Surgery theory today — what it is and where it's going

Jonathan Rosenberg*

Introduction

This paper is an attempt to describe for a general mathematical audience what surgery theory is all about, how it is being used today, and where it might be going in the future. I have not hesitated to express my personal opinions, especially in Sections 1.2 and 4, though I am well aware that many experts would have a somewhat different point of view. Why such a survey now? The main outlines of surgery theory on compact manifolds have been complete for quite some time now, and major changes to this framework seem unlikely, even though better proofs of some of the main theorems and small simplifications here and there are definitely possible. On the other hand, when it comes to applications of surgery theory, there have been many important recent developments in different directions, and as far as I know this is the first attempt to compare and contrast many of them.

To keep this survey within manageable limits, it was necessary to leave out a tremendous amount of very important material. So I needed to come up with selection criteria for deciding what to cover. I eventually settled on the following:

1. My first objective was to get across the major ideas of surgery theory in a non-technical way, even if it meant skipping over many details and definitions, or even oversimplifying the statements of major theorems.

2. My second objective was to give the reader some idea of the many areas in which the theory can be applied.

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3. Finally, in the case of subjects covered elsewhere (and more expertly) in these volumes, I included a pointer to the appropriate article(s) but did not attempt to go into details myself.

I therefore beg the indulgence of the experts for the fact that some topics are covered in reasonable detail and others are barely mentioned at all. I also apologize for the fact that the bibliography is very incomplete, and that I did not attempt to discuss the history of the subject or to give proper credit for the development of many important ideas. To give a complete history and bibliography of surgery would have been a very complicated enterprise and would have required a paper at least three times as long as this one.

I would like to thank Sylvain Cappell, Karsten Grove, Andrew Ranicki, and Shmuel Weinberger for many helpful suggestions about what to include (or not to include) in this survey. But the shortcomings of the exposition should be blamed only on me.

1 What is surgery?

1.1 The basics

Surgery is a procedure for changing one manifold into another (of the same dimension n) by excising a copy of $S^r \times D^{n-r}$ for some r, and replacing it by $D^{r+1} \times S^{n-r-1}$, which has the same boundary, $S^r \times S^{n-r-1}$. This seemingly innocuous operation has spawned a vast industry among topologists. Our aim in this paper is to outline some of the motivations and achievements of surgery theory, and to indicate some potential future developments.

The classification of surfaces is a standard topic in graduate courses, so let us begin there. A surface is a 2-dimensional manifold. The basic result is that compact connected oriented surfaces, without boundary, are classified up to homeomorphism by the genus g (or equivalently, by the Euler characteristic $\chi = 2-2g$). Recall that a surface of genus g is obtained from the sphere S^2 by attaching g *handles*. The effect of a surgery on $S^0 \times D^2$ is to attach a handle, and of a surgery on $S^1 \times D^1$ is to remove a handle. (See the picture on the next page.) Thus, from the surgery theoretic point of view, the genus g is the minimal number of surgeries required either to obtain the surface from a sphere, or else, starting from the given surface, to remove all the handles and reduce to the sphere S^2. There is a similar surgery interpretation of the classification in the nonorientable case, with S^2 replaced by the projective plane $\mathbb{RP}^2$.

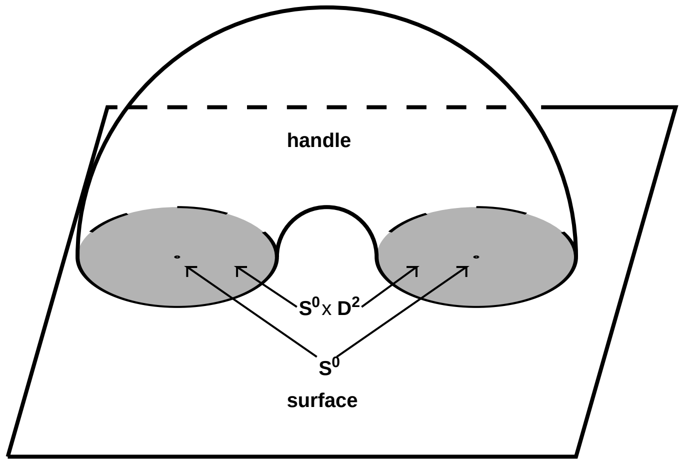

Figure 1. Surgery on an embedded $S^0 \times D^2$.

In dimension $n = 2$, one could also classify manifolds up to homeomorphism by their fundamental groups, with $2g$ the minimal number of generators (in the orientable case). But for every $n \geq 4$, *every* finitely presented group arises as the fundamental group of a compact n-manifold.[1] It is not possible to classify finitely presented groups. Indeed, the problem of determining whether a finite group presentation yields the trivial group or not, is known to be undecidable. Thus there is no hope of a complete classification of *all* n-manifolds for $n \geq 4$. Nevertheless, in many cases it is possible to use surgery to classify the manifolds within a given homotopy type, or even with a fixed fundamental group (such as the trivial group).

Just as for surfaces, high-dimensional manifolds are built out of handles. (In the smooth category, this follows from Morse theory [14]. In the topological category, this is a deep result of Kirby and Siebenmann [11].) Again, each handle attachment or detachment is the result of a surgery. That is why surgery plays such a major role in the classification of manifolds. But since the same manifold may have many quite different handle decompositions, one needs an effective calculus for keeping track of the effect of many surgeries. This is what usually goes under the name of *surgery*

[1] This fact is easy to prove using surgery. Suppose one is given a group presentation $\langle x_1, \ldots, x_k \mid w_1, \ldots, w_s \rangle$. Start with the manifold $M_1 = (S^1 \times S^{n-1}) \# \cdots \# (S^1 \times S^{n-1})$ (k factors), whose fundamental group is a free group on k generators x_i. Then for each word w_j in the generators, represent this word by an embedded circle (this is possible by the [easy] Whitney embedding theorem since $n \geq 3$). This circle has trivial normal bundle since M_1 is orientable, so perform a surgery on a tube $S^1 \times D^{n-1}$ around the circle to kill off w_j. The restriction $n \geq 4$ comes in at this point since it means that the copies of S^{n-2} introduced by the surgeries do not affect π_1. The final result is an n-manifold M with the given fundamental group.

theory.

1.2 Successes

Surgery theory has had remarkable successes. Here are some of the highlights:

- the discovery and classification of exotic spheres (see [107] and [94]);

- the characterization of the homotopy types of differentiable manifolds among spaces with Poincaré duality of dimension ≥ 5 (Browder and Novikov; see in particular [33] for an elementary exposition);

- Novikov's proof of the topological invariance of the rational Pontrjagin classes (see [110]);

- the classification of "fake tori" (by Hsiang-Shaneson [82] and by Wall [25]) and of "fake projective spaces" (by Wall [25], also earlier by Rothenberg [unpublished] in the complex case): manifolds homotopy-equivalent to tori and projective spaces;

- the disproof by Siebenmann [11] of the manifold Hauptvermutung, the [false] conjecture that homeomorphic piecewise linear manifolds are PL-homeomorphic [21];

- Kirby's proof of the Annulus conjecture and the work of Kirby and Siebenmann characterizing which topological manifolds (of dimension > 4) admit a piecewise linear structure [11];

- the characterization (work of Wall, Thomas, and Madsen [101]) of those finite groups that can act freely on spheres (the "topological space form problem" — see Section 3.5 below);

- the construction and partial classification (by Cappell, Shaneson, and others) of "nonlinear similarities" (see 3.4.5 below), that is, linear representations of finite groups which are topologically conjugate but not linearly equivalent;

- Freedman's classification of all simply-connected *topological* 4-manifolds, up to homeomorphism [63]. (This includes the 4-dimensional topological Poincaré conjecture, the fact that all 4-dimensional homotopy spheres are homeomorphic to S^4, as a special case.) For a survey of surgery theory as it applies to 4-manifolds, see [90].

- the proof of Farrell and Jones [55] of topological rigidity of compact locally symmetric spaces of non-positive curvature.

The main drawback of surgery theory is that it is necessarily quite complicated. Fortunately, one does not need to know everything about it in order to use it for many applications.

1.3 Dimension restrictions

As we have defined it, surgery is applicable to manifolds of all dimensions, and works quite well in dimension 2. The surgery theory novice is therefore often puzzled by the restriction in many theorems to the case of dimension ≥ 5. In order to do surgery on a manifold, one needs an embedded product of a sphere (usually in a specific homology class) and a disk. By the Tubular Neighborhood Theorem, this is the same as finding an embedded sphere with a trivial normal bundle. The main tool for constructing such spheres is the [strong] Whitney embedding theorem [143], which unfortunately fails for embeddings of surfaces into [smooth] 4-manifolds.[2] This is the main source of the dimensional restrictions. Thus Smale was able to prove the h-cobordism theorem in dimensions ≥ 5, a recognition principle for manifolds, as well as the high-dimensional Poincaré conjecture, by repeated use of Whitney's theorem (and its proof). (See [15] for a nice exposition.) The h-cobordism theorem was later generalized by Barden, Mazur, and Stallings [88] to the s-cobordism theorem for non-simply connected manifolds. This is the main tool, crucial for future developments, for recognizing when two seemingly different homotopy-equivalent manifolds are isomorphic (in the appropriate category, TOP, PL, or DIFF). The s-cobordism theorem is known to fail for 3-manifolds (where the cobordisms involved are 4-dimensional), at least in the category TOP [39], and for 4-manifolds, at least in the category DIFF (by Donaldson or Seiberg-Witten theory). Nevertheless, Freedman ([63], [8]) was able to obtain remarkable results on the *topological* classification of 4-manifolds by proving a version of Whitney's embedding theorem in the 4-dimensional topological category, with some restrictions on the fundamental group. This in turn has led [64] to an s-cobordism theorem for 4-manifolds in TOP, provided that the fundamental groups involved have subexponential growth.

[2]The "easy" Whitney embedding theorem, usually proved in a first course on differential topology, asserts that if M^m is a smooth compact manifold, then embeddings are dense in the space of smooth maps from M into any manifold N^n of dimension $n \geq 2m+1$. The "hard" embedding theorem, which is considerably more delicate, improves this by asserting in addition that any map $M^m \to N^{2m}$ is homotopic to an embedding, provided that $m \neq 2$ and N is simply connected. This fails for smooth manifolds when $m=2$, since it is a consequence of Donaldson theory that some classes in π_2 of a simply connected smooth 4-manifold may not be represented by smoothly embedded spheres. In fact, the "hard" embedding theorem also fails in the topological locally flat category when $m=2$.

2 Tools of surgery

2.1 Fundamental group

The first topic one usually learns in algebraic topology is the theory of the fundamental group and covering spaces. In surgery theory, this plays an even bigger role than in most other areas of topology. Proper understanding of manifolds requires taking the fundamental group into account everywhere. As we mentioned before, any finitely presented group is the fundamental group of a closed manifold, but many interesting results of surgery theory only apply to a limited class of fundamental groups.

2.2 Poincaré duality

Any attempt to understand the structure of manifolds must take into account the structure of their homology and cohomology. The main phenomenon here is Poincaré[-Lefschetz] duality. For a compact oriented manifold M^n, possibly with boundary, this asserts that the cap product with the fundamental class $[M, \partial M] \in H_n(M, \partial M; \mathbb{Z})$ gives an isomorphism

$$H^j(M; \mathbb{Z}) \xrightarrow{\cong} H_{n-j}(M, \partial M; \mathbb{Z}) . \qquad \text{(eq. 2.1)}$$

This algebraic statement has important geometric content — it tells homologically how submanifolds of M intersect.

For surgery theory, one needs the generalization of Poincaré duality that takes the fundamental group π into account, using homology and cohomology with coefficients in the group ring $\mathbb{Z}\pi$. Or for work with non-orientable manifolds, one needs a still further generalization involving a twist by an orientation character $w : \pi \to \mathbb{Z}/2$. The general form is similar to that in equation (eq. 2.1): one has a fundamental class $[M, \partial M] \in H_n(M, \partial M; \mathbb{Z}, w)$ and an isomorphism

$$H^j(M; \mathbb{Z}\pi) \xrightarrow{\cong} H_{n-j}(M, \partial M; \mathbb{Z}\pi, w) . \qquad \text{(eq. 2.2)}$$

2.3 Hands-on geometry

One of Wall's great achievements ([25], Chapter 5), which makes a general theory of non-simply connected surgery possible, is a characterization of when homology classes up to the middle dimension, in a manifold of dimension ≥ 5, can be represented by spheres with trivial normal bundles. This requires several ingredients. First is the Hurewicz theorem, which says that a homology class in the smallest degree where homology is nontrivial comes from the corresponding homotopy group, in other words, is represented by a map from a sphere. The next step is to check that this map is homotopic to an embedding, and this is where [143] comes in. The third step requires keeping track of the normal bundle, and thus leads us to the next major tool:

2.4 Bundle theory

If X is a compact space such as a manifold, the m-dimensional real vector bundles over X are classified up to isomorphism by the homotopy classes of maps from X into $BO(m)$, the limit (as $k \to \infty$) of the Grassmannian of m-dimensional subspaces of $\mathbb{R}^{m+k}$. Identifying bundles which become isomorphic after the addition of trivial bundles gives the classification up to *stable isomorphism*, and amounts to replacing $BO(m)$ by $BO = \varinjlim BO(m)$. This has the advantage that $[X, BO]$, the set of homotopy classes of maps $X \to BO$, is given by $\widetilde{KO}(X)$, a cohomology theory. A basic fact is that if m exceeds the dimension of X, then one is already in the *stable range*, that is, the isomorphism classification of rank-m bundles over X coincides with the stable classification. Furthermore, if X is a manifold, then all embeddings of X into a Euclidean space of sufficiently high dimension are isotopic, by the [easy] Whitney embedding theorem, and so the normal bundle of X (for an *arbitrary* embedding into a Euclidean space or a sphere) is determined up to stable isomorphism. Thus it makes sense to talk about the *stable normal bundle*, which is stably an inverse to the tangent bundle (since the direct sum of the normal and tangent bundles is the restriction to X of the tangent bundle of Euclidean space, which is trivial).

Now consider a sphere S^r embedded in a manifold M^n. If $2r < n$, then the normal bundle of S^r in M^n has dimension $m = n - r > r$ and so is in the stable range, and hence is trivial if and only if it is stably trivial. Furthermore, since the tangent bundle of S^r is stably trivial, this happens exactly when the restriction to S^r of the stable normal bundle of M^n is trivial. If $2r = n$, i.e., we are in the middle dimension, then things are more complicated. If M is oriented, then the Euler class of the normal bundle of S^r becomes relevant.

2.5 Algebra

Poincaré duality, as discussed above in Section 2.2, naturally leads to the study of quadratic forms over the group ring $\mathbb{Z}\pi$ of the fundamental group π. These are the basic building blocks for the definition of the surgery obstruction groups $L_n(\mathbb{Z}\pi)$, which play a role in both the existence problem (when is a space homotopy-equivalent to a manifold?) and the classification problem (when are two manifolds isomorphic?). For calculational purposes, it is useful to define the L-groups more generally, for example, for arbitrary rings with involution, or for certain categories with an involution.

The groups that appear in surgery theory are then important special cases, but are calculated by relating them to the groups for other situations (such as semisimple algebras with involution over a field). In fact the surgery obstruction groups for finite fundamental groups have been completely calculated this way, following a program initiated by Wall (e.g., [136]). For more details on the definition and calculation of the surgery obstruction groups by algebraic methods, see the surveys [118] and [80].

Algebra also enters into the theory in one more way, via Whitehead torsion (see the survey [106]) and algebraic K-theory. The key issue here is distinguishing between homotopy equivalence and *simple* homotopy equivalence, the kind of homotopy equivalence between complexes that can be built out of elementary contractions and expansions. These two notions coincide for simply connected spaces, but in general there is an obstruction to a homotopy equivalence being simple, called the Whitehead torsion, living in the Whitehead group $\mathrm{Wh}(\pi)$ of the fundamental group π of the spaces involved.[3] This plays a basic role in manifold theory, because of the basic fact that if M^n is a manifold with dimension $n \geq 5$ and fundamental group π, then any element of $\mathrm{Wh}(\pi)$ can be realized by an h-cobordism based on M, in other words, by a manifold W^{n+1} with two boundary components, one of which is equal to M, such that the inclusion of either boundary component into W is a homotopy equivalence. In fact, this is just one part of the celebrated *s-cobordism theorem* [88], which also asserts that the h-cobordisms based on M, up to isomorphism (diffeomorphism if one is working with smooth manifolds, homeomorphism if one is working with smooth manifolds), are in bijection with $\mathrm{Wh}(\pi)$ via the Whitehead torsion of the inclusion $M^n \hookrightarrow W^{n+1}$. The identity element of $\mathrm{Wh}(\pi)$ of course corresponds to the cylinder $W = M \times [0, 1]$. By the topological invariance of Whitehead torsion [43], any homeomorphism between manifolds is necessarily a *simple* homotopy equivalence, so $\mathrm{Wh}(\pi)$ is related to the complexity of the family of homeomorphism classes of manifolds homotopy equivalent to M. In addition, the Whitehead group is important for understanding "decorations" on the surgery obstruction groups, a technical issue we won't attempt to describe here at all.

2.6 Homotopy theory

Homotopy theory enters into surgery theory in a number of different ways. For example it enters indirectly via bundle theory, as indicated in Section 2.4 above. More interestingly, it turns out that surgery obstruction groups can be described as the homotopy groups of certain infinite loop spaces,

[3] The Whitehead group $\mathrm{Wh}(\pi)$ is defined to be the abelianization of the general linear group $GL(\mathbb{Z}\pi) = \varinjlim GL(n, \mathbb{Z}\pi)$, divided out by the "uninteresting" part of this group, generated by the units $\pm 1 \in \mathbb{Z}$ and the elements of π.

related to classifying spaces such as G/O, the study of which becomes important in the most comprehensive approaches to the subject. For this point of view, see [13], [19], [21], and [26].

2.7 Analysis on manifolds

While surgery theory in principle provides an algebraic scheme for classifying manifolds, it is rarely sufficiently explicit so that one can begin with pure algebra and deduce interesting geometric consequences. Usually one has to use the correspondence between geometry and algebra in both directions. One way of using the geometry is through analysis, more specifically, the index theory of certain geometrically defined elliptic differential operators, such as the signature operator. For details of how this matches up with surgery theory, see [121] and [120].

2.8 Controlled topology

Another tool which is not needed for the "classical" theory of surgery, but which is playing an increasingly important role in current work, is controlled topology, by which we mean topology in which one keeps track of "how far" things are allowed to move. This idea, introduced into surgery theory by Chapman, Ferry, and Quinn, has played an important role in the work of many surgery theorists, and is especially important in dealing with non-compact manifolds. But as an example of how it can be applied to compact manifolds, suppose one has a homeomorphism $h\colon M_1^n \to M_2^n$ between compact smooth manifolds, and one wants to know how the smooth invariants (for example, the Pontrjagin classes) of the two manifolds M_1 and M_2 can differ from one another. One way of approaching this, which can be used to prove Novikov's theorem that h^* preserves *rational* Pontrjagin classes, is to observe that we can approximate h as well as we like by a smooth map h'. Now h' will not necessarily be invertible in DIFF (otherwise M_1 and M_2 would be diffeomorphic), but it is a homotopy equivalence. In fact, given $\varepsilon > 0$, we can choose h' and k and homotopies from $k \circ h'$ to id_{M_1} and from $k \circ h'$ to id_{M_1} which move points by no more than ε (with respect to choices of metrics). Or in other words, we can approximate h by a *controlled homotopy equivalence* in the category DIFF. In the other direction, in dimension $n > 4$, Chapman and Ferry showed that any controlled homotopy equivalence is homotopic to a homeomorphism [44]. For more on controlled surgery, see [59], [111], and [112].

3 Areas of application

3.1 Classification of manifolds

The most important application of surgery theory, the one for which the theory was invented, is the classification of manifolds and manifold structures. This begins with the existence problem for manifold structures: when is a given finite complex X homotopy equivalent to a manifold? An obvious prerequisite is that X satisfy Poincaré duality for some dimension n, in the generalized sense of equation (eq. 2.2) above. When this is the case, we call X an n-dimensional *Poincaré space* or *Poincaré complex*. This insures that X has a "homotopy-theoretic stable normal bundle," the Spivak spherical fibration ν. The Browder-Novikov solution to the existence problem, as systematized in [25], then proceeds in two more steps. First one must check if the Spivak fibration is the reduction of a genuine bundle ξ (in the appropriate category, TOP, PL, or DIFF). If it isn't, then X is not homotopy equivalent to a manifold. If it is, then given ξ reducing to ν, one finds by transversality a *degree-one normal map* $(M, \eta) \to (X, \xi)$, in other words, a manifold M with stable normal bundle η, together with a degree-one map $M \to X$ covered by a bundle map $\eta \to \xi$. The gadget $(M, \eta) \to (X, \xi)$ is also called a *surgery problem*. One needs to check whether it is possible to do surgery on M, keeping track of the bundle data as one goes along, in order to convert M to a manifold N (simple) homotopy equivalent to X. Here one needs an important observation of Browder and Novikov (which follows easily from Poincaré duality): for a degree-one map of Poincaré spaces, the induced map on homology is split surjective. So it is enough to try to kill off the homology kernel. This is done working up from the bottom towards the middle dimension, at which point an obstruction appears, the *surgery obstruction* $\sigma((M, \eta) \to (X, \xi))$ of the surgery problem, which lies in the group $L_n(\mathbb{Z}\pi)$, π the fundamental group of X.

Uniqueness of manifold structures is handled by the relative version of the same construction. Given a simple homotopy equivalence of n-dimensional manifolds $h\colon M \to X$, one must check if the stable normal bundle of X pulls back under h to the stable normal bundle of M. If it doesn't, h cannot be homotopic to an isomorphism. If it does, M and N are normally cobordant, and one attempts to do surgery on a cobordism W^{n+1} between them in order to convert W to an s-cobordism (a cobordism for which the inclusion of either boundary component is a simple homotopy equivalence). Again a surgery obstruction appears, this time in $L_{n+1}(\mathbb{Z}\pi)$. If the obstruction vanishes and we can convert W to an s-cobordism, the s-cobordism theorem says that the map $M \to X$ is homotopic to an isomorphism (again, in the appropriate category). The upshot of this analysis is best formulated in terms of the *surgery exact sequence* of Sullivan and

Wall,

$$\cdots \xrightarrow{\sigma} L_{n+1}(\mathbb{Z}\pi) \overset{\omega}{\dashrightarrow} \mathcal{S}(X) \longrightarrow \mathcal{N}(X) \xrightarrow{\sigma} L_n(\mathbb{Z}\pi). \qquad \text{(eq. 3.1)}$$

discussed in greater detail in [118] and [33]. This long exact sequence relates three different items:

1. the *structure set* $\mathcal{S}(X)$ of the Poincaré complex X, which measures the number of distinct manifolds (up to the appropriate notion of isomorphism) in the simple homotopy class of X
2. normal data $\mathcal{N}(X)$, essentially measuring the possible characteristic classes of the normal or the tangent bundle of manifolds in the simple homotopy type of X; and
3. the Wall surgery groups $L_n(\mathbb{Z}\pi)$, depending only on the fundamental group π of X and the dimension n (modulo 4) (plus the orientation character w, in the non-orientable case).

The map σ sends a surgery problem to its surgery obstruction. Note incidentally that as $\mathcal{S}(X)$ is simply a set, not a group, the meaning of the exact sequence is that for $x \in \mathcal{N}(X)$, $\sigma(x) = 0$ if and only if $x \in \text{im}\,(\mathcal{S}(X))$, and ω denotes an *action* of $L_{n+1}(\mathbb{Z}\pi)$ on $\mathcal{S}(X)$ such that if a, $b \in \mathcal{S}(X)$, a and b map to the same element of $\mathcal{N}(X)$ if and only if there is a $c \in L_{n+1}(\mathbb{Z}\pi)$ such that $\omega(c, a) = b$.

3.2 Similarities and differences between categories: TOP, PL, and DIFF

At this point it is necessary to say something about the different categories of manifolds. So far we have implicitly been working in the category DIFF of smooth manifolds, since it is likely to be more familiar to most readers than the categories TOP and PL of topological and piecewise linear manifolds. However, surgery works just as well, and in fact in some ways better, in the other categories. We proceed to make this precise.

In the smooth category, except in low dimensions, most closed manifolds have non-trivial structure sets (or in other words, there are usually plenty of non-diffeomorphic manifolds of the same homotopy type). This phenomenon first showed up in the work of Milnor and Milnor-Kervaire on exotic spheres (see [107], [94]). From the point of view of the surgery exact sequence (eq. 3.1), it is due to the rather complicated nature of the normal data term, $\mathcal{N}(X) = [X, G/O]$, and its relationship with the J-homomorphism $BO \to BG$.

In the piecewise linear category, things tend to be somewhat simpler, as one can already see from looking at homotopy spheres. In the category

DIFF, the homotopy spheres of a given dimension $n > 4$, up to isomorphism, form a finite abelian group Θ_n under the operation of connected sum $\#$, and the order of Θ_n is closely related to the Bernoulli numbers. (See [94] for more details.) But in the PL category, Smale's proof [15] of the h-cobordism theorem shows that all homotopy spheres of a fixed dimension $n > 4$ are PL isomorphic to one another. What accounts for this is the "Alexander trick," the fact that if two disks D^n are glued together by a PL isomorphism of their boundaries, then one can extend the gluing map (by linear rescaling) to all of one of the disks, and thus the resulting homotopy sphere is standard. From the point of view of the exact sequence (eq. 3.1), we can explain this by noting that the normal data term $\mathcal{N}(X) = [X, G/\mathrm{PL}]$ is smaller than in the category DIFF. In fact, after inverting 2, it turns out (a theorem of Sullivan) that G/PL becomes homotopy equivalent to a more familiar space, the classifying space BO for real K-theory [13]. This fact is not obvious, of course; it is itself a consequence of surgery theory.

In the category TOP of topological manifolds, the work of Kirby and Siebenmann [11] makes it possible to carry over everything we have done so far. In fact, their work shows that (in dimensions $\neq 4$), there is very little difference between the categories PL and TOP. What difference there is comes from Rochlin's Theorem in dimension 4, which says that a smooth (or PL) spin manifold of dimension 4 must have signature divisible by 16. (For present purposes, we can define "spin" in the PL and TOP categories to mean that the first two Stiefel-Whitney classes vanish.) In contrast, the work of Freedman [63] shows there are closed spin 4-manifolds in TOP with signature 8. This difference (between 8 and 16) accounts for a single $\mathbb{Z}/2$ difference between the homotopy groups of $B\mathrm{PL}$ and $B\mathrm{TOP}$: $\mathrm{TOP}/\mathrm{PL} \simeq K(\mathbb{Z}/2, 3)$. This turns out to be just enough of a difference to make surgery work even better in TOP than in PL.

To explain this, we need not only the surgery obstruction groups, but also the surgery spectra $\mathbb{L}(\mathbb{Z}\pi)$, of which the surgery obstruction groups are the homotopy groups. These spectra are discussed in detail in [19]; suffice it to say here that they are constructed out of parameterized families of surgery problems. Then the fact that surgery works so well in the category TOP may be summarized by saying that in this category, the normal data term $\mathcal{N}(X)$ is basically just the homology of X with coefficients in $\mathbb{L}(\mathbb{Z})$. Furthermore, obstruction theory gives us a classifying map $X \xrightarrow{c} B\pi$ for the universal cover of X, and the obstruction map σ in the exact sequence (eq. 3.1) is the composite of c_* with the map induced on homotopy groups by an *assembly map* $B\pi_+ \wedge \mathbb{L}(\mathbb{Z}) \to \mathbb{L}(\mathbb{Z}\pi)$. This point of view then makes it possible (when $\mathcal{S}^{\mathrm{TOP}}(X)$ is non-empty) to view the structure set $\mathcal{S}^{\mathrm{TOP}}(X)$ as the zero-th homotopy group of still another spectrum, and thus to put a group structure on $\mathcal{S}^{\mathrm{TOP}}(X)$. (See [19], §18.) When this is done, ω in the exact sequence (eq. 3.1) becomes a group homomorphism, and the whole exact sequence becomes an exact sequence of abelian groups.

3.3 Immediate consequences

This is a good point to give some concrete examples of immediate consequences of the surgery classification of manifolds. Some of these follow from the general form of the theory, and do not require any specific calculations. For example, we have (in any of the three categories DIFF, PL, and TOP):

Proposition 3.1 *Suppose* $f\colon (M,\eta) \to (X,\xi)$ *is a surgery problem, that is, a degree-one normal map, in any of the categories* DIFF, PL, *or* TOP. *(Here* M^n *is a compact manifold and* X^n *is a Poincaré complex. We allow the case where* M *and* X *have boundaries, in which case all constructions are to be done rel boundaries.) Then the surgery obstruction of* $f \times \mathrm{id}\colon M \times \mathbb{CP}^2 \to X \times \mathbb{CP}^2$ *is the same as for* f*, and the surgery obstruction of* $f \times \mathrm{id}\colon M \times S^k \to X \times S^k$ *vanishes for* $k > 1$*. In particular, if* $k > 1$ *and* $n + k \geq 5$*, then* $f \times \mathrm{id}\colon M \times S^k \to X \times S^k$ *is normally cobordant to a simple homotopy equivalence, so* $X \times S^k$ *has the simple homotopy type of a compact manifold. And if* $n \geq 5$ *and* $f \times \mathrm{id}\colon M \times \mathbb{CP}^2 \to X \times \mathbb{CP}^2$ *is normally cobordant to a simple homotopy equivalence, then the same is true for* f*.*

Proof. The first statement is the geometric meaning of the periodicity of the surgery obstruction groups. The second statement is a special case of a *product formula* for surgery obstructions, in view of the fact that all signature invariants vanish for a sphere. But it can also be proved directly, using surgery on f below the middle dimension and the fact that a sphere has no homology except in dimension 0 and in the top dimension. (See [33], §1, proofs of Propositions 1.2 and 1.4, for the trick.) □

Remark. The statement of Proposition 3.1 is *false* if we replace S^k, $k > 1$, by S^1. The reason is that taking a product with S^1 has the effect[4] of replacing the fundamental group π by $\pi \times \mathbb{Z}$, and simply shifting the original surgery obstruction up by one in dimension.

Other simple examples of applications of the classification theory that make use of a few elementary facts about G/O, *etc.*, are the following:

Theorem 3.2 *Let* CAT *be* DIFF *or* PL *and let* M^n *be a closed* CAT *manifold of dimension* $n \geq 5$*. Then there are only finitely many* CAT *isomorphism classes of manifolds homeomorphic to* M^n*.*

[4] modulo a "decoration" nuance, which we're ignoring here

Remark. This is definitely false in dimension 4, as follows from Donaldson theory or Seiberg-Witten theory.

Proof. The issue here is to look at the commutative diagram of exact surgery sequences (the top sequence only of pointed sets, the bottom one, as we've explained in §3.2, of groups)

$$\begin{array}{ccccccccc} \mathcal{N}^{\mathrm{CAT}}(M \times I;\, \partial) & \xrightarrow{\sigma} & L_{n+1}(\mathbb{Z}\pi) & \overset{\omega}{\dashrightarrow} & \mathcal{S}^{\mathrm{CAT}}(M) & \longrightarrow & \mathcal{N}^{\mathrm{CAT}}(M) & \xrightarrow{\sigma} & L_n(\mathbb{Z}\pi) \\ \downarrow & & \| & & \downarrow & & \downarrow & & \| \\ \mathcal{N}^{\mathrm{TOP}}(M \times I;\, \partial) & \xrightarrow{\sigma} & L_{n+1}(\mathbb{Z}\pi) & \xrightarrow{\omega} & \mathcal{S}^{\mathrm{TOP}}(M) & \longrightarrow & \mathcal{N}^{\mathrm{TOP}}(M) & \xrightarrow{\sigma} & L_n(\mathbb{Z}\pi), \end{array}$$

and to show that the preimage in $\mathcal{S}^{\mathrm{CAT}}(M)$ of the basepoint in $\mathcal{S}^{\mathrm{TOP}}(M)$ is finite. But the maps $\mathcal{N}^{\mathrm{CAT}}(M) \to \mathcal{N}^{\mathrm{TOP}}(M)$ and $\mathcal{N}^{\mathrm{CAT}}(M \times I;\, \partial) \to \mathcal{N}^{\mathrm{TOP}}(M \times I;\, \partial)$ are finite-to-one since M has finite homotopy type and since the homotopy groups of TOP/CAT are finite (see §3.2 above). So the result follows from diagram chasing. □

Proposition 3.3 *For any $n \geq 3$, there are infinitely many distinct manifolds with the homotopy type of $\mathbb{CP}^n$ (in any of the categories* DIFF, PL, *or* TOP*).*

Proof. Fix a category CAT, one of DIFF, PL, or TOP. We need to show that $\mathcal{S}(\mathbb{CP}^n)$ is infinite. Now $L_k(\mathbb{Z})$ is $\mathbb{Z}$ in dimensions divisible by 4, $\mathbb{Z}/2$ in dimensions 2 mod 4, 0 in odd dimensions. Since the dimension k of $\mathbb{CP}^n$ is even, $L_{k+1}(\mathbb{Z}) = 0$ and $\mathcal{S}(\mathbb{CP}^n)$ can be identified with the kernel (in the sense of maps of pointed sets) of the surgery obstruction map $\sigma\colon \mathcal{N}(\mathbb{CP}^n) \to L_k(\mathbb{Z})$. Now (see §3.2 above for the topological category and [139] for the argument needed to make this work smoothly as well) modulo finite ambiguities, $\mathcal{N}(\mathbb{CP}^n)$ is just $\widetilde{KO}(\mathbb{CP}^n)$, which has rank $\left[\frac{n}{2}\right]$, and σ is given by the formula for the signature coming from the Hirzebruch signature formula. If $n \geq 4$, then $\widetilde{KO}(\mathbb{CP}^n)$ has rank bigger than 1, and if $n = 3$, then $L_k(\mathbb{Z}) = \mathbb{Z}/2$ is finite. So in either case, the kernel of σ is infinite. □

3.4 Classification of group actions

Surgery theory is particularly useful in classifying and studying group actions on manifolds. Depending on what hypotheses one wants to impose, one obtains various generalizations of the fundamental exact sequence (eq. 3.1) in the context of G-manifolds, G some compact Lie group. A few key references are [34], [4], [17], [24], and Parts II and III of [26]. Many of the early references may also be found in §4.6 of [33]. While there is no room here to go into great detail, we will discuss a few cases:

3.4.1 Free actions

If a compact Lie group G acts freely on a connected manifold M, then the quotient space $N = M/G$ is itself a manifold, and there is a fibration

$$G \to M \to N.$$

Thus the fundamental group π of N fits into an exact sequence

$$\pi_1(G) \to \pi_1(M) \to \pi \to \pi_0(G) \to 1. \qquad \text{(eq. 3.2)}$$

Say for simplicity that we take G to be finite, so $G = \pi_0(G)$ and $\pi_1(G) = 1$. One can attempt to classify the free actions of G on M by classifying such group extensions (eq. 3.2), and then, for a fixed such extension, classifying those manifolds N having M as the covering space corresponding to the map $\pi \twoheadrightarrow G$. Note that in this context there is a *transfer map* $\mathcal{S}(N) \to \mathcal{S}(M)$ defined by lifting to the covering space. It will often happen that there are many manifolds homotopy equivalent to N, but non-isomorphic to it, that also have M as the covering space corresponding to G. Such manifolds give elements of the kernel of this transfer map. In section 3.5 below, we shall have more to say about the important special case where M is a sphere.

3.4.2 Semi-free actions

After free actions, the simplest actions of a compact group G on a manifold M are those which are *semi-free*, that is, trivial on a submanifold M^G and free on $M \setminus M^G$. For such an action, the quotient space M/G is naturally a stratified space with two manifold strata, the closed stratum M^G and the open stratum $\left(M \setminus M^G\right)/G$. A discussion of this case from the point of view of stratified spaces may be found in [26], §13.6. Here is a sample result ([138], Theorem A) about semi-free actions of a finite group G: A PL locally flat submanifold Σ^n of S^{n+k} for $k > 2$ is the fixed set of an orientation-preserving semifree PL locally linear G-action on S^{n+k} if and only if Σ is a $\mathbb{Z}/|G|$ homology sphere, $\mathbb{R}^k$ has a free linear representation of G, and certain purely algebraically describable conditions hold for the torsion in the homology of Σ.

3.4.3 Gap conditions

An annoying but sometimes important part of equivariant surgery theory involves what are called *gap conditions*. When a compact group G acts on a manifold M, these are lower bounds on the possible values of the codimension of M^K in M^H, for subgroups $H \subset K$ of G for which $M^K \neq M^H$. Let's specialize now to the case where G is finite. Roughly speaking, there are three kinds of equivariant surgery theory:

1. Surgery without any gap conditions. This is very complicated and not much is known about it.

2. Surgery with a "small" gap condition, the condition that no fixed set component be of codimension < 3 in another. Such a condition is designed to eliminate some fundamental group problems, due to the fact that if M^K has codimension 2 in M^H, there is no way to control the fundamental group of the complement $M^H \setminus M^K$. When M^K can have codimension 1 in M^H, then things are even worse, since one can't even control the number of components of $M^H \setminus M^K$.

3. Surgery with a "large" gap condition, the condition that each fixed set have more than twice the dimension of any smaller fixed set.

For each of cases (2) and (3), there are analogues of the major concepts of non-equivariant surgery theory: normal cobordism, surgery obstruction groups, and a surgery exact sequence. However, there are several ways to set things up, depending on whether one considers equivariant maps (as in most references) or *isovariant* maps (equivariant maps that preserve isotropy groups) as in [34], and depending on whether one tries to do surgery up to equivariant homotopy equivalence (as in [3]) or only up to *pseudoequivalence* (as in [2]). (A map is defined to be a pseudoequivalence if it is equivariant and if, *non-equivariantly* it is a homotopy equivalence.) The big advantage of the large gap condition is that when this condition is satisfied, then one can show [46] that any equivariant homotopy equivalence can be homotoped equivariantly to an isovariant one. For a detailed study of the differences between gap conditions (2) and (3), see [140].

3.4.4 Differences between categories

In the context of group actions, the differences between different categories of manifolds become more pronounced than in the non-equivariant situation studied in §3.2 above. Aside from the smooth and PL categories, the most studied category is that of *topological locally linear actions*, meaning actions on topological manifolds M for which each point $x \in M$ has a G_x-invariant neighborhood equivariantly homeomorphic to a linear action of G_x on $\mathbb{R}^n$. If one studies topological actions with no extra conditions at all, then actions can be very pathological, and the fixed set for a subgroup can be a completely arbitrary compact metrizable space of finite dimension. In particular, it need not be a manifold, and need not even have finite homotopy type. Various properties of the smooth, PL, and topological locally linear categories are discussed in [4], [24], [26], and [83]; they are too complicated to discuss here.

3.4.5 Nonlinear similarity

One of the most dramatic applications of surgery to equivariant topology (even though the original work on this problem only uses *non*-equivariant surgery theory) is to the *nonlinear similarity problem.* This goes back to an old question of de Rham: if G is a finite group and if $\rho_1\colon G \to O(n)$, $\rho_2\colon G \to O(n)$ are two linear (orthogonal) representations of G on Euclidean n-spaces V_1 and V_2, respectively, does a topological conjugacy between ρ_1 and ρ_2 imply that the two representations are linearly equivalent? Here a *topological conjugacy* means a homeomorphism $h\colon V_1 \to V_2$ conjugating ρ_1 to ρ_2. If such a homeomorphism exists, it restricts to a homeomorphism $V_1^{\rho_1(G)} \to V_2^{\rho_2(G)}$, so these two subspaces must have the same dimension. Since we may compose with translation in V_2 by $h(0) \in V_2^{\rho_2(G)}$, there is no loss of generality in assuming that $h(0) = 0$. Now if such an h were to exist and be a diffeomorphism, then the differential of h at the origin would be an invertible linear intertwining operator between ρ_1 and ρ_2, so this problem is only interesting if we allow h to be non-smooth.

One special case is worthy of note: if G is cyclic, if $\mathbb{R}^n$ carries a G-invariant complex structure, and if G acts freely on the complement of the origin, then $S^{n-1}/\rho_1(G)$ and $S^{n-1}/\rho_2(G)$ are lens spaces, and so the question essentially comes down to the issue of whether homeomorphic lens spaces must be diffeomorphic. The answer is "yes," as can be shown using the topological invariance of simple homotopy type [43] together with Reidemeister torsion [106]. The next important progress was made by Schultz [125] and Sullivan (independently) and then by Hsiang-Pardon [81] and Madsen-Rothenberg [100] (again independently). The upshot of this work is that if $|G|$ is of odd order, then topological conjugacy implies linear conjugacy. Then in [38], Cappell and Shaneson showed that for G cyclic of order divisible by 4, there are indeed examples of topological conjugacy between linearly inequivalent representations. This work has been refined over the last two decades, and a summary of some of the most recent work may be found in [40].

While it would be impractical to go into much detail, we can at least sketch some of the key ideas that go into these results. First let's consider the theorems that give *constraints* on existence of nonlinear similarities. Suppose $h\colon V_1 \to V_2$ is a topological conjugacy between representations ρ_1 and ρ_2, say with $h(0) = 0$ (no loss of generality). Then we glue together V_1 and $(V_2 \setminus \{0\}) \cup \{\infty\}$, using h to identify $V_1 \setminus \{0\} \subset V_1$ with $V_2 \setminus \{0\} \subset (V_2 \setminus \{0\}) \cup \{\infty\}$. The result is a copy of S^n equipped with a topological (locally linear) action of G and with ρ_1 as the isotropy representation at one fixed point (0 in V_1), and ρ_2^*, the contragredient of ρ_2, as the isotropy representation at another fixed point (the point at infinity in V_2). Results such as those of Hsiang-Pardon and Madsen-Rothenberg can then be deduced from a suitable version of the G-signature theorem applied to this situation. (Of course, the classical G-signature theorem doesn't apply here, since the group action is not smooth, so proving such a G-signature theorem is not so easy.) One convenient formulation of what comes out, sufficient to give the Hsiang-Pardon and Madsen-Rothenberg results and much more, is the following:

Theorem 3.4 ([123], Theorem 3.3) *Let ρ be a finite-dimensional representation of a finite group G, and let $\gamma \in G$ be of order k. Define the "renormalized Atiyah-Bott number" $AB(\gamma, \rho)$ to be 0 if -1 is an eigenvalue of $\rho(\gamma)$. If this is not the case, suppose that after discarding the $+1$-eigenspace of $\rho(\gamma)$, $\rho(\gamma)$ splits as a direct sum of n_j copies of counterclockwise rotation by $2\pi j/k$, $0 < j < k$, and define in this case*

$$AB(\gamma, \rho) = \prod_{0<j<k} \left(\frac{\zeta^j + 1}{\zeta^j - 1} \right)^{n_j},$$

where $\zeta = e^{2\pi i/k}$. Then the numbers $AB(\gamma, \rho)$, $\gamma \in G$, are oriented topological conjugacy invariants of ρ, and up to sign are topological conjugacy invariants (even if one doesn't require orientation to be preserved).

Next we'll give a rough idea of how Cappell and Shaneson constructed non-trivial nonlinear similarities between representations ρ_1 and ρ_2 of a cyclic group G of order $4q$ with generator γ_0, in the case where $\rho_j(\gamma_0)$ has eigenvalue -1 with multiplicity 1 and all its other eigenvalues are primitive $4q$th roots of unity. Let V_j be the representation space on which ρ_j acts. Then V_j has odd dimension $2k+1$, and we may write it as

$$\begin{aligned} \mathbb{R}^{2k+1} \quad &\cong \quad \{0\} \cup \left(S^{2k-1} \times [-1,1] \times (0,\infty)\right) \bigcup_{S^{2k-1}\times\{\pm 1\}\times(0,\infty)} \\ &\qquad \left(D^{2k} \times \{-1,1\} \times (0,\infty)\right), \end{aligned}$$

where the factor $(0,\infty)$ at the end represents the radial coordinate. Here ρ_j acts by a free linear representation on S^{2k-1}, for which the quotient is a lens space L_j with fundamental group of order $4q$, γ_0 acts by multiplication by -1 on $[-1,1]$, and γ_0 acts trivially on $(0,\infty)$. So the idea is to choose L_1 and L_2 so that they are homotopy equivalent but not diffeomorphic, but so that their non-trivial double covers $\widetilde{L}_j$, which are lens spaces with fundamental group of order $2q$, are isomorphic to one another. (This is possible using the known classification theorems for lens spaces, as found in [106] for example.) Then if E_j denotes the non-trivial $[-1,1]$-bundle over L_j (obtained by dividing $S^{2k-1} \times [-1,1]$ by the group action), one can arrange for E_1 and E_2 to be h-cobordant. (This takes a pretty complicated calculation. First one needs to make them normally cobordant, and then one needs to show that the surgery obstruction to converting a normal cobordism to an h-cobordism vanishes.) Then it turns out that E_1 and E_2 become homeomorphic after crossing with $(0, \infty)$. Lifting back to the universal covers, one gets equivariant homeomorphisms

$$S^{2k-1} \times [-1,1] \times (0,\infty) \to S^{2k-1} \times [-1,1] \times (0,\infty)$$

and

$$D^{2k} \times \{-1,1\} \times (0,\infty) \to D^{2k} \times \{-1,1\} \times (0,\infty),$$

which patch together to give the desired nonlinear similarity.

3.5 The topological space form problem

As we mentioned above, one of the successes of surgery theory is the classification of those finite groups G that can act freely on spheres. This subject begins with the observation that if G acts freely on S^n, then the (Tate) cohomology groups of G must be periodic with period $n+1$ ([42], Chapter XVI, §9, Application 4). One of the great classical theorems on cohomology of finite groups ([42], Chapter XII, Theorem 11.6) then says that this happens (for some n) if and only if every abelian subgroup of G is cyclic, or equivalently, if and only if every Sylow subgroup of G is either cyclic or else a generalized quaternion group.

This then raises an obvious question. Suppose G has periodic cohomology. Then does G act freely on a finite CW complex X with the homotopy type of S^n, and if so, can this space X be chosen to be S^n itself? The first part of this question was answered by Swan [131], who showed that, yes, G acts freely on a finite CW complex X with the homotopy type of S^n. The argument for this has nothing to do with surgery; rather, it requires showing that the trivial G-module $\mathbb{Z}$ has a periodic resolution by finitely generated free $\mathbb{Z}G$-modules. (Initially, one only gets such a resolution by finitely generated *projective* $\mathbb{Z}G$-modules, so that it would appear that a finiteness obstruction comes in (see [60]), but one can kill off the obstruction at the expense of possibly increasing the period of the resolution.)

Then one has to determine if X can be taken to be a sphere. One case is classical: if every subgroup of G of order pq (p and q primes) is cyclic, then it is known by a theorem of Zassenhaus that G acts freely and *linearly* (and thus certainly smoothly) on some sphere [146]. Milnor [105] showed, however, that in order for G to act freely on a manifold which is a homology sphere (even in the topological category), it is necessary that all subgroups of order $2p$, p an odd prime, be cyclic rather than dihedral. The argument for this is remarkably elementary, and again doesn't use surgery. An alternative argument using equivariant bordism and equivariant semicharacteristics, again fairly elementary, was given by R. Lee [98]. The point of Lee's proof is basically to show that if a closed oriented manifold M^{2n+1} has as fundamental group the dihedral group $G = D_{2p}$ of order p (p an odd prime), then the formal sum, in an appropriate Grothendieck group, of the G-modules $H_{2i}(M^{2n+1}; FG)$, F a suitable finite field of characteristic 2, if non-zero, has to involve the non-abelian representations of G. This clearly gives a contradiction if the universal cover of M is a homology sphere, since then $H_*(M^{2n+1}; FG)$ is identified with the homology of a sphere, which is only non-zero in bottom and top degree, and the action of G has to be trivial.

The papers [134] and [101] then showed that the condition of Milnor is sufficient as well as necessary for G to act smoothly and freely on a sphere. The method of proof is to go back to Swan's argument in [131] and show that there is a simple Poincaré space with fundamental group G for which the universal cover is (homotopy-theoretically) a sphere, and that its Spivak fibration admits a PL bundle reduction [134], in fact, a smooth bundle reduction [101]. Finally [101], the full power of Wall's surgery theory is used to show that the surgery obstruction vanishes, and thus that there is a manifold with fundamental group G whose universal cover is a homotopy sphere.

3.6 Algebraic theory of quadratic forms

While the main idea of surgery theory is usually to reduce manifold theory to algebra, there are cases where it can be used in the opposite direction, to obtain information about the theory of quadratic forms from geometry. We give just one example. When π is an infinite group with some "non-positive curvature" properties, for example, the fundamental group of a hyperbolic manifold, then various geometrical or analytical techniques can be used to prove the Novikov conjecture or sometimes even the Borel rigidity conjecture for π. (See [61], [45], and [129] for surveys of the literature, which is quite extensive.) This in turn, from the surgery exact sequence (eq. 3.1), implies significant information about the stable classification of quadratic forms over $\mathbb{Z}\pi$.

3.7 Submanifolds, fibrations, and embeddings

Surgery theory can deal not only with individual manifolds, but also with questions concerning how one manifold can embed in another. There is an extensive literature on such problems, but we will only mention a few examples. For instance, suppose M is a manifold, and suppose that from a homotopy point of view, M looks like the total space of a fibration $F \to M \to B$. Then can M be made into the total space of a genuine manifold fiber bundle with base and fiber homotopy equivalent to B and F? Or suppose $X \hookrightarrow Y$ is a Poincaré embedding. That means that Y is a Poincaré complex, say of dimension n, X is a Poincaré complex of dimension $n-q$ with a mapping to Y, and we have subspaces $S \subset C \subset Y$ with the following properties:

1. There is a spherical fibration $S^{q-1} \to S \xrightarrow{p} X$, with S a Poincaré complex of dimension $n-1$. (S is the homotopy analogue of the boundary of a tubular neighborhood of a submanifold X in Y of codimension q. The map p corresponds to the retraction of this tubular neighborhood onto the submanifold X.)

2. (C, S) is a Poincaré pair of dimension n. That is, C has the Poincaré duality properties of an n-dimensional manifold with boundary S. The diagram

$$\begin{array}{ccccc} S & \hookrightarrow & C & \hookrightarrow & Y \\ \big\downarrow p & & & \nearrow & \\ X & & & & \end{array}$$

is homotopy commutative.

3. Up to simple homotopy equivalence, Y is the union of C and the mapping cylinder of p, joined along S. (The mapping cylinder of p is the homotopy analogue of the closed tubular neighborhood of the submanifold. This says that C plays the role of the complement of an open tubular neighborhood of X in Y.)

Now suppose M is a manifold and $h\colon M \to Y$ is a homotopy equivalence. Then can h be homotoped to a map h' so that $h'^{-1}(X)$ is a genuine submanifold N of M (of codimension q) and h' restricted to N is a homotopy equivalence $N \to X$? When this is the case, h is said to be *splittable* along X.

There is an extensive literature on questions such as these but we content ourselves here with a few representative examples.

For the fiber bundle problem, the first case to be studied, but still one of the most important, is whether a certain manifold fibers over S^1. In other words, one is given a compact manifold M^n and a map $f\colon M^n \to S^1$ with f_* surjective on π_1, and one wants to know if one can change f within its homotopy class to the projection map p of a fiber bundle $N^{n-1} \to M^n \xrightarrow{p} S^1$. The map f defines an infinite cyclic covering $\widetilde{M} \to M$, and if the desired fiber bundle exists, then $\widetilde{M}$ must be isomorphic (in the appropriate category) to $N \times \mathbb{R}$ with N a compact manifold. So first one must check if the finiteness obstruction vanishes (so that $\widetilde{M}$ is homotopically finite and is equivalent to a finite Poincaré complex), and then one must solve a surgery problem to see if $\widetilde{M}$ can be realized as cylinder $N \times \mathbb{R}$. The solution to the problem was found by Farrell ([50], [51]) (following earlier work by Browder and Levine in the simply connected case), who found that if $\widetilde{M}$ is indeed homotopically finite, the necessary and sufficient condition for a positive solution to fiber bundle problem is the vanishing of a Whitehead torsion obstruction in the Whitehead group of $\pi_1(M)$.

For the splitting problem, there are essentially three cases.

1. When X is of codimension one, the issues involved are somewhat similar to what arises in the problem of fibering over a circle, and the key result (in the categories TOP and PL) is due to Cappell [36]. A special case of this concerns the following question. Suppose M^n is a closed manifold that looks homotopy-theoretically like a connected sum. (Since we are assuming $n > 2$, that means in particular that $\pi_1(M)$ must be the free product of the fundamental groups of the prospective summands.) Then does M have a splitting of the form $M \cong M_1 \# M_2$? (This corresponds to the case where $X = S^{n-1}$ and is "two-sided" and separating in Y.) Cappell discovered that when $n \geq 5$, the answer to this question is not always "yes," but that the only obstruction to a positive answer is an algebraic one related to the fundamental groups involved.[5] The obstruction group vanishes when $\pi_1(M)$ has no 2-torsion, so in this case one indeed has a splitting $M \cong M_1 \# M_2$. Incidentally, the dimension restriction is necessary, for it follows from Donaldson theory that there are many simply connected PL 4-manifolds (a K3 surface, for example) which are homotopy theoretically connected sums, but which do not split as connected sums in the PL category. (In dimension 4, the PL and DIFF categories are equivalent.)

2. When X is of codimension two, the splitting problem is closely related to the classification of knots; see [18], §§7.8–7.9, [22], and the survey [96] in this collection for more details.

3. When X is of codimension 3 or more and Y is of dimension 5 or more, the splitting problem always has a positive answer in the PL category, provided that the obvious necessary condition (that X have the simple homotopy type of a PL manifold) is satisfied, as shown by Wall in [25], Corollary 11.3.1. In the smooth case one needs a little more, since the spherical fibration $S \xrightarrow{p} X$ must come from a rank q vector bundle, but the "expected results" are still true.

[5] In the category DIFF, the result is the same as long as one allows generalized connected sums along a (possibly exotic) separating homotopy sphere.

3.8 Detection on submanifolds

The study of submanifolds can also be turned around, and we can ask to what extent surgery obstructions are determined by what happens on submanifolds. This in turn is related to another problem: How much of the Wall surgery groups $L_n(\mathbb{Z}\pi)$ arises from surgery obstructions of degree-one normal maps between closed manifolds? To be more precise, the issue is basically how much information about submanifolds of M is needed either

1. to compute the surgery obstruction of a surgery problem $(N^n, \eta) \to (M^n, \xi)$, or how to best understand *geometrically* the obstruction map $\sigma : \mathcal{N}(M) \to L_n(\mathbb{Z}\pi)$; or
2. to determine when a class in $\mathcal{S}(M^n)$ is trivial, or in other words, when a homotopy equivalence $N \xrightarrow{h} M$ is homotopic to an isomorphism.

These issues often go under the general name of "oozing," which is supposed to suggest how simply connected surgery obstructions on submanifolds "ooze up" to give obstructions on a larger manifold, not usually simply connected.

The first major result along these lines was the characteristic variety theorem of Sullivan (see Sullivan's 1967 notes, republished in [21], pp. 69–103). It says (*roughly* that the answer to question (2) can be formulated in terms of simply connected surgery obstructions (signatures and Arf-Kervaire invariants) to splitting along certain (possibly singular) submanifolds of M. This theorem is also related to various formulas for characteristic classes found in [109].

As far as question (1) is concerned, the basic question was whether surgery obstructions can be computed from simply connected splitting obstructions on submanifolds of bounded codimension. For general fundamental groups this is certainly not the case (see [137]), but it was expected by the experts (the "oozing conjecture") that this would be the case for manifolds with finite fundamental group. This issue has now been settled. Codimension 2 manifolds *do not* suffice; Cappell and Shaneson [37] showed that if M^3 is the usual quaternionic lens space (the quotient of S^3 by the linear action of the quaternion group Q_8 of order 8) and $K^{4k+2} \xrightarrow{\kappa} S^{4k+2}$ is the "Kervaire problem" (a simply connected surgery problem representing the generator of $L_{4k+2}(\mathbb{Z}) \cong \mathbb{Z}/2$), then

$$\sigma(M^3 \times K^{4k+2} \xrightarrow{\mathrm{id}\times\kappa} M^3 \times S^{4k+2}) \neq 0 \text{ in } L_{4k+5}(\mathbb{Z}Q_8),$$

even though the obstruction here comes from the Arf invariant on the codimension 3 manifold K^{4k+2}. But codimension 3 manifolds *do* suffice ([79], [103]).[6]

[6]For those who know what this means: at least for the h decoration.

3.9 Differential geometry

Surgery theory becomes especially interesting when applied to certain problems in differential geometry. We begin with Riemannian geometry. Recall that a *Riemannian metric* on a manifold is a smoothly varying choice of inner products on tangent spaces. This makes it possible to measure lengths of curves, and thus to define geodesics (curves which locally minimize length), and also to measure angles between intersecting curves. The most important intrinsic geometric invariants of a Riemannian manifold are those having to do with curvature. The *sectional curvature* of a Riemannian manifold M (also called the Gaussian curvature if $\dim M = 2$) at a point $p \in M$ in the direction of some 2-plane P in the tangent space T_pM through p measures how the sum of the angles of a small geodesic triangle differs from π, if one vertex of the triangle is at p, and the incident sides there lie in the plane P. The *Ricci curvature* (which is a tensor) and *scalar curvature* (the trace of the Ricci curvature) at p are then obtained from various averages of the sectional curvatures there. As such, the standard curvature invariants are defined locally, but global bounds on curvature (for a closed or complete manifold) have implications for global topology. We mention just a few prominent examples: the Gauss-Bonnet Theorem for closed surfaces M, which says that $\int_M K\,dA = 2\pi\,\chi(M)$, where K is the Gaussian curvature, $\chi(M)$ is the Euler characteristic, and dA is the Riemannian area measure; Myers' Theorem, that any complete manifold of Ricci curvature $\geq c > 0$ is closed and has finite fundamental group; the Cartan-Hadamard Theorem, that any complete manifold of non-positive sectional curvature is aspherical[7], with universal cover diffeomorphic to Euclidean space (and with covering map the exponential map from the tangent space at a basepoint); and the generalized Gauss-Bonnet Theorem of Chern and Allendoerfer, expressing the Euler characteristic as a multiple of the integral of the Pfaffian of the curvature form. Global consequences of positivity of the scalar curvature are discussed in [122].

3.9.1 Rigidity theorems for Riemannian manifolds

One place where surgery can be of particular help in Riemannian geometry is in the study of *rigidity theorems*, results that say that two Riemannian manifolds sharing a very specific geometric property must be homeomorphic, diffeomorphic, etc. Such theorems abound in Riemannian geometry. Classical examples (proved without using surgery) are *sphere theorems*, such as the fact that a complete simply connected manifold with sectional curvature K satisfying $\frac{1}{4} < K \leq 1$ must be the union of two disks glued together via a diffeomorphism of their boundaries, and thus a homotopy sphere. Another famous examples is the Mostow Rigidity Theorem, which

[7] That is, all its higher homotopy groups π_j, $j > 1$, vanish.

says that two irreducible locally symmetric spaces of dimension ≥ 3 and non-compact type, with finite volume and with isomorphic fundamental groups, must be isometric to one another. Mostow's Theorem helped to motivate the Borel Conjecture, that two compact aspherical manifolds[8] with isomorphic fundamental groups are homeomorphic.

Here is a brief [very incomplete] list of a number of rigidity theorems proved using a combination of surgery theory and Riemannian geometry:

1. the work of Farrell and Hsiang [52] on the Novikov Conjecture. This was very influential in its time but has now been superseded by the work of Farrell and Jones cited below.

2. Kasparov's proof ([85], [86]) of the Novikov Conjecture for arbitrary discrete subgroups of Lie groups. This has been improved by Kasparov and Skandalis [87] to give the Novikov Conjecture for "bolic" groups, by weakening nonpositive curvature in Riemannian geometry to a rough substitute in the geometry of metric spaces.

3. the work of Farrell and Jones ([5], [6]) on topological rigidity of manifolds of nonpositive curvature. This includes for example:

   Theorem 3.5 ([55]) *Let M and N be closed aspherical topological manifolds of dimensions $\neq 3, 4$. If M is a smooth manifold with a nonpositively curved Riemannian metric and if $\pi_1(M) \xrightarrow{\cong} \pi_1(N)$ is an isomorphism, then this isomorphism is induced by a homeomorphism between M and N.*

4. the work of Farrell and Jones [54] on pseudoisotopies of manifolds of nonpositive curvature. This gives substantial information about the homotopy types of the diffeomorphism groups of these manifolds.

5. examples, constructed by Farrell and Jones ([53], [56], [57]), of manifolds of nonpositive curvature which are homeomorphic but not diffeomorphic.

6. the theorem of Grove and Shiohama [76] that a complete connected Riemannian manifold with dimension ≤ 6, with sectional curvature $\geq \delta > 0$ and with diameter $> \pi/2\sqrt{\delta}$, is diffeomorphic to a standard sphere.

7. work of Grove-Peterson-Wu [75] (see also the work of Ferry [58]) showing that for any integer n, any real number k and positive numbers D and v, the class of closed Riemannian n-manifolds M with sectional

[8] Recall that locally symmetric spaces of non-compact type are included here, by the Cartan-Hadamard Theorem.

curvature $K_M \geq k$, diameter $d_M \leq D$ and volume $V_M \geq v$ contains at most finitely many homeomorphism types when $n \neq 3$, and only finitely many diffeomorphism types if, in addition, $n \neq 4$. (There is a similar result for manifolds with injectivity radius $i_M \geq i_0 > 0$ and volume $V_M \leq v$.)

8. work of Ferry and Weinberger [62] growing out of work on the Novikov Conjecture. This includes the very interesting result that if M^n is an irreducible compact locally symmetric space of noncompact type (with $n > 4$), then the natural forgetful map $\mathrm{Diff}(M) \to \mathrm{Homeo}(M)$ has a continuous splitting.

9. the "packing radius" sphere theorem of Grove and Wilhelm [77], stating that for $n \geq 3$, a closed Riemannian n-manifold M with sectional curvature ≥ 1 and $(n-1)$-packing radius $> \frac{\pi}{4}$ is diffeomorphic to S^n.[9]

10. an improvement of the classical sphere theorem due to Weiss [142], showing that if M^n is a complete simply connected manifold with sectional curvature K satisfying $\frac{1}{4} < K \leq 1$, then not only is M a homotopy sphere, but M has "Morse perfection n," which rules out some of the exotic sphere possibilities for M. See also [78] for further developments. (In the other direction, Wraith [144] has constructed metrics of positive *Ricci* curvature on all homotopy spheres that bound parallelizable manifolds. A few exotic spheres are known to admit metrics of nonnegative sectional curvature (see [71], [119], and work in progress by Grove and Ziller), but the sectional curvatures of the metrics constructed to date are not strictly positive, let alone $\frac{1}{4}$-pinched.)

11. work of Brooks, Perry, and Petersen [31] showing that given a sequence of isospectral manifolds of dimension n for which either all the sectional curvatures are negative or there exists a uniform lower bound on the sectional curvatures, then the sequence contains only finitely many homeomorphism types, and if $n \neq 4$, at most finitely many diffeomorphism types.

12. recent theorems of Belegradek [29] showing that in many cases, given a group π, an integer n larger than the homological dimension of π, and real numbers $a < b < 0$, there are only finitely many diffeomorphism types of complete Riemannian n-manifolds with curvature $a \leq K \leq b$ and fundamental group π. The manifolds involved here are noncompact, and usually have infinite volume.

[9] The $(n-1)$-packing radius is defined to be half the maximum, over all configurations of $(n-1)$ points in M, of the minimum distance between points.

What most of these references have in common is that a geometric assumption, usually based on curvature bounds, is used to deduce some consequences that, while sometimes rather technical and not always directly interesting in themselves, can be plugged into the "surgery machine" to deduce the desired rigidity theorem.

3.9.2 Surgery and positive scalar or Ricci curvature

Surgery enters into differential geometry in another somewhat different way as well: through "surgery theorems" that say that under appropriate hypotheses, a certain geometrical structure on one manifold may be transported via a surgery to some other manifold. In this subsection we will discuss application of this principle to positive scalar or Ricci curvature, in section 3.9.4 we will discuss conformal geometry, and in section 3.9.5 we will discuss application to the study of symplectic or contact structures.

So far the most remarkable and useful surgery theorem is the theorem of [73] and [124] regarding positive scalar curvature. (See also [122] for an exposition and for a correction to one point in the Gromov-Lawson proof.) This says that if M_1^n is a compact manifold (not necessarily connected) with a Riemannian metric of positive scalar curvature, and if M_2^n can be obtained from M_1 by surgery on a sphere of codimension ≥ 3, then M_2 can also be given a metric of positive scalar curvature. This result is so powerful that, when combined with known index obstructions to positive scalar curvature based on the Dirac operator, it has made complete classification of the manifolds admitting positive scalar curvature metrics feasible in many cases. See [122] for a detailed exposition.

In the case of positive Ricci curvature, a surgery theorem as general as this could not be true, for surgery on $S^0 \hookrightarrow S^n$ results in a manifold with infinite fundamental group, which cannot have a metric of positive Ricci curvature by Myers' Theorem. Nevertheless, there is no known reason why surgery on a sphere of dimension ≥ 1 and codimension ≥ 3 in a manifold of positive Ricci curvature cannot result in a manifold of positive Ricci curvature, and in fact there is some positive evidence for this in [126] and [145]. But Stolz in [130], based upon both heuristics of Dirac operators on loop spaces and upon calculations with homogeneous spaces and complete intersections, has conjectured that the Witten genus vanishes for spin manifolds with positive Ricci curvature and with vanishing $\frac{p_1}{2}$.[10] If this is the case, then Stolz has shown [130] that there are simply connected closed manifolds with positive scalar curvature metrics but without metrics of positive Ricci curvature, and thus a surgery theorem this general for positive Ricci curvature cannot hold. So perhaps it should be necessary to

[10] For spin manifolds M, the first Pontrjagin class p_1 is always divisible by 2, and there is an integral characteristic class $\frac{p_1}{2} \in H^4(M;\mathbb{Z})$ which when multiplied by 2 gives p_1.

restrict to surgeries of some greater codimension.

3.9.3 Surgery and the Yamabe invariant

The problem of prescribing scalar curvature on a manifold also has a quantitative formulation in terms of the so-called *Yamabe invariant.* If M^n is a closed manifold and we fix a Riemannian metric g on M, then by the solution of the Yamabe problem, it is always possible to make a (pointwise) conformal change in the metric, i.e., to multiply g by a positive real-valued function, so as to obtain a metric with constant scalar curvature and total volume 1. The minimum possible value of the scalar curvature of such a metric is an invariant of the conformal class of the original metric, known as the *Yamabe constant* of the conformal class. The *Yamabe invariant* $Y(M)$ of M is then defined as the supremum, taken over all conformal classes of metrics on M, of the various Yamabe constants. It is bounded above by a universal constant depending only on n, namely $n(n-1)(\operatorname{vol} S^n(1))^{2/n}$ (the scalar curvature of a round n-sphere of unit volume), and is closely related to the question of determining what real-valued functions can be scalar curvatures of Riemannian metrics on M with volume 1 [92]. Note that $Y(M) > 0$ if and only if M admits a metric of positive scalar curvature. It is known that "most" closed 4-manifolds have negative Yamabe invariant [95]. In a counterpart to the surgery theorem of [73] and [124], it is shown in [115] that if M' can be obtained from M by surgeries in codimension ≥ 3 and if $Y(M) \leq 0$, then $Y(M') \geq Y(M)$. This fact has been applied in [113] to obtain exact calculations of $Y(M)$ for some 4-manifolds M, and in [114] to show that $Y(M) \geq 0$ for every simply connected closed n-manifold M^n with $n \geq 5$.

3.9.4 Surgery and conformal geometry

A *conformal structure* on manifold M^n is an equivalence class of Riemannian structures, in which two metrics are identified if angles (but not necessarily distances) are preserved. For oriented 2-manifolds, this is the same thing as a complex analytic structure. A conformal structure is called *conformally flat* if each point in M has a neighborhood conformally equivalent to Euclidean n-space $\mathbb{R}^n$. (This is true for the standard round metric on S^n, for example.) An immersion $M^n \looparrowright \mathbb{R}^{n+k}$ is called *conformally flat* if the standard flat metric on $\mathbb{R}^{n+k}$ pulls back to a conformally flat structure on M. One of the important problems in conformal geometry is the classification of conformally flat manifolds and conformally flat immersions into Euclidean space. For immersions of hypersurfaces, i.e., immersions in codimension $k = 1$, a complete classification has been given (begun in [93], completed in [41]) using the idea of conformal surgery. The final result is:

Theorem 3.6 **([41])** *A compact connected manifold M^n has a conformal immersion into $\mathbb{R}^{n+1}$ if and only if M can be obtained from S^n by adding finitely many 1-handles, i.e., by doing surgery on a finite set of copies of S^0 in S^n. In particular, any such M has free fundamental group.*

3.9.5 Surgery and symplectic and contact structures

A *symplectic structure* on an even-dimensional manifold M^{2n} is given by a closed 2-form ω such that $\omega^n = \omega \wedge \omega \wedge \cdots \wedge \omega$ (n factors) is everywhere non-zero, i.e., is a volume form. A *contact structure* on an odd-dimensional manifold M^{2n+1} is a maximally non-integrable subbundle ξ of TM of codimension 1, and thus is locally given by $\xi = \ker \alpha$, where α is a 1-form such that $\alpha \wedge (d\alpha)^n$ is a volume form. Symplectic and contact structures arise naturally in classical mechanics, and there is a close link between them.

The problem of determining what manifolds admit symplectic or contact structures is not so easy, though there are some obvious necessary conditions. If M^{2n} is a closed connected manifold which admits a symplectic form ω, then if $[\omega] \in H^2(M; \mathbb{R})$ denotes its de Rham class, $[\omega]^n$ must generate $H^{2n}(M; \mathbb{R}) \cong \mathbb{R}$. In particular, M is oriented, and ω gives a reduction of the structure group of TM from $GL(2n, \mathbb{R})$ to $Sp(2n, \mathbb{R})$, which has maximal compact subgroup $U(n)$; thus ω defines an isomorphism class of almost complex structures J on M. The most familiar examples of symplectic manifolds are Kähler, in other words, admit a Riemannian metric g and an integrable (and parallel) almost complex structure J for which $\omega(X, Y) = g(JX, Y)$ for all vector fields X and Y. However, it is known that there are plenty of symplectic manifolds without Kähler structures [135]. A promising line of attack in constructing symplectic structures is therefore to start with the standard examples and try construct new ones using fiber bundles, "blow-ups," and surgery methods. (See [102] for a detailed exposition.) In particular, "symplectic surgery" has been studied in [70] and [132]. With it Gompf has proved [70] that every finitely presented group is the fundamental group of a compact symplectic 4-manifold, even though there are constraints on the fundamental groups of Kähler manifolds. It is not always possible to put a symplectic structure on the connected sum of two symplectic manifolds, since in dimension 4, Taubes [133] has shown using Seiberg-Witten theory that a closed symplectic manifold cannot split as a connected sum of two manifolds each with $b_1^+ > 0$. Gompf's "symplectic connected sum" construction is therefore somewhat different: if M_1^{2n}, M_2^{2n}, and N^{2n-2} are symplectic and one has symplectic embeddings $N \hookrightarrow M_1$, $N \hookrightarrow M_2$ whose normal bundle Euler classes are negatives of one another, then Gompf's $M_1 \#_N M_2$ is obtained by joining the complements of tubular neighborhoods of N in M_1 and in M_2 along their common boundary (a sphere bundle over N).

Surgery has also played an important role in a number of other problems connected with symplectic geometry: the theory of Lagrangian embeddings[11] (see, e.g., [99], [49], and [116]) and Eliashberg's topological classification [47] of Stein manifolds.[12] A Stein manifold of complex dimension n is known to admit a proper Morse function with all critical points of index $\leq n$ (so that, roughly speaking, M is the thickening of an n-dimensional CW-complex), and Eliashberg showed that for $n > 2$, a $2n$-dimensional almost complex manifold admits a Stein structure exactly when it satisfies this condition.

A contact structure on an odd-dimensional manifold appears at first sight to be a very "flabby" object. If we consider only contact structures ξ for which TM/ξ is orientable (this is only a slight loss of generality, and turns out to be automatic if M is orientable and n even), then every contact structure ξ is defined by a global 1-form α such that $\alpha \wedge (d\alpha)^n \neq 0$ everywhere, and α is determined by ξ up to multiplication by an everywhere non-zero real function. Note also that as $d\alpha$ defines a symplectic structure on ξ, α defines an *almost contact structure* on M, that is, an isomorphism class of reductions of the structure group of TM from $GL(2n+1, \mathbb{R})$ to $1 \times U(n)$. So a natural question is whether an odd-dimensional manifold always admits a contact structure within every homotopy class of almost contact structures. When $n = 1$, i.e., $\dim M = 3$, the answer is known to be "yes," though Eliashberg showed that there are basically two distinct types of contact structure, "tight" and "overtwisted." Furthermore, if M is a closed oriented 3-manifold, then every class in $H^2(M; \mathbb{Z})$ is the Euler class of an overtwisted contact structure, but only finitely many homology classes in $H^2(M; \mathbb{Z})$ can be realized as the Euler class of a tight contact structure. (For surveys, see [69] and [48].) In higher dimensions, it is not known if every manifold with an almost contact structure admits a contact structure, though the experts seem to doubt this. And it is known that S^{2n+1} has at least two non-isomorphic contact structures in the homotopy class of the standard almost contact structure ([48], Theorem 3.1).

Nevertheless, in many cases one can construct contact structures in a given homotopy class of almost contact structures through a process of "contact surgery." The key tools for doing this may be found in [141] and in [47]. These references basically prove that if (M_1^{2n+1}, ξ_1) is a contact manifold and M_2^{2n+1} can be obtained from M_1 by surgery on $S^k \subset M_1^{2n+1}$,

[11] If (M^{2n}, ω) is a symplectic manifold, an embedding $f\colon N^n \hookrightarrow M^{2n}$ is called *Lagrangian* if $f^*\omega = 0$. Aside from the obvious bundle-theoretic consequence, that ω induces an isomorphism between the cotangent bundle of N and the normal bundle for the embedding, this turns out to put considerable constraints on isotopy class of the embedding.

[12] A complex manifold M is called a *Stein manifold* if $H^j(M, \mathcal{S}) = 0$ for all $j > 0$ for any coherent analytic sheaf $\mathcal{S}$ on M (though it is enough to assume this for $j = 1$), or equivalently, if M has a proper holomorphic embedding into some $\mathbb{C}^k$, that is, M is an affine subvariety of $\mathbb{C}^k$. An open subset of $\mathbb{C}^n$ is a Stein manifold if and only if it is a domain of holomorphy.

then M_2 also admits a contact structure ξ_2 (in the corresponding homotopy class of almost contact structures), provided that S^k is tangent to the contact structure ξ_1, and has trivial "conformal symplectic normal" (CSN) bundle. Since ξ_1 is maximally non-integrable, the first condition (S^k tangent to ξ_1) forces TS^k to be isotropic in ξ_1 for the symplectic form $d\alpha$ on ξ_1, α a 1-form defining ξ_1. In other words, if $(TS^k)^\perp$ denotes the orthogonal complement of TS^k in ξ_1, which has rank $2n-k$, then $TS^k \subseteq (TS^k)^\perp$, so $k \leq n$. The CSN bundle is then $(TS^k)^\perp/TS^k$, and a trivialization of this bundle determines a homotopy class of almost contact structures on M_2. Applications of this theorem may be found in [27], [65], [66], [67], and [68]. Some of the results are that:

1. Every finitely presented group is the fundamental group of a closed contact manifold of dimension $2n+1$, for any $n > 1$ [27].

2. Every simply connected spin^c 5-manifold admits a contact structure in every homotopy class of almost contact structures [65]. (The spin^c condition is necessary for existence of an almost contact structure.)

3. Every closed spin 5-manifold with fundamental group of odd order and with periodic cohomology admits a contact structure [68].

3.10 Manifold-like spaces

While the original applications of surgery theory were to the classification and study of manifolds, in recent years surgery has also been applied quite successfully to spaces which are not manifolds but which share some of the features of manifolds. We list just a few examples:

1. **Poincaré spaces:** Poincaré spaces have already appeared in this survey; they are spaces with the homotopy-theoretic features of manifolds. Thus for example it makes sense to talk about the bordism theory Ω_*^P defined like classical oriented bordism Ω_*, but using oriented Poincaré complexes in place of oriented smooth manifolds. Since Poincaré complexes do not satisfy transversality, this theory does *not* agree with the homology theory defined by the associated Thom spectrum MSG (whose homotopy groups are all finite), but the two are related by an exact sequence where the relative groups are the Wall surgery groups. The proof uses surgery on Poincaré spaces, and may be found in [97], or in slightly greater generality, in [9]. Another interesting issue is the extent to which Poincaré spaces can be built up by pasting together manifolds with boundary, using homotopy equivalences (instead of diffeomorphisms or homeomorphisms) between boundary components. It turns out that all Poincaré spaces can be pieced together this way (at least if one avoids the usual

problems with dimensions 3 and 4), and that the minimal number of manifold pieces required is an interesting invariant. See [84] and [9] for more details, as well as [91] for a survey of several other issues about Poincaré spaces.

2. **Stratified spaces:** *Stratified spaces* are locally compact spaces X which are not themselves manifolds but which have a filtration $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_k = X$ by closed subspaces such that each $X_j \setminus X_{j-1}$ is a manifold and the strata fit together in a suitable way. There are many different categories of such spaces, depending on the exact patching conditions assumed. But two important sets of examples motivate most of theory: algebraic varieties over $\mathbb{R}$ or $\mathbb{C}$, and quotients of manifolds by actions of compact Lie groups. Surgery theory has been very effective in classifying and studying such spaces. There is no room to go into details here, but see [26] and [83] for surveys.

3. **ENR homology manifolds:** Still another way to weaken the definition of a manifold is to consider ***homology** n-**manifolds***, spaces X with the property that for every $x \in X$, $H_j(X, X \setminus \{x\}; \mathbb{Z}) = \begin{cases} 0, & j \neq n, \\ \mathbb{Z}, & j = n. \end{cases}$ In order for such a space to look more like a topological manifold, it is natural to assume also that it is an ENR (Euclidean neighborhood retract). So a natural question is: is every ENR homology n-manifold M homeomorphic to a topological n-manifold? It has been known for a long time that the answer to this question is "no" (the simplest counterexample is the suspension of the Poincaré homology 3-sphere), so to make the question interesting, let's throw in the additional assumption that M has the "disjoint disks property." Then M has (at least) a very weak kind of transversality, and is thus quite close to looking like a manifold. Does this make it a manifold? This question has a long history, and the surprising answer of "no," due to Bryant, Ferry, Mio, and Weinberger [35], is discussed in this collection in [108].

3.11 Non-compact manifolds

Almost all the applications of surgery theory which we have discussed so far are for compact manifolds, but surgery can also be used to study non-compact manifolds as well. Here we just mention a few cases:

1. Siebenmann's characterization [127] of when a non-compact manifold X^n (without boundary) is the interior of some compact manifold W^n with boundary. Obvious necessary conditions are that X have finite homotopy type and have finitely many ends. Furthermore, the fundamental group "at infinity" in each end must be finitely presented,

and the Wall finiteness obstruction (see [60]) of the end must vanish. Siebenmann's Theorem ([127] or [26], §§1.5–1.6) says that these obvious necessary conditions are sufficient if $n \geq 6$.

2. Siebenmann's characterization of when a non-compact manifold W with boundary is an open collar of its boundary, or in other words, when $W \cong \partial W \times [0, \infty)$. It turns out ([128], Theorem 1.3) that necessary and sufficient conditions when $\dim M \geq 5$ (in any of the three categories TOP, PL, or DIFF) are that $(W, \partial W)$ is $(n-2)$-connected, W has one end, and π_1 is "essentially constant at ∞" with "$\pi_1(\infty)$"$\cong \pi_1(W)$. An alternative statement is that $W \cong \partial W \times [0, \infty)$ if and only if $(W, \partial W)$ is $(n-2)$-connected and W is proper homotopy equivalent to $\partial W \times [0, \infty)$. An elegant application ([128], Theorem 2.7) is a characterization of $\mathbb{R}^n$: if X^n is a noncompact oriented n-manifold, $n \geq 5$, then $X^n \cong \mathbb{R}^n$ (in any of the three categories TOP, PL, or DIFF) if and only if there exists a degree-1 proper map $\mathbb{R}^n \to X^n$.

3. Classification in a proper homotopy type. Surgery theory can be used to classify noncompact manifolds with a given *proper* homotopy type. For example, Siebenmann's Theorem 2.7 in [128] can be restated as saying that a non-compact n-manifold of dimension ≥ 5 is isomorphic to $\mathbb{R}^n$ if and only if it has the proper homotopy type of $\mathbb{R}^n$. Similarly much of the proof of the Farrell Fibration Theorem in section 3.7 above may be interpreted as a classification of manifolds with the proper homotopy type of $N \times \mathbb{R}$, for some compact manifold N.

4. There is a close connection between the classification of compact manifolds with fundamental group $\mathbb{Z}^n$ and the classification of noncompact manifolds with a proper map to $\mathbb{R}^n$, which played a vital role in Novikov's proof of the topological invariance of rational Pontrjagin classes (see [110]).

5. Finally (and probably most importantly), controlled surgery classifies noncompact manifolds in various "bounded" and "controlled" categories. See [111] and [112] for surveys and references.

4 Future directions

So where is surgery theory heading today? A glance at the dates on the papers in the bibliography to this article shows that history has proved wrong those who felt that surgery is a dead subject. At the risk of being another false prophet, I would predict that future development of the subject, at least over the next ten years, will lie mostly in the following areas:

- **Surgery in dimension 4.** Some very basic (and very hard!) questions remain concerning surgery in the topological category in dimension 4. (See [117].) In particular, is the surgery exact sequence valid without any restriction on fundamental groups? We can probably expect more work on this question, and also on the question of whether the smooth s-cobordism theorem is valid for 4-dimensional s-cobordisms (between 3-manifolds).

- **Differential geometry.** One of the areas of application of surgery theory that is developing most rapidly is that of applications to differential geometry. I would expect to see further growth in this area, especially in the areas of application to positive Ricci curvature (section 3.9.2 above) and to symplectic and contact geometry (section 3.9.5 above). In these areas what we basically have now are a lot of tantalizing examples, but very little in the way of definitive results, so there is lots of room for innovative new ideas.

- **Coarse geometry.** Still another area of very rapid current development is the study of "behavior at infinity" of noncompact manifolds. Especially fruitful ideas in this regard have been the "macroscopic" or "asymptotic" notions of Gromov [72] in geometry and geometric group theory and Roe's notion of "coarse geometry" [23]. But the Gromovian approach to geometry has not yet been fully integrated with surgery theory. The author expects a synthesis of these subjects to be a major theme in coming years. Ideas of what we might expect may be found in the work of Attie on classification of manifolds of bounded geometry [28] and in the work of Block and Weinberger [30].

- **Manifold-like spaces.** Last but not least, I think we can expect much more work on surgery theory applied to manifold-like spaces which are not manifolds (section 3.10 above). While outlines of basic surgery theories for stratified and singular spaces are now in place, major applications are only beginning to be developed. When it comes to homology manifolds, the situation is even more mysterious, due to the fact that all current arguments for "constructing" exotic ENR homology manifolds are basically non-constructive. It is also not clear if these spaces are homogeneous (like manifolds) or not. (See [108] for a discussion of some of the key unsolved problems.) So we can expect to see much further investigation of these topics.

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Department of Mathematics
University of Maryland
College Park, MD 20742, USA
email: `jmr@math.umd.edu`