU}$, where

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{U} = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -26 \end{bmatrix}$$

(i) Obtain the matrix $\mathbf{A}$.

(ii) Solve $\mathbf{Ax} = \mathbf{b}$, where $\begin{bmatrix} 4 & 1 & -3 & 4 \end{bmatrix}^T$, by means of the two triangular systems of equations

$$\mathbf{Ly} = \mathbf{b} \qquad \text{and} \qquad \mathbf{Ux} = \mathbf{y}$$

S **4.7**. **(i)** Determine the LU factorization of

$$\mathbf{A} = \begin{bmatrix} 2 & 7 & -7 & -3 \\ 1 & 5 & -5 & -3 \\ 2 & 3 & 1 & 3 \\ 1 & 4 & -2 & 2 \end{bmatrix}$$

(ii) Solve $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{b} = [10 \;\; 5 \;\; -6 \;\; 3]^T$, by means of the two triangular systems

$$\mathbf{Ly} = \mathbf{b} \qquad \text{and} \qquad \mathbf{Ux} = \mathbf{y}$$