The signature operator at 2

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Abstract

It is well known that the signature operator on a manifold defines a K-homology class which is an orientation after inverting 2. Here we address the following puzzle: what is this class localized at 2, and what special properties does it have? Our answers include the following:

- the K-homology class Δ_M of the signature operator is a bordism invariant;
- the reduction mod 8 of the K-homology class of the signature operator is an oriented homotopy invariant;
- the reduction mod 16 of the K-homology class of the signature operator is *not* an oriented homotopy invariant.

0 Introduction

The motivation for this paper comes from a basic question, of how to relate index theory (studied analytically) with geometric topology. More specifically, if M is a manifold (say smooth and closed), then the machinery of Kasparov theory ([5], [12], [13]) associates a K-homology class to any elliptic differential

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operator on M. If M is oriented, then in particular one can do this construction with the signature operator (with respect to some choice of Riemannian metric), and it is easy to show [3] that the class Δ_M so obtained is independent of the choice of metric. It is thus some invariant of the diffeomorphism type of M, and it should be possible to relate it to more familiar topological invariants. *Rationally*, Δ_M is computed by its Chern character, which the Atiyah-Singer index theorem shows to be the Poincaré dual of the (total) $\mathcal{L}$-class. This is the Atiyah-Singer L-class, not the Hirzebruch L-class, but the two only differ by certain powers of 2. [4] So, in particular, one can recover from Δ_M all the rational Pontrjagin classes of M. But when we localize at 2, these powers of 2 really matter, and it is not so clear what Δ_M encodes. The purpose of this paper is to take a first step toward solving this puzzle.

The main results of this paper are Theorem 2, which says that Δ_M is a bordism invariant, and Theorem 11, which says that the reduction of Δ_M mod 8 is an oriented homotopy invariant. On the other hand, a specific calculation in Proposition 17 shows that Δ_M is not an oriented homotopy invariant mod 16.

Below we use the following notation. We denote homotopy functors by regular italic or Greek letters, and we denote spectra by boldface letters. In particular we distinguish between a spectrum and the associated homology theory. Thus the spectra of topological K-theory and of L-theory are denoted by $\mathbf{K}$, $\mathbf{KO}$, $\mathbf{L}_\bullet$, $\mathbf{L}^\bullet$, etc. The spectra of oriented smooth and topological bordism Ω, Ω^{Top} are denoted by $\mathbf{MSO}$, $\mathbf{MSTop}$. The Eilenberg-Mac Lane spectrum corresponding to ordinary homology with coefficients G is denoted $\mathbf{H}(G)$, or $\mathbf{H}$ if $G = \mathbb{Z}$. If $\mathbf{A}$ denotes a spectrum, the corresponding homology theory and cohomology theory are denoted $H_*(\_\_; \mathbf{A})$ and $H^*(\_\_; \mathbf{A})$. Thus, for example, $H_*(\_\_; \mathbf{H}(G)) = H_*(\_\_; G)$. We write $\mathbb{Z}_{(2)}$ for $\mathbb{Z}$ localized at 2, i.e., for $\mathbb{Z}[\frac{1}{3}, \frac{1}{5}, \ldots] \subset \mathbb{Q}$. The notation $\mathbf{A}_{(2)}$ denotes the spectrum $\mathbf{A}$ localized at 2. Note that since $\mathbb{Z}_{(2)}$ is flat over $\mathbb{Z}$, $H_*(\_\_; \mathbf{A}_{(2)})$ coincides with $H_*(\_\_; \mathbf{A})_{(2)}$.

1 Basic properties of the invariant

Definition and Notations 1. Let M^n be a closed smooth oriented manifold. Fix a Riemannian metric on M. Then using this data, one can define the *signature operator* D_M on M, which is a self-adjoint elliptic operator. When the dimension $n = 2k$ of M is even, D_M is given by the de Rham operator $d + d^*$ on the total exterior algebra complex $\bigwedge^* T^*_{\mathbb{C}} M$, together with a certain $\mathbb{Z}/2$-grading on this bundle manufactured out of the Hodge $*$-operator [3]. More specifically, the grading operator τ (whose ± 1 eigenspaces are the even and

[3] This is because a homotopy of metrics gives a homotopy of operators, and one divides out by homotopy in defining the Kasparov groups.

[4] The Hirzebruch L-class is attached to the power series $x \coth x$, whereas the $\mathcal{L}$-class is attached to the power series $x \coth \frac{x}{2}$.

odd subbundles for the grading) is given on p-forms by $i^{p(p-1)+k}*$, and $d + d^*$ anticommutes with τ, so that it interchanges the even and odd subbundles. There is an equivalent approach using Clifford algebras [15, Ch. II, Example 6.2]. By means of the usual identification of the exterior algebra and Clifford algebra (as vector spaces, of course, not as algebras), we can view D_M as being given by the Dirac-type operator on $\mathrm{Cliff}_{\mathbb{C}} M$, the complexified Clifford algebra bundle of the tangent bundle (with connection and metric coming from the Riemannian connection and metric), with grading operator τ given by the "complex volume element" [15, pp. 33–34 and 135–137], a parallel section of $\mathrm{Cliff}_{\mathbb{C}} M$ which in local coordinates is given by $i^k e_1 \cdots e_n$, where $e_1, \ldots, e_n$ are a local orthonormal frame for the tangent bundle.

When the dimension $n = 2k + 1$ of M is odd, $\tau = i^{k+1} e_1 \cdots e_n$ acting on $\mathrm{Cliff}_{\mathbb{C}} M$ by Clifford multiplication still satisfies $\tau^2 = 1$, but the Dirac-type operator commutes with τ. Furthermore, if σ is the usual grading operator on $\mathrm{Cliff}_{\mathbb{C}} M$ (which is $(-1)^p$ on products $e_{i_1} \cdots e_{i_p}$), then τ and the Dirac-type operator both anticommute with σ. So we define the signature operator in this case to be the restriction of the Dirac-type operator to the $+1$ eigenspace of τ. (See also [20, Remark following Definition 2.1].) From a slightly fancier point of view, we consider the Dirac-type operator on $\mathrm{Cliff}_{\mathbb{C}} M$, with the grading given by σ, but with the extra action of the Clifford algebra $C_1 = \mathrm{Cliff}_{\mathbb{C}} \mathbb{R}$, where the odd generator of C_1 acts by τ. By means of Kasparov's model of K-homology [12], [13], D_M defines a class

$$\Delta_M \in \begin{cases} K_0(M), & n \text{ even}, \\ K_1(M), & n \text{ odd}, \end{cases}$$

which is independent of the choice of Riemannian metric (since a homotopy of metrics gives a homotopy of operators). (Recall that a class in $K_0(M)$ is defined by a graded Hilbert space equipped with a $*$-representation of $C(M)$, together with an odd operator "essentially commuting" with the action of $C(M)$. It is easiest to use the Baaj-Julg model [4] in which the operator is unbounded and self-adjoint, with compact resolvent, and "essentially commuting" means there is a dense subalgebra of $C(M)$ (in this case $C^\infty(M)$) that preserves the domain of the operator and has bounded commutator with it. A class in $K_1(M)$ is similarly defined by a graded Hilbert space with commuting actions of C_1 and of $C(M)$, and with a C_1-linear odd operator "essentially commuting" with the action of $C(M)$.) By Bott periodicity, we will identify the group in which Δ_M lives with the group $K_n(M)$.

The class Δ_M has been studied by many authors, and in $K_n(M)[\frac{1}{2}]$, it is an orientation class, basically agreeing with Sullivan's $K[\frac{1}{2}]$-orientation for topological manifolds. (See for example [16, Ch. 4] for the theory of the Sullivan orientation and [11], [8], [9], [17], and [25] for the connections with the signature operator.) Our purpose here is to study the behavior of Δ_M in K-theory *localized at 2*, where it definitely is *not* an orientation class.

Theorem 2. *Let M^n be a closed oriented manifold, let X be any finite CW complex, and let $f : M \to X$ be a continuous map. Then $f_*(\Delta_M) \in K_n(X)$ is a bordism invariant of the pair (M, f). In other words, if M_1 and M_2 are closed oriented n-manifolds with maps $f_i : M_i \to X$, W^{n+1} is a compact oriented manifold with boundary with $\partial W = M_1 \amalg (-M_2)$, and if $f : W \to X$ restricts to f_i on M_i, then $(f_1)_*(\Delta_{M_1}) = (f_2)_*(\Delta_{M_2})$.*

PROOF. We use the fact, pointed out for example in [20, p. 290], that the signature operator on W defines a class $\Delta_{(W,\partial W)}$ in the relative K-homology group $K_{n+1}(W, \partial W)$, and that $\partial\Delta_{(W,\partial W)} = k(\Delta_{M_1} - \Delta_{M_2})$ in $K_n(\partial W) = K_n(M_1) \oplus K_n(M_2)$, where

$$k = \begin{cases} 1, & n \text{ even}, \\ 2, & n \text{ odd}. \end{cases}$$

(The reason for the extra factor of 2 when n is odd will be elucidated in the course of the proof of Lemma 6 below.) First suppose n is even, and consider the commutative diagram

$$\begin{array}{ccc} K_{n+1}(W, \partial W) & \xrightarrow{f_*} & K_{n+1}(X, X) = 0 \\ \partial \downarrow & & \partial \downarrow \\ K_n(M_1) \oplus K_n(M_2) & \xrightarrow{(f_1)_* + (f_2)_*} & K_n(X). \end{array}$$

Chasing $\Delta_{(W,\partial W)}$ both ways around the diagram, we see

$$(f_1)_*(\Delta_{M_1}) - (f_2)_*(\Delta_{M_2}) = 0,$$

as desired. The general structure of this argument comes from [6] and [7].

Now suppose n is odd. The situation is harder because of the factor of 2; the above argument only shows that $(f_1)_*(2\Delta_{M_1}) - (f_2)_*(2\Delta_{M_2}) = 0$, i.e., that $f_*(2\Delta_M)$ is a bordism invariant. This is not good enough for us since we will be concerned below with 2-primary torsion. However, we can use a variant of the trick in [20, §4] for getting around this. As pointed out there, we can split $D_{(W,\partial W)}$ as a direct sum of two operators E_1 and E_2, each with "boundary" $D_{\partial W}$, provided that W admits an everywhere non-vanishing vector field v which on ∂W is normal to the boundary, pointing inward. (See also [15, Ch. IV, proof of Theorem 2.7].) Then the argument just given will prove that $f_*(\partial[E_1]) = 0$, or that $(f_1)_*(\Delta_{M_1}) - (f_2)_*(\Delta_{M_2}) = 0$. The only problem is that there is an obstruction to the existence of v; a necessary and sufficient condition for such a vector field v to exist (assuming that W is connected) is that $\chi(W) = 0$. First we dispose of one exceptional case: if $n = 1$, then a closed n-manifold M is just a disjoint union of finitely many copies of S^1. Furthermore $\Omega_1(X) = H_1(X)$ and Δ_{S^1} is the usual generator of $K_1(S^1)$. Hence the theorem just asserts in this case that given a disjoint union M of finitely many (oriented) copies of S^1 and given a map $f : M \to X$, then f_* of the orientation class in $K_1(M)$ is just the image of f_* of the orientation class in $H_1(M)$ under the canonical map $H_1 \to K_1$, which is clear. So we may suppose $n \geq 3$. If we replace W by $W' = W \# N$, where N is a closed oriented $(n+1)$-manifold (we form the connected sum away from the boundary), we can extend f over W', and (since W is even-dimensional) $\chi(W)$ is replaced by $\chi(W) + \chi(N) - 2$.

If $n + 1 = \dim W$ is divisible by 4, we can make $\chi(N)$ whatever we want (by taking a connected sum of copies of $\mathbb{CP}^{\frac{n+1}{2}}$, which has odd Euler characteristic $\frac{n+3}{2}$, and with copies of $S^2 \times S^{n-1}$ and of $S^1 \times S^n$, which have Euler characteristic 4 and 0, respectively), so taking $\chi(N) = 2 - \chi(W)$ reduces us to the case where the vector field v exists.

If $n + 1 = \dim W$ is congruent to 2 mod 4, then there is still a further complication since we can only make $\chi(N)$ an arbitrary *even* integer. If $\chi(W)$ is even, then again taking $\chi(N) = 2 - \chi(W)$ reduces us to the case where the vector field v exists. If $\chi(W)$ is odd, punch out a small disk from W to obtain W' with $\partial W' = M_1 \amalg (-M_2) \amalg S^n$ and with $\chi(W')$ even. By the case we just handled, we know $(f_1)_*(\Delta_{M_1}) - (f_2)_*(\Delta_{M_2}) + f_*(\Delta_{S^n}) = 0$. However, by construction, f is null-homotopic when restricted to S^n, so $f_*(\Delta_{S^n})$ factors through $K_1(\mathrm{pt}) = 0$. So again $(f_1)_*(\Delta_{M_1}) - (f_2)_*(\Delta_{M_2}) = 0$. □

Corollary 3. *For each $n \geq 0$, the map $(f : M \to X) \rightsquigarrow f_*(\Delta_M)$ defines a natural transformation of homotopy functors $s_n : \Omega_n \to K_n$, from oriented bordism to K-homology.*

PROOF. Theorem 2 shows we have a well-defined map $\Omega_n(X) \to K_n(X)$ for every finite CW complex X. Naturality is obvious. □

Remark 4. Caution: the natural transformations $\{s_n : \Omega_n \to K_n\}_{n\geq 0}$ do **not** give a natural transformation of homology theories $\Omega \to K$, hence do not come from a map of spectra $\mathbf{MSO} \to \mathbf{K}$. However, there **is** a map of spectra $\tilde{s} : \mathbf{MSO} \to \mathbf{K}[\frac{1}{2}]$ (the spectrum on the right is K-theory with the prime 2 inverted) defined by the natural transformations of homotopy functors $2^{-\lfloor n/2 \rfloor} s_n : \Omega_n \to K_n[\frac{1}{2}]$. To see this, note that $\{s_n : \Omega_n \to K_n\}_{n\geq 0}$ would be a map of homology theories if and only if the diagrams

$$\begin{array}{ccc} \Omega_{n+1}(X \times [0, 1], X \times \{0, 1\}) & \xrightarrow{s_{n+1}} & K_{n+1}(X \times [0, 1], X \times \{0, 1\}) \\ \cong\big\downarrow & & \cong\big\downarrow \\ \Omega_n(X) & \xrightarrow{s_n} & K_n(X) \end{array}$$

were commutative for all n. By definition of s_n, this would be tantamount to showing that for all closed oriented n-manifolds M^n, the composite

$$K_{n+1}(M \times [0, 1], M \times \{0, 1\}) \xrightarrow{\partial} K_n(M \times \{0, 1\}) \xrightarrow{\text{proj}} K_n(M),$$

which is an isomorphism, would take $\Delta_{(M\times[0,1],\,M\times\{0,1\})}$ to Δ_M. But as we saw in the proof of Theorem 2, this is true for n even but false for n odd. However,

$$\begin{array}{ccc} \Omega_{n+1}(X\times[0,\,1],\,X\times\{0,\,1\}) & \xrightarrow{2^{-\lfloor (n+1)/2\rfloor}s_{n+1}} & K_{n+1}(X\times[0,\,1],\,X\times\{0,\,1\})[\tfrac{1}{2}] \\ \cong\Big\downarrow & & \cong\Big\downarrow \\ \Omega_n(X) & \xrightarrow{2^{-\lfloor n/2\rfloor}s_n} & K_n(X)[\tfrac{1}{2}] \end{array}$$

is commutative for all n, because if n is even, $\lfloor (n+1)/2\rfloor = \lfloor n/2\rfloor = n/2$, and if n is odd, $2^{\lfloor (n+1)/2\rfloor} = 2\cdot 2^{\lfloor n/2\rfloor}$ and we've corrected for the extra factor of 2.

Theorem 5. *After localization at 2, the natural transformation $s_n : \Omega_n \to K_n$ of Corollary 3 factors through $\bigoplus_{0\le k\le\lfloor n/4\rfloor} H_{n-4k}(\_\_;\,\mathbb{Z}_{(2)})$.*

Before starting on the proof we need to study how the signature operator on a product manifold is related to the signature operators on the factors.

Lemma 6. *Let M^m and N^n be closed manifolds. Then $\Delta_{M\times N} = \Delta_M \boxtimes \Delta_N$ if mn is even, and $\Delta_{M\times N} = 2\Delta_M\boxtimes\Delta_N$ if mn is odd. Here $\boxtimes$ denotes the external Kasparov product $K_m(M)\otimes K_n(N) \to K_{m+n}(M\times N)$, m and n interpreted mod 2.*

PROOF of Lemma 6. Choose Riemannian metrics on M and N, and give $M\times N$ the product metric. We use the Clifford algebra point of view given in Definition 1. Observe that $\mathrm{Cliff}_{\mathbb{C}}(M\times N)$, with its usual parity grading, naturally splits as the graded tensor product $\mathrm{Cliff}_{\mathbb{C}} M \widehat{\otimes} \mathrm{Cliff}_{\mathbb{C}} N$ [15, Ch. I, §1], and that the Dirac-type operator $D_{M\times N}$ on $\mathrm{Cliff}_{\mathbb{C}}(M\times N)$ splits as $D_M\widehat{\otimes}1 + 1\widehat{\otimes}D_N$, which matches perfectly with the Baaj-Julg "unbounded" version ([4] or [5, §17.11]) of the Kasparov product $\boxtimes$. So the whole issue is to see what happens to the gradings. Let τ_M and τ_N be the "complex volume elements" in $\mathrm{Cliff}_{\mathbb{C}} M$ and $\mathrm{Cliff}_{\mathbb{C}} N$, respectively, as in Definition 1. If $e_1,\,\dots,\,e_m$ and $f_1,\,\dots,\,f_n$ are local orthonormal frames for the tangent bundles of M and N, respectively, then

$$\tau_M = i^{\lfloor m/2\rfloor} e_1\cdots e_m, \quad \tau_N = i^{\lfloor n/2\rfloor} f_1\cdots f_n,$$

and

$$\tau_{M\times N} = i^{\lfloor (m+n)/2\rfloor} e_1\cdots e_m f_1\cdots f_n.$$

The cases where mn is even are straightforward now, so we only consider the harder case where m and n are both odd. In this case, τ_M and τ_N are both odd Clifford elements, and

$$\tau_M\tau_N = -\tau_N\tau_M, \qquad \tau_{M\times N} = i\tau_M\tau_N.$$

Now $\mathrm{Cliff}_{\mathbb{C}}(M\times N)$ comes with the action of $C_1\widehat{\otimes}C_1 = C_2$ defined by τ_M and τ_N, and we see that the external Kasparov product of D_M and D_N is the class in $KK(C(M\times N),\,C_2) = K_2(M\times N)$ defined by $\mathrm{Cliff}_{\mathbb{C}}(M\times N)$ with the

Dirac-type operator and this C_2-action. To compare this with $\Delta_{M\times N}$, we need to apply the Bott periodicity isomorphism

$$KK(C(M\times N),\, C_2) \cong KK(C(M\times N),\, \mathbb{C}),$$

which comes from the Morita equivalence between $C_2 \cong M_2(\mathbb{C})$ (with the standard even grading) and $\mathbb{C}$. This isomorphism is obtained by cutting down by a rank-one idempotent in C_2, for which the obvious choice is $(1+\tau_{M\times N})/2$. So the upshot is that $\Delta_{M\times N} \cong 2\cdot(D_M \boxtimes D_N)$ in this case. □

PROOF of Theorem 5. We use the fact [26, Lemma, p. 209], basically due to Wall, that $\mathbf{MSO}_{(2)}$ splits as a sum of (shifted) Eilenberg-Mac Lane spectra for the groups $\mathbb{Z}_{(2)}$ and $\mathbb{Z}/2$. Thus for any X,

$$\Omega_n(X)_{(2)} \cong \bigoplus_{0\le j\le n} H_{n-j}(X;\, (\Omega_j)_{(2)}).$$

For each summand of $\mathbb{Z}_{(2)}$ in $(\Omega_j)_{(2)}$, the associated summand of

$$H_{n-j}(X;\, (\Omega_j)_{(2)})$$

corresponds to bordism classes of the form $M^{n-j}\times N^j \xrightarrow{f} X$, where the map f collapses the second factor N^j to a point. Let's compute s_n on this class. By Lemma 6, $\Delta_{M\times N} = \Delta_M \boxtimes \Delta_N$ (or twice this, if M and N are both odd-dimensional), where $\boxtimes$ denotes the external Kasparov product. Since f factors as $f|_M \times c$, where c is the "collapse map" $N \to \mathrm{pt}$, we have $f_*(\Delta_M \boxtimes \Delta_N) = (f|_M)_*(\Delta_M) \otimes c_*(\Delta_N)$, where $\otimes$ again denotes a Kasparov product. But $c_*(\Delta_N) \in K_j(\mathrm{pt})$ vanishes if j is odd and is just the signature of N if j is even. So

$$s_n\Big(M^{n-j}\times N^j \xrightarrow{f} X\Big) = s_{n-j}(M^{n-j} \xrightarrow{f} X)\cdot \text{signature}\, N.$$

For the $\mathbb{Z}/2$ summands in $(\Omega_j)_{(2)}$, things are a bit more complicated. If a homology class in $H_{n-j}(X;\,\mathbb{Z}/2)$ is the reduction of an integral class, then again the associated bordism classes are of the form $M^{n-j}\times N^j \xrightarrow{f} X$ as above. However, one also has homology classes in $H_{n-j}(X;\,\mathbb{Z}/2)$ which are not reductions of integral homology classes. The associated bordism classes can be represented by bordism Toda brackets or Massey products, in the sense of [1]. Choose $P^{n-j-1} \xrightarrow{f} X$ representing the Bockstein of the given class in $H_{n-j}(X;\,\mathbb{Z}/2)$, and N^j representing a $\mathbb{Z}/2$ summand in Ω_j. By [2, Prop. 4 and Prop. 5], N may be chosen to have an orientation-reversing involution r.[5] Then our class of order 2 in $H_i(X, \Omega_{n-i})$ corresponds to a Toda bracket

[5] Anderson shows that torsion generators in Ω_* may be chosen to be total spaces $\mathbb{P}(\lambda\oplus(2k+1)\theta)$ of $\mathbb{RP}^{2k+1}$ bundles (for varying k) coming from real vector bundles

$\langle P, 2, N\rangle$, which we can realize as follows. Let $F: V \to X$ bound two copies of $f: P \to X$. Now $N \times I$ bounds $N \coprod -N$. So glue $V \times N$ to $P \times N \times I$ via the usual gluing on one copy of $P \times N$, $\mathrm{id} \times r$ on the other. The result is a fiber bundle $N \to E \to M$, with $M = V \cup_{P \times \{0,1\}} P \times I$ non-orientable and the map $E \to X$ factoring through M. Note that since $r^2 = \mathrm{id}$, E has a double cover of the form $\widetilde{M} \times N$, with the covering map the quotient map for the involution $\phi \times r$, where $M = \widetilde{M}/\phi$ and the map $E \to M$ is just projection onto the first factor $(\widetilde{M} \times N)/(\phi \times r) \to \widetilde{M}/\phi = M$.

Now fix metrics on $\widetilde{M}$ and N for which ϕ and r are isometries, and consider the signature operator element on E. We are "almost" in the situation of Lemma 6, but there are complications due to the fact that ϕ and r reverse orientation (so that M itself doesn't carry a signature operator, just a "twisted" signature operator, with the twist given by the orientation line bundle). The signature operator of E can be viewed as acting on sections of $\mathrm{Cliff}_{\mathbb{C}}\, \widetilde{M} \widehat{\otimes} \mathrm{Cliff}_{\mathbb{C}}\, N$ which are invariant under the involution induced by $\phi \times r$. Since the map $E \to X$ factors through M, it will be enough to show that the class in $K_*(M)$, defined by the signature operator on E, is 0. This class is given by the graded Hilbert space

$$L^2(\mathrm{Cliff}_{\mathbb{C}}\, \widetilde{M})^{\phi_*} \widehat{\otimes} L^2(\mathrm{Cliff}_{\mathbb{C}}\, N)^{r_*} \oplus L^2(\mathrm{Cliff}_{\mathbb{C}}\, \widetilde{M})^{\phi_*\text{-odd}} \widehat{\otimes} L^2(\mathrm{Cliff}_{\mathbb{C}}\, N)^{r_*\text{-odd}},$$

the operator $D_{\widetilde{M}} \widehat{\otimes} 1 \oplus 1 \widehat{\otimes} D_N$, and the complex volume form $\tau_{\widetilde{M} \times N}$, which up to a power of i is $\tau_{\widetilde{M}} \cdot \tau_N$. Since we are restricting the class in $K_*(E)$ to an element of $K_*(M)$, there is no loss of generality in replacing $D_{\widetilde{M}} \widehat{\otimes} 1 \oplus 1 \widehat{\otimes} D_N$ with $D_{\widetilde{M}} \widehat{\otimes} 1$ and replacing $L^2(\mathrm{Cliff}_{\mathbb{C}}\, N)$ by the finite-dimensional kernel of D_N on this Hilbert space, which we can identify with the de Rham cohomology of N. Thus our class is now given by the graded Hilbert space

$$L^2(\mathrm{Cliff}_{\mathbb{C}}\, \widetilde{M})^{\phi_*} \widehat{\otimes} H^*(N)^{r_*} \oplus L^2(\mathrm{Cliff}_{\mathbb{C}}\, \widetilde{M})^{\phi_*\text{-odd}} \widehat{\otimes} H^*(N)^{r_*\text{-odd}}, \qquad (1)$$

multiplication by functions in $C^\infty(M)$, the operator $D_{\widetilde{M}} \widehat{\otimes} 1$, and the complex volume form $\tau_{\widetilde{M} \times N}$. Note that since r and ϕ are orientation-reversing isometries, ϕ_* anticommutes with $\tau_{\widetilde{M}}$, and similarly r_* anticommutes with τ_N. Since r_* and τ_N anticommute, they generate a complex Clifford algebra isomorphic to M_2 acting on $H^*(N)$, and so the two eigenspaces of τ_N or of r_* acting on $H^*(N)$ each have the same dimension.

There are now various subcases, depending on the parities of the dimensions of M and N, just as in the proof of Lemma 6, but the differences among them are the same as before, so we content ourselves with writing out the details of the cases where $\dim M$ and $\dim N$ are both even or both odd. Since $D_{\widetilde{M}}$ commutes with ϕ_*, the two summands in (1) are both invariant under $D_{\widetilde{M}}$ (as

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$\lambda \oplus (2k+1)\theta$. Here λ is a non-trivial real line bundle and $(2k+1)\theta$ is a trivial $\mathbb{R}^{2k+1}$-bundle. The orientation-reversing involution can be chosen as the projectivization of the vector bundle automorphism given by -1 on λ and $+1$ on $(2k+1)\theta$.

well as multiplication by functions in $C^\infty(M)$), but are interchanged by $\tau_{\widetilde{M}\times N}$. If the second tensor factors were absent (i.e., we had just $L^2(\mathrm{Cliff}_{\mathbb{C}}\,\widetilde{M})^{\phi_*} \oplus L^2(\mathrm{Cliff}_{\mathbb{C}}\,\widetilde{M})^{\phi_*\text{-odd}}$ with multiplication by functions in $C^\infty(M)$, the operator $D_{\widetilde{M}}$, and grading given by $\tau_{\widetilde{M}}$), the corresponding K-homology class would be the class of the twisted signature operator on M. But because of the second factors, this class is multiplied by an integer, namely the signature of N, which is 0. Now consider the case where $\dim M$ and $\dim N$ are both odd. In this case, the K-homology class is just an integer multiple of what we'd have if N were replaced by S^1 and r by complex conjugation (on the unit circle in the complex plane). Then (1) would reduce to

$$L^2(\mathrm{Cliff}_{\mathbb{C}}\,\widetilde{M})^{\phi_*}\widehat{\otimes}\mathbb{C}^{\text{even}} \oplus L^2(\mathrm{Cliff}_{\mathbb{C}}\,\widetilde{M})^{\phi_*\text{-odd}}\widehat{\otimes}\mathbb{C}^{\text{odd}},$$

where $\mathbb{C}^{\text{even}}$ and $\mathbb{C}^{\text{odd}}$ denote a copy of $\mathbb{C}$ in even (resp., odd) degree. The two eigenspaces of $\tau_{\widetilde{M}\times N}$ would then be identical as $C^\infty(M)$-modules, or more precisely, the Kasparov module has the form

$$\left(\mathcal{H}\oplus\mathcal{H},\ \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix}\right),$$

where $\mathcal{H}$, one of the eigenspaces of $\tau_{\widetilde{M}\times N}$, is a Hilbert space module for $C^\infty(M)$, and T is a self-adjoint operator on $\mathcal{H}$ with compact resolvent, commuting up to bounded operators with the $C^\infty(M)$-action. So again the class would be trivial, since it is a Kasparov product of the class in $K_1(M)$ represented by $(\mathcal{H}, T)$ with a (trivial) class in $KK^1(\mathbb{C},\mathbb{C}) = 0$. The subcases where one dimension is even and one is odd are similar to the cases we've considered, and thus in all cases, the $\mathbb{Z}/2$ summands in Ω_* don't contribute.

Since $\Omega_j \otimes_{\text{signature}} \mathbb{Z}$ is $\mathbb{Z}$ for j divisible by 4 and is 0 otherwise, we obtain the desired factorization. □

Theorem 7. *There are natural transformations*

$$\mathcal{S}_n : H_n(\_\_;\mathbb{Z}_{(2)}) \to K_n(\_\_)_{(2)} = H_n(\_\_;\mathbf{K}_{(2)}),$$

such that, after localization at 2, the natural transformation $s_n : \Omega_n \to K_n$ of Corollary 3 factors through the natural transformation

$$\bigoplus_{0\le k\le \lfloor n/4\rfloor} H_{n-4k}(\_\_;\mathbb{Z}_{(2)}) \xrightarrow{\bigoplus_{0\le k\le \lfloor n/4\rfloor}\mathcal{S}_{n-4k}} K_n(\_\_)_{(2)} = H_n(\_\_;\mathbf{K}_{(2)}).$$

(Here we are implicitly using Bott periodicity to view $\mathcal{S}_{n-4k}$ as a map into K_n.) For the reasons discussed in Remark 4, the maps $\mathcal{S}_n$ do **not** *give a natural transformation of homology theories from ordinary homology to K-homology.*

PROOF. This is partially contained in Theorem 5, but we need to construct the natural transformations $\mathcal{S}_n$ and see that they have the right properties.

To do this, choose a natural transformation of homology theories $\rho : \mathbf{H}_{(2)} \to \mathbf{MSO}_{(2)}$ that splits the natural orientation map $\mathcal{O} : \mathbf{MSO} \to \mathbf{H}$ after localizing. (Localizing at 2 is essential here; there is no integral splitting map, since for odd primes p, $\mathbf{MSO}_{(p)}$ is built out of Brown-Peterson spectra, not Eilenberg-Mac Lane spectra.) Then let $\mathcal{S}_n = s_n \circ \rho$. We just need to see that the factorization of Theorem 5 indeed comes from $\bigoplus_k \mathcal{S}_{n-4k}$. By Theorem 5 and its proof, it's enough to check this on the the product of the image of ρ_{n-4k} with a $4k$-manifold of signature 1, say $\mathbb{CP}^{2k}$, but this case is immediate from the first calculation in the proof of Theorem 5. □

Theorem 8. *The map s_n factors through the (real) symmetric L-theory orientation $\sigma_{\mathbb{R}}$. This is the natural transformation of homology theories*

$$\Omega_n(\_\_) \xrightarrow{\sigma_{\mathbb{R}}} H_n(\_\_; \mathbf{L}^\bullet(\mathbb{R}))$$

obtained from the integral symmetric L-theory orientation, described in [21, §7.1] *and in* [22, §§16–17] *as a natural transformation of homology theories:*

$$\Omega_n(\_\_) \xrightarrow{\sigma} H_n(\_\_; \mathbf{L}^\bullet(\mathbb{Z})),$$

followed by the obvious change-of-rings map

$$H_n(\_\_; \mathbf{L}^\bullet(\mathbb{Z})) \to H_n(\_\_; \mathbf{L}^\bullet(\mathbb{R})).$$

(*Note that $\sigma_{\mathbb{R}}(M)$ maps under symmetric L-theory assembly to the symmetric signature of Mishchenko.*)

PROOF. One could perhaps approach the relationship between the s_n and $\sigma_{\mathbb{R}}$ directly, using Hutt's idea [10] for describing the latter in terms of cobordism classes of complexes of sheaves satisfying Poincaré duality, together with the description of the signature operator class in [11] or [17]. But this would be technically complicated (indeed, this is why [10] has not been published), and here we can get away with something simpler. We consider the maps s_n localized both away from 2 and at 2. The map $\mathbf{L}^\bullet(\mathbb{Z})[\frac{1}{2}] \to \mathbf{L}^\bullet(\mathbb{R})[\frac{1}{2}]$ is a homotopy equivalence, and $\mathbf{L}^\bullet(\mathbb{R})[\frac{1}{2}] \cong \mathbf{KO}[\frac{1}{2}]$ (see [16, pp. 83–85] and [23]). Our previously constructed map of homology theories $\mathbf{MSO} \to \mathbf{KO}[\frac{1}{2}]$, given by the maps $2^{-\lfloor \frac{n}{2} \rfloor} s_n$, coincides with $\sigma_{\mathbb{R}}$, since both maps do the same thing on coefficient groups, sending $[M^n] \in \Omega_n$ to $2^{-\lfloor \frac{n}{2} \rfloor}\text{signature}(M)$. (See [16, pp. 83–85].) Hence it is clear that s_n factors though $\sigma_{\mathbb{R}}$ after localizing away from 2; in fact, s_n is nothing but $\sigma_{\mathbb{R}}$ followed by the natural transformation (of functors but not of homology theories) $KO[\frac{1}{2}]_* \to KO[\frac{1}{2}]_*$ which is multiplication by $2^{\lfloor \frac{n}{2} \rfloor}$ in degree n.

Localized at 2, **MSO** and the L-theory spectra $\mathbf{L}^\bullet(\mathbb{Z})$ and $\mathbf{L}^\bullet(\mathbb{R})$ are of generalized Eilenberg-Mac Lane type ([27]; this can also be deduced from the results in [16], Ch. 7). The natural transformation $\sigma_{\mathbb{R}}$, since it comes from the symmetric signature, sends (with the notation of the proof of Theorem 5)

$$\left(M^{n-j} \times N^j \xrightarrow{f} X\right) \mapsto \sigma(M^{n-j} \xrightarrow{f} X) \cdot \text{signature}\, N.$$

Note that the connective spectrum $\mathbf{L}^\bullet(\mathbb{R})_{(2)}\langle 0\rangle$ is a direct summand in $\mathbf{L}^\bullet(\mathbb{R})_{(2)}$, and $\sigma_{\mathbb{R}}$ is a split surjection of homology theories onto $\mathbf{L}^\bullet(\mathbb{R})_{(2)}\langle 0\rangle$. So comparison with the above calculation of what s_n does on the same generators shows that s_n localized at 2 is $\sigma_{\mathbb{R}}$ followed by $\bigoplus_j \mathcal{S}_{n-4j}$ (in the notation of Theorem 7).

Now consider the pullback diagram of functors

$$\begin{array}{ccc} \Omega_n(\_\_) & \longrightarrow & \Omega_n(\_\_)_{(2)} \\ \downarrow & & \downarrow \\ \Omega_n(\_\_)[\frac{1}{2}] & \longrightarrow & \Omega_n(\_\_) \otimes \mathbb{Q}. \end{array}$$

This square maps under $\sigma_{\mathbb{R}}$ to a corresponding square

$$\begin{array}{ccc} H_n(\_\_; \mathbf{L}^\bullet(\mathbb{R})) & \longrightarrow & H_n(\_\_; \mathbf{L}^\bullet(\mathbb{R})_{(2)}) \\ \downarrow & & \downarrow \\ H_n(\_\_; \mathbf{L}^\bullet(\mathbb{R})[\frac{1}{2}]) & \longrightarrow & H_n(\_\_; \mathbf{L}^\bullet(\mathbb{R}) \otimes \mathbb{Q}). \end{array}$$

Recall that we showed that s_n localized at two is $\sigma_{\mathbb{R}}$ followed by $\bigoplus_j \mathcal{S}_{n-4j}$, and that s_n inverting two is $\sigma_{\mathbb{R}}$ followed by multiplication by $2^{\lfloor \frac{n}{2} \rfloor}$. These two agree rationally, so s_n factors as $\sigma_{\mathbb{R}}$ followed by the pullback of the natural transformations $\bigoplus_j \mathcal{S}_{n-4j}$ and multiplication by $2^{\lfloor \frac{n}{2} \rfloor}$. □

We can get some more information about the maps $\mathcal{S}_n : H_n(\_\_; \mathbb{Z}_{(2)}) \to K_n(\_\_)_{(2)}$ as follows. Consider a closed connected oriented n-manifold M^n, $n = 4k$, and let $c : M \to \text{pt}$ be the "collapse map." Chasing the commutative diagram

$$\begin{array}{ccccc} \Omega_{4k}(M)_{(2)} & \longrightarrow & \bigoplus_{j=0}^k H_{4j}(M; \mathbb{Z}_{(2)}) & \xrightarrow{\bigoplus_{j=0}^k \mathcal{S}_{4j}} & K_0(M)_{(2)} \\ {\scriptstyle c_*}\downarrow & & {\scriptstyle c_*}\downarrow & & {\scriptstyle c_*}\downarrow \\ \Omega_{4k}(\text{pt})_{(2)} & \longrightarrow & H_0(\text{pt}; \mathbb{Z}_{(2)}) = \mathbb{Z}_{(2)} & \xrightarrow{\mathcal{S}_0} & K_0(\text{pt})_{(2)} = \mathbb{Z}_{(2)}, \end{array}$$

we see that $[M \to M]$ in the upper left maps to Δ_M in the upper right and down to $c_*(\Delta_M) = \text{signature}\,(M)$ in the lower right. (A basic principle of Kasparov theory is that for any elliptic operator such as the signature operator, the image under c_* of its K-homology class is its index.) One the other hand, $c_*([M \to M]) = [M \to \text{pt}]$ in the lower left, which maps to signature (M) in $H_0(\text{pt})$. From this one can see that $\mathcal{S}_0 : H_0(\text{pt}; \mathbb{Z}_{(2)}) \to K_0(\text{pt})_{(2)}$ is the identity map $\mathbb{Z}_{(2)} \to \mathbb{Z}_{(2)}$, that the map $\Omega_{4k}(M)_{(2)} \to H_0(M; \mathbb{Z}_{(2)})$ can be identified with the signature, and that the image of $\mathcal{S}_j$, $j > 0$, lies in $\widetilde{K}_0(M)_{(2)}$.

More generally, consider a closed oriented n-manifold M^n. The canonical generator $[M]$ of $H_n(M; \mathbb{Z})$ is the top-degree part of the homology class corresponding to the bordism class of the identity map $M \to M$, so $\mathcal{S}_n([M]) \equiv \Delta_M$ modulo the image of $\mathcal{S}_{n-4} \oplus \cdots$. Let $f : M^n \to S^n$ be a map of degree 1. Then f induces an isomorphism on H_n (by definition!) and also induces a map

$$\kappa : K_n(M) \to H_n(M^n; \mathbb{Z})$$

via the composite

$$\kappa : K_n(M) \twoheadrightarrow \widetilde{K}_n(M) \xrightarrow{f_*} \widetilde{K}_n(S^n) \cong H_n(S^n; \mathbb{Z}) \xrightarrow{\cong} (f_*)^{-1} H_n(M^n; \mathbb{Z}). \quad (2)$$

(Here the isomorphism $\widetilde{K}_n(S^n) \cong H_n(S^n; \mathbb{Z})$ is not quite the Chern character (which involves denominators!) but instead comes from the degeneration of the Atiyah-Hirzebruch spectral sequence.) One can also view κ as the map induced by collapsing the $(n-1)$-skeleton of a suitable CW decomposition of M.

Proposition 9. *Let M^n be a closed oriented n-manifold, and let $\mathcal{S}_n$ be as defined in Theorem 7 and κ as defined in* (2)*, localized at* 2*. Then $\kappa \circ \mathcal{S}_n$ is multiplication by $2^{\lfloor n/2 \rfloor}$ on $H_n(M)$.*

PROOF. Since $\mathcal{S}_n$ is a natural transformation, $\kappa \circ \mathcal{S}_n$ mapping $H_n(M)$ to itself must be multiplication by a constant, and it is enough to compute for a sphere S^n. For $n = 0$ or 1, Δ_{S^n} is the usual orientation class in $K_n(S^n)$. For n even, the Clifford algebra of $\mathbb{C}^n$ is isomorphic to a matrix algebra, and D_{S^n} is basically the Dirac operator with coefficients in the dual of the (complex) spinor bundle, which has dimension $\sqrt{2^n} = 2^{n/2}$. Since the Dirac operator gives an orientation for K-homology, the result is correct in this case. For n odd, the Clifford algebra of $\mathbb{C}^n$ splits as a sum of two matrix algebras each of dimension 2^{n-1}, and D_{S^n} is basically the Dirac operator with coefficients in a spinor bundle of dimension $\sqrt{2^{n-1}} = 2^{\lfloor n/2 \rfloor}$, so again the calculation is correct. □

Another important fact about the element $f_*(\Delta_M)$ associated to a bordism class $[f : M^n \to X]$, which is true integrally (in other words, without having to localize either at or away from 2), is the following.

Theorem 10. *Let M^n be a closed oriented n-manifold, let π be any countable group, and let $f : M \to B\pi$ be any map. Then $f_*(\Delta_M) \in K_n(B\pi)$ is an oriented homotopy invariant of M provided* **either** *that the assembly map $K_*(B\pi) \to K_*(C^*(\pi))$ is injective* (*the "Strong Novikov Conjecture"*) **or** *the assembly map $H_*(B\pi; \mathbf{L}^\bullet(\mathbb{R})) \to L^\bullet_*(\mathbb{R}\pi)$ is injective* (*a weak form of the "Integral Novikov Conjecture"*)*. By "oriented homotopy invariant," we mean that* *if $N^n \xrightarrow{h} M^n$ is an orientation-preserving homotopy equivalence of manifolds, then $f_*(\Delta_M) = (f \circ h)_*(\Delta_N)$.*

PROOF. This was proved in [14, §9, Theorem 2] and in [11] when the C^*-algebraic assembly map is injective. However, injectivity of the C^*-algebraic assembly map only implies the Integral Novikov Conjecture in L-theory after localizing away from 2 [23, Corollary 2.10], and there is no known implication in the other direction, so another argument is needed if we assume instead the injectivity of the L-theoretic assembly map. However, the image of the symmetric signature $\sigma_{\mathbb{R}}(M \xrightarrow{f} B\pi) \in H_n(B\pi; \mathbf{L}^\bullet(\mathbb{R}))$ in $L_n^\bullet(\mathbb{R}\pi)$ is a homotopy invariant, so that $\sigma_{\mathbb{R}}(M \xrightarrow{f} B\pi)$ is itself a homotopy invariant when the L-theoretic assembly map is injective. But $f_*(\Delta_M)$ is the image of $\sigma_{\mathbb{R}}(M \xrightarrow{f} B\pi)$ under a natural transformation, by Theorem 8. □

Theorem 11. *Let M^n be a closed oriented n-manifold. Then the image of Δ_M in $K_n(M; \mathbb{Z}/8)$ is an oriented homotopy invariant of M. In other words, if $N^n \xrightarrow{h} M^n$ is an orientation-preserving homotopy equivalence of manifolds, then $h_*(\Delta_N) = \Delta_M$ in $K_n(M; \mathbb{Z}/8)$.*

PROOF. We make use of Theorem 8, which factors s_n through

$$\overbrace{\Omega_n(\_\_) \xrightarrow{\sigma} H_n(\_\_; \mathbf{L}^\bullet(\mathbb{Z})) \to H_n(\_\_; \mathbf{L}^\bullet(\mathbb{R}))}^{\sigma_{\mathbb{R}}}.$$

By surgery theory, the homotopy equivalence h defines a class

$$[h] \in H_n(M; \mathbf{L}_\bullet(\mathbb{Z})),$$

and $\sigma_{\mathbb{R}}(M) - h_*(\sigma_{\mathbb{R}}(N)) \in H_n(M; \mathbf{L}^\bullet(\mathbb{R}))$ is the image of $[h]$ under symmetrization $L_\bullet(\mathbb{Z}) \to L^\bullet(\mathbb{Z})$ followed by the change-of-rings map $L^\bullet(\mathbb{Z}) \to L^\bullet(\mathbb{R})$. The symmetrization map is multiplication by 8 on homotopy groups in degrees divisible by 4 [21, §4.3], [22, pp. 12–13], so $\sigma_{\mathbb{R}}(M) - h_*(\sigma_{\mathbb{R}}(N)) \in H_n(M; \mathbf{L}^\bullet(\mathbb{R}))$ is divisible by 8 and maps to 0 in $K_n(M; \mathbb{Z}/8)$. □

Remark 12. Note that we didn't make full use of the assumption that h was a homotopy equivalence here. We would have gotten the same conclusion if it was only a degree-1 normal map (in the sense of surgery theory).

2 Examples and calculations

If M^n is a closed manifold, the image of Δ_M in $H_*(M; \mathbb{Q})$ under the Chern character only differs from the Poincaré dual of the L-class by certain powers of 2 (explained by Theorem 7). So Δ_M is completely computed rationally in terms of the Pontrjagin classes. In fact, Δ_M is basically the same as the Sullivan orientation in $KO[\frac{1}{2}]_n$ except for powers of 2. So calculations of our invariants are only interesting in the presence of 2-torsion. That makes it quite natural to compute them for real projective spaces and lens spaces for cyclic 2-groups and quaternion groups. Calculation for such manifolds is expedited by the following.

Lemma 13. *Let M^n be a closed manifold equipped with a spin^c structure, and let $\not{D}_M$ be the corresponding Dirac operator. Then in $K_n(M)$, $\Delta_M = [\not{D}_M] \cap [\overline{E}]$, where $[E] \in K^0(M)$ is the class of the complex spinor bundle E and $[\overline{E}] \in K^0(M)$ is the class of the dual bundle.* (*Note that the complex Clifford algebra bundle of M is isomorphic to* $\operatorname{End}(E) \cong E \otimes \overline{E}$ *when n is even and to a direct sum of two copies of* $\operatorname{End}(E)$ *when n is odd. The rank of E or $\overline{E}$ is $2^{\lfloor n/2 \rfloor}$.*)

PROOF. This is just a restatement of the relationship between the Dirac and signature operators, as explained in [15]. □

Remark 14. It is important to note in Lemma 13 that if M^n is a spin^c manifold, the Dirac operator $\not{D}_M$ defines a Poincaré duality isomorphism between $K^0(M)$ and $K_n(M)$ which depends on the choice of spin^c structure. The class $[E] \in K^0(M)$ will also vary with the spin^c structure. However, $\Delta_M \in K_n(M)$ only depends on the orientation of M, not on the spin^c structure. (If we fix the orientation of the manifold M and assume that M admits a spin^c structure, then the group $H^2(M; \mathbb{Z})$ acts freely [6] on the set of spin^c structures compatible with this orientation. Identify $H^2(M; \mathbb{Z})$ with the group of isomorphism classes $[L]$ of line bundles on M, the group operation being tensor product. Then if we operate on the spin^c structure by the class $[L]$, $[\not{D}_M]$ is multiplied by $[L]$, while $[E]$ is also multiplied by $[L]$, so $[\overline{E}]$ is multiplied by $[L]^{-1}$ and $\Delta_M = [\not{D}_M] \cap [\overline{E}]$ remains unchanged.)

Example 15. Consider a cyclic group $G = C_r$ of order $r = 2^k$ acting linearly on $\mathbb{C}^n$ with the action free away from the origin. We identify G with the group of rth roots of unity. The action is the restriction of an action of the circle group S^1 by a direct sum of characters $t^{j_1}, \ldots, t^{j_n}$, where $j_1, \ldots, j_n$ are relatively prime mod r and t is the canonical generator of $R(S^1) \cong \mathbb{Z}[t, t^{-1}]$. The action of G is free on the unit sphere $S(\mathbb{C}^n) \cong S^{2n-1}$ and the quotient space $M = S(\mathbb{C}^n)/G$ is an orientable lens space of dimension $2n-1$ with fundamental group G. Since the action of G on $\mathbb{C}^n$ is complex linear, G preserves the canonical spin^c structure on $S(\mathbb{C}^n)$ and M is a spin^c manifold. (This is also clear from the fact that $H^3(M; \mathbb{Z})$ is torsion-free.) (*Caution*: the manifold M admits $2r$ different spin^c structures compatible with its usual orientation, since $H^2(M; \mathbb{Z}) \cong G$ and $H^1(M; \mathbb{Z}/2) \cong \mathbb{Z}/2$. They differ from one another

[6] In fact, the group $H^1(M; \mathbb{Z}/2) \times H^2(M; \mathbb{Z})$ acts simply transitively. The action of $H^1(M; \mathbb{Z}/2)$ corresponds to twisting by real line bundles, which also doesn't change the class Δ_M.

simply by tensoring with flat real and complex line bundles. But there is a canonical choice of spin^c structure coming from the unique spin^c structure on $S(\mathbb{C}^n)$. This is the one we will use.) First we compute $K^0(M)$. This is most easily computed as $K^0_G(S(\mathbb{C}^n))$, which in turn is obtained from the $R(S^1)$-module $K^0_{S^1}(S(\mathbb{C}^n))$ by dividing out by the additional relation $t^r = 1$. From the inclusion of $S(\mathbb{C}^n)$ in the unit disk $D(\mathbb{C}^n)$, we have the exact sequence of $R(S^1)$-modules

$$K^0_{S^1}(D(\mathbb{C}^n),\, S(\mathbb{C}^n)) \longrightarrow K^0_{S^1}(D(\mathbb{C}^n)) \longrightarrow K^0_{S^1}(S(\mathbb{C}^n)) \longrightarrow 0.$$

Here the quotient map is not just a map of $R(S^1)$-modules but also a map of rings (with respect to the cup product). Since $D(\mathbb{C}^n)$ is equivariantly contractible, its equivariant K-theory is $R(S^1)$, and equivariant Bott periodicity gives an isomorphism of $K^0_{S^1}(D(\mathbb{C}^n),\, S(\mathbb{C}^n))$ with $R(S^1)$ via the alternating sum of the exterior powers of $t^{j_1} + \cdots + t^{j_n}$. So

$$K^0_{S^1}(S(\mathbb{C}^n)) \cong \mathbb{Z}[t,\, t^{-1}]/\prod_{m=1}^{n}\Big(t^{j_m} - 1\Big).$$

In particular, when $j_1 = \cdots = j_m = 1$ and $r = 2$, we obtain the standard calculation of $K^0(\mathbb{RP}^{2n-1})$ as

$$\mathbb{Z}[t,\, t^{-1}]/\Big((t-1)^n,\, t^2 - 1\Big) = \mathbb{Z}[u]/\Big(u^n,\, u(u+2)\Big) \cong \mathbb{Z} \oplus (\mathbb{Z}/2^{n-1})u,$$

where u corresponds to $t-1$ (note that t corresponds to a non-trivial flat line bundle, 1 to the trivial line bundle), and $u^2 = -2u$.

Now, as a class in $K^0_{S^1}(S(\mathbb{C}^n))$, the complexified tangent bundle of $S(\mathbb{C}^n)$ is given by the image of $t^{j_1}+\cdots+t^{j_n}+t^{-j_1}+\cdots+t^{-j_n}-1 \in K^0_{S^1}(D(\mathbb{C}^n)) = R(S^1)$ (since on addition of the normal line bundle, which is trivial, one obtains the sum of the restrictions of the holomorphic and anti-holomorphic tangent bundles of $\mathbb{C}^n$). So the complex spinor bundle E, which has rank 2^{n-1}, has K-theory class:

$$\frac{1}{2}\prod_{m=1}^{n}\Big(t^{j_m} + 1\Big).$$

Here the division by 2 has a well-defined meaning in $K^0_{S^1}(S(\mathbb{C}^n))$, which is torsion-free as an abelian group, and then one can specialize from S^1 to G. For example, in the case of $\mathbb{RP}^{2n-1}$, this becomes

$$\frac{1}{2}\Big(t+1\Big)^n = \frac{1}{2}\Big(u+2\Big)^n$$

in $\mathbb{Z}[t,\, t^{-1}]/\Big((t-1)^n\Big) = \mathbb{Z}[u]/\Big(u^n\Big)$, which works out to

$$\frac{1}{2}\sum_{j=0}^{n-1}\binom{n}{j}u^j 2^{n-j} = \sum_{j=0}^{n-1}\binom{n}{j}u^j 2^{n-j-1}.$$

When one then adds the relation $u^2 = -2u$, this becomes

$$\begin{aligned} 2^{n-1} + nu2^{n-2} + \sum_{j=2}^{n-1} \binom{n}{j} (-2)^{j-1} u 2^{n-j-1} \\ = 2^{n-1} + 2^{n-2} u \sum_{j=1}^{n-1} \binom{n}{j} (-1)^{j-1} \\ = 2^{n-1} + 2^{n-2} u \big(1 + (-1)^n\big), \end{aligned}$$

which simplifies simply to 2^{n-1} since $2^{n-1}u = 0$. What this means is that in $\mathbb{RP}^{2n-1}$, Δ_M is simply $2^{n-1}[\not{D}]$. From this we can deduce:

Proposition 16. *If M is a closed manifold with the homotopy type of $\mathbb{RP}^{2n-1}$, then Δ_M is 2^{n-1} times a K-theory fundamental class, and is an oriented homotopy invariant.*

PROOF. We have checked this for $M = \mathbb{RP}^{2n-1}$ itself. Now if

$$M^{2n-1} \xrightarrow{h} \mathbb{RP}^{2n-1}$$

is an orientation-preserving homotopy equivalence, $h_*(\Delta_M)$ and $\Delta_{\mathbb{RP}^{2n-1}}$ have the same image in $H_{2n-1}(\mathbb{RP}^{2n-1}) \cong \mathbb{Z}$ by Proposition 9, so their difference lies in the torsion subgroup of $K_{2n-1}(\mathbb{RP}^{2n-1})$, which as we have seen is cyclic of order 2^{n-1}. However, by Theorem 7, this difference lies in the image of the odd-dimensional homology of $\mathbb{RP}^{2n-1}$ not in top degree, which is all torsion of exponent 2. So $h_*(\Delta_M) - \Delta_{\mathbb{RP}^{2n-1}}$ is therefore either 0 or the unique element of $K_{2n-1}(\mathbb{RP}^{2n-1})$ of order 2. The latter possibility is ruled out by the proof of Theorem 11, since the symmetrization map

$$H_*(\mathbb{RP}^{2n-1}; \mathbf{L}_\bullet(\mathbb{Z})) \to H_*(\mathbb{RP}^{2n-1}; \mathbf{L}^\bullet(\mathbb{Z}))$$

is multiplication by 8 and thus 0 on all the 2-torsion in $H_{2n-1-4j}(\mathbb{RP}^{2n-1}; \mathbb{Z})$. □

The fact that this is somewhat special is indicated by the following example.

Proposition 17. *For 5-dimensional lens spaces (this is corresponds to the case of $n = 3$ above), Δ_M is not necessarily 4 times a K-theory fundamental class, and is* **not** *an oriented homotopy invariant, even mod 16.*

PROOF. Retain the same notation as above and take $r = |G| = 2^k$ with k large (or at least ≥ 3). Then the 5-dimensional lens space M is classified by the triple (j_1, j_2, j_3), where j_1, j_2, j_3 are odd and defined modulo r. Also, without loss of generality we may take $j_1 = 1$ (otherwise change generators of G). The oriented homotopy type of M is determined by $j_1 j_2 j_3 \in (\mathbb{Z}/r)^\times$, modulo multiplication by s^3 for $s \in (\mathbb{Z}/r)^\times$ [19, Theorem VI]. Since $(\mathbb{Z}/r)^\times$

has order 2^{k-1}, which is a positive power of 2, and since 3 is relatively prime to 2, s^3 runs through all of $(\mathbb{Z}/r)^\times$ as s runs through $(\mathbb{Z}/r)^\times$, and hence all 5-dimensional lens spaces with fundamental group G are homotopy equivalent. However, there are many diffeomorphism classes of such lens spaces (see [18, Theorem 12.7] for the exact classification theorem).

To compute the structure of $K^0(M)$, it suffices to take $j_1 = j_2 = j_3 = 1$ (since all other lens spaces with the same dimension and fundamental group have the same homotopy type). Calculation just as in Example 15 gives

$$\begin{aligned} K^0(M) &\cong \mathbb{Z}[t,\, t^{-1}]/\left((t-1)^3,\, t^r - 1\right) \\ &= \mathbb{Z}[u]/\left(u^3,\, (u+1)^r - 1\right) \\ &= \mathbb{Z}[u]/\left(u^3,\, ru + \tbinom{r}{2}u^2\right), \end{aligned}$$

with again $u = t-1$. Since $\binom{r}{2} = 2^{k-1}(2^k - 1)$, we see that

$$2^{k+1}u = 2\cdot 2^k u = -2^k(2^k-1)u^2 = \left(-(2^k-1)u\right)\left(2^k u\right) = 2^{k-1}(2^k-1)^2u^3 = 0,$$

so u has additive order 2^{k+1} and

$$\widetilde{K^0}(M) \cong (\mathbb{Z}/2^{k+1})u \oplus (\mathbb{Z}/2^{k-1})(2u + (2^k-1)u^2).$$

Next we compute the class of the spinor bundle E. If $j_1 = j_2 = j_3 = 1$, we see (just as in Example 15) that $[E]$ is the image of

$$\tfrac{1}{2}\left(t+1\right)^3 = \tfrac{1}{2}\left(u+2\right)^3 \in K^0_{S^1}(S(\mathbb{C}^n)) = \mathbb{Z}[t,\, t^{-1}]/\left((t-1)^3\right) = \mathbb{Z}[u]/(u^3).$$

This is of course just

$$\tfrac{1}{2}\left(2^3 + 3\cdot 2^2u + 3\cdot 2u^2 + u^3\right) = 4 + 6u + 3u^2.$$

Note that in $K^0(M)$, this is not only not divisible by 4, but not divisible by 2. So Δ_M is not 4 times a K-theory fundamental class; in fact, it is not even divisible by 2.

On the other hand, suppose $k = 4$, $r = 2^k = 16$, let M be the standard lens space above, let $\mathbb{C}^{n\prime}$ be $\mathbb{C}^n$ with the S^1-action given by $j_1 = 1$, $j_2 = 3$, and $j_3 = 11$, and let M' be the associated lens space. The numbers j_2 and j_3 were chosen so that $j_1j_2j_3 \equiv 1 \pmod{16}$, so that

$$f : (z_1,\, z_2,\, z_3) \mapsto \left(z_1,\, z_2^3,\, z_3^{11}\right)$$

induces an oriented G-homotopy equivalence $S(\mathbb{C}^n) \to S(\mathbb{C}^{n\prime})$ and an oriented homotopy equivalence $M \to M'$. Then $[E_{M'}]$ is the image of

$$\tfrac{1}{2}\left(t+1\right)\left(t^3+1\right)\left(t^{11}+1\right) \in K^0_{S^1}(S(\mathbb{C}^{n\prime})) = \mathbb{Z}[t,\, t^{-1}]/\left((t-1)(t^3-1)(t^{11}-1)\right).$$

Let $u = t - 1$, $v = t^3 - 1$, $w = t^{11} - 1$. Then in $K^0_{S^1}(S(\mathbb{C}^{n\prime}))$, $uvw = 0$ and

$$\begin{aligned}\tfrac{1}{2}\Big(t+1\Big)\Big(t^3+1\Big)\Big(t^{11}+1\Big) &= \tfrac{1}{2}\Big(u+2\Big)\Big(v+2\Big)\Big(w+2\Big) \\ &= \tfrac{1}{2}\Big(uvw + 2uv + 2uw + 2vw + 4u + 4v + 4w + 8\Big) \\ &= uv + uw + vw + 2u + 2v + 2w + 4.\end{aligned}$$

But $f_* : K^0_G(S(\mathbb{C}^n)) \to K^0_G(S(\mathbb{C}^{n\prime}))$ is a ring isomorphism sending t to t, and hence $32u = 0$, $16u + 120u^2 = 0$, and $u^3 = 0$ in $K^0_G(S(\mathbb{C}^{n\prime}))$, as well as in $K^0_G(S(\mathbb{C}^n))$. So

$$v = (u+1)^3 - 1 = 3u + 3u^2 = 3u(1+u),$$

$$w = (u+1)^{11} - 1 = 11u + 55u^2 = 11u(1+5u),$$

and

$$\begin{aligned}&uv + uw + vw + 2u + 2v + 2w + 4 \\ &\qquad = 3u^2(1+u) + 11u^2(1+5u) + 33u^2(1+u)(1+5u^2) \\ &\qquad\quad + 2u + 6u(1+u) + 22u(1+5u) + 4 \\ &\qquad = 4 + 30u + 163u^2 = 4 - 2u + 3u^2,\end{aligned}$$

which is different from what we obtained for M. Hence $f_*(\Delta_M) \neq \Delta_{M'}$, so Δ_M is not a homotopy invariant. Note, incidentally, that $f_*([E_M])$ and $[E_{M'}]$ differ by $8u$, so our calculation doesn't contradict Theorem 11. □

The above examples show that any formula for the image of Δ_M in $K_n(M; \mathbb{Z}/8)$ must be fairly complicated. But in a sequel paper we will give a simple formula for the image of Δ_M in $K_n(M; \mathbb{Z}/2)$.

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