# Metrics of positive scalar curvature and connections with surgery

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January 2, 1998

## 1 Introduction

This chapter discusses the connection between geometry of Riemannian metrics of positive scalar curvature and surgery theory. While this is quite a deep subject which has attracted quite a bit of recent attention, the most surprising aspect of this whole area remains the original discovery of Gromov-Lawson and of Schoen-Yau from about 20 years ago—namely, that there *is* a connection between positive scalar curvature metrics and surgery. The Surgery Theorem of Gromov-Lawson and Schoen-Yau remains the most important result in this subject. We discuss it and its variants at length in Section 3. Then in Section 4, we discuss the status of the so-called Gromov-Lawson Conjecture, which relates the existence of positive scalar curvature metrics to index theory and $KO$-homology. This is preparatory to Section 5, which explains the parallels between the classification of positive scalar curvature metrics and the classification of manifolds via Wall's surgery theory. In the final section, Section 6, we discuss a number of open problems.

All manifolds in this paper will be assumed to be smooth ($C^\infty$). For simplicity, we restrict attention to compact manifolds, although there are also plenty of interesting questions about complete metrics of positive scalar curvature on non-compact manifolds. At some points in the discussion, however, it will be necessary to consider manifolds with boundary.

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*Partially supported by NSF Grant # DMS-96-25336 and by the General Research Board of the University of Maryland.

†Partially supported by NSF Grant # DMS-95-04418.

## 2 Background and Preliminaries

One of the most important problems in global differential geometry is to study how curvature relates to topology, or to phrase things differently, to study what constraints topology places on curvature. This problem can be asked in several different contexts. When applied to vector bundles with a connection, it gives rise to Chern-Weil theory and the theory of characteristic classes. Here we will instead ask about the scalar curvature of a Riemannian manifold. The scalar curvature is the weakest curvature invariant one can attach (pointwise) to a Riemannian $n$-manifold. Its value at any point can be described in several different ways:

1. as the trace of the Ricci tensor, evaluated at that point.
2. as twice the sum of the sectional curvatures over all 2-planes $e_i \wedge e_j$, $i < j$, in the tangent space to the point, where $e_1, \ldots, e_n$ is an orthonormal basis.
3. up to a positive constant depending only on $n$, as the leading coefficient in an expansion telling how volumes of small geodesic balls differ from volumes of corresponding balls in Euclidean space. Positive scalar curvature means balls of radius $r$ for small $r$ have a smaller volume than balls of the same radius in Euclidean space; negative scalar curvature means they have larger volume.

In the special case $n = 2$, the scalar curvature is just twice the Gaussian curvature.

We can now state the basic problems we will consider in this paper:

**Problems 2.1**

1. If $M^n$ is a closed $n$-manifold, when can $M$ be given a Riemannian metric for which the scalar curvature function is everywhere strictly positive? (For simplicity, such a metric will henceforth be called a metric of positive scalar curvature.)
2. If $M^n$ is a closed manifold which admits at least one Riemannian metric of positive scalar curvature, what is the topology of the space $\mathfrak{R}^+(M)$ of all such metrics on $M$? In particular, is this space connected?
3. If $M^n$ is a compact manifold with boundary, when does $M$ admit a Riemannian metric of positive scalar curvature which is a product metric on a collar neighborhood $\partial M \times [0, a]$ of the boundary? When this is the case, what is the topology of the space of all such metrics? Of the space of all such metrics extending a fixed metric in $\mathfrak{R}^+(\partial M)$?

A few comments on these problems are in order. With regard to question (1), the reader might well ask what is special about positivity. Why not ask about metrics of *negative* scalar curvature, or of vanishing scalar curvature, or of non-negative scalar curvature? More generally, we could ask which smooth functions on a manifold $M$ are realized as the scalar curvature function of some metric on $M$. It is a remarkable result of Kazdan and Warner that the answer to this question *only* depends on which of the following classes the manifold $M$ belongs to:

1. Closed manifolds admitting a Riemannian metric whose scalar curvature function is non-negative and not identically 0.

2. Closed manifolds admitting a Riemannian metric with non-negative scalar curvature, but not in class (1).

3. Closed manifolds not in classes (1) or (2).

All three classes are non-empty if $n \geq 2$. For example, it is easy to see from the Gauss-Bonnet-Dyck Theorem[1] that if $n = 2$, class (1) consists of $S^2$ and $\mathbb{RP}^2$; class (2) consists of $T^2$ and the Klein bottle; and class (3) consists of surfaces with negative Euler characteristic.

**Theorem 2.2 (Trichotomy Theorem, [KW1], [KW2])** *Let $M^n$ be a closed connected manifold of dimension $n \geq 3$.*

1. *If $M$ belongs to class (1), every smooth function is realized as the scalar curvature function of some Riemannian metric on $M$.*

2. *If $M$ belongs to class (2), then a function $f$ is the scalar curvature of some metric if and only if either $f(x) < 0$ for some point $x \in M$, or else $f \equiv 0$. If the scalar curvature of some metric $g$ vanishes identically, then $g$ is Ricci flat.* (I.e., *not only does the scalar curvature vanish identically, but so does the Ricci tensor.*)

3. *If $M$ belongs to class (3), then $f \in C^\infty(M)$ is the scalar curvature of some metric if and only if $f(x) < 0$ for some point $x \in M$.*

We note that the Theorem shows that deciding whether a manifold $M$ belong to class (1) is equivalent to solving Problem 2.1.1. Futaki [Fu] has shown that – at least for simply connected manifolds – class (2) consists of very special manifolds admitting metrics with restricted holonomy groups.

As further justification for our concentrating on *positive* scalar curvature in Problem 2.1.2, one has the following fairly recent result:

[1]The point is that for any choice of Riemannian metric, the integral of the scalar curvature with respect to the measure defined by the metric is $4\pi$ times the Euler characteristic. Dyck's role in this is explained in the interesting article by D. Gottlieb, "All the way with Gauss-Bonnet and the sociology of mathematics," Amer. Math. Monthly **103** (1996), no. 6, 457–469.

**Theorem 2.3 ([Loh])** *The space $\mathfrak{R}^-(M)$ of negative scalar curvature metrics on $M$ is contractible, for any closed manifold $M^n$ of dimension $n \geq 3$.*

Finally, one might ask the reason for the Riemannian product boundary condition in Problem 2.1.3. The first part of the answer comes from the fact that without a boundary condition, *any* manifold with non-empty boundary admits a metric of positive scalar curvature. (In fact, Gromov [Gr], Theorem 4.5.1, even showed it admits a metric of positive *sectional* curvature, a much stronger condition.) The second part of the answer is that there are other interesting boundary conditions one could impose that are relevant to the study of positive scalar curvature, such as positive mean curvature on the boundary (see [GL1], Theorem 5.7, for example), but we have tried to limit attention to the simplest such condition. Often one can reduce to this condition anyway—see [Gaj1], Theorem 5.

# 3 The Surgery Theorem and its Variants

The connection between positive scalar curvature metrics and surgery begins with:

**Theorem 3.1 (Surgery Theorem, [GL2], Theorem A and [SY])** *Let $N^n$ be a closed manifold, not necessarily connected, with a Riemannian metric of positive scalar curvature, and let $M^n$ be obtained from $N$ by a surgery of codimension $q \geq 3$. Then $M$ can be given a metric of positive scalar curvature.*

*Proof.* We give the argument of Gromov-Lawson, just briefly sketching their initial reduction of the problem (which is explained well in their paper), but going over the crucial "bending argument" in detail. (The reason for this is that it appears there is a mistake in [GL2] on page 428—in the displayed formula on the middle of that page, there is a factor of $\sin^2\theta_0$ missing, and thus the argument at the bottom of page 428 doesn't work as stated.)

Suppose $S^p$ is an embedded sphere in $N$ of codimension $q = n - p \geq 3$, with trivial normal bundle. By using the exponential map on the normal bundle of $S^p$, we may assume that we have an embedding of $S^p \times D^q(\bar{r})$ into $N$ for some $\bar{r} > 0$ (the radius of a "good tubular neighborhood of $S^p$") so that the sphere on which we will do surgery is $S^p \times \{0\}$, the radial coordinate $r$ on $D^q(\bar{r})$ measures distances from $S^p \times \{0\}$, and such that curves of the form $\{y\} \times \ell$, where $\ell$ is a ray in $D^q(\bar{r})$ starting at the origin, are geodesics. However, we know nothing about the restriction of the metric on $N$ to the sphere $S^p \times \{0\}$.

The key idea of the proof is to choose a suitable $C^\infty$ curve $\gamma$ (with endpoints) in the $t$-$r$ plane, and to consider

$$T = \{(y, x, t) \in \left(S^p \times D^q(\bar{r})\right) \times \mathbb{R} : (t, r = \|x\|) \in \gamma\}$$

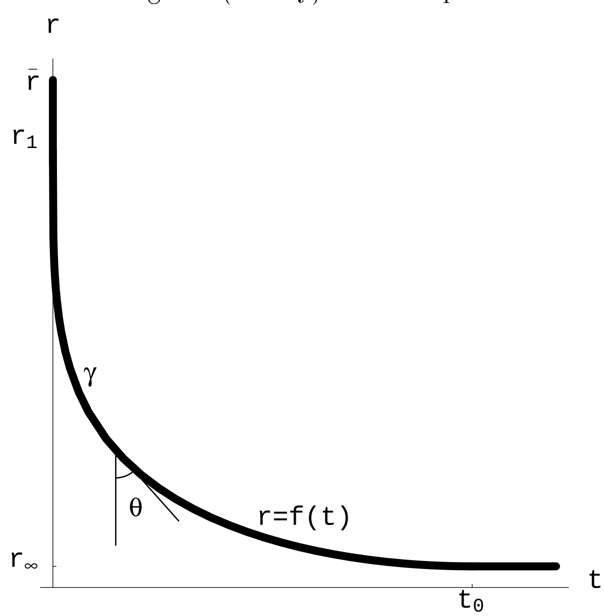
with the induced metric, where $\mathbb{R}$ is given the Euclidean metric and $\big(S^p \times D^q(\bar{r})\big) \times \mathbb{R}$ is given the metric of the Riemannian product $N \times \mathbb{R}$. We choose the curve $\gamma$ to satisfy the following constraints:

1. $\gamma$ lies in the region $0 < r \le \bar{r}$ of the $t$-$r$ plane.

2. $\gamma$ begins at one end with a vertical line segment $t = 0$, $r_1 \le r \le \bar{r}$. This guarantees that near one of the two components of $\partial T$, $T$ is isometric to a portion of $N$.

3. $\gamma$ ends with a horizontal line segment $r = r_\infty > 0$, with $r_\infty$ very small. This guarantees that near the other component of $\partial T$, $T$ is isometric to the Riemannian product of a line segment with $S^p \times S^{q-1}(r_\infty)$, where the metric on $S^p \times S^{q-1}(r_\infty)$ (not in general a product metric) is induced by the embedding $S^p \times S^{q-1}(r_\infty) \subset S^p \times D^q(\bar{r}) \subset N$.

4. In the region $r_\infty < r < r_1$, $\gamma$ is the graph of a function $r = f(t)$ which is decreasing and (weakly) concave upward.




5. $\gamma$ is chosen so that the scalar curvature of $T$ is everywhere positive. This is the hard part. The Gauss curvature equation says that the sectional curvature of a hypersurface, evaluated on a plane spanned by two principal directions for the second fundamental form, is the corresponding sectional curvature of the ambient manifold, plus the product of the two principal curvatures. So, summing the sectional

curvatures over all the two-planes spanned by pairs of principal directions, one derives for small $r > 0$ the formula:

$$\begin{aligned}\kappa_T = \kappa_N + O(1)\sin^2\theta + (q-1)(q-2)\frac{\sin^2\theta}{r^2}\\ -(q-1)\frac{k\sin\theta}{r} - O(r)(q-1)k\sin\theta,\end{aligned} \tag{3.1}$$

where $\kappa_T$ and $\kappa_N$ are the scalar curvatures of $T$ and $N$, respectively, where $k$ is the curvature of $\gamma$ (as a curve in the Euclidean plane), and where $\theta$ is the angle between $\gamma$ and a vertical line. (See figure above.)

Assume for the moment that we have constructed $\gamma$ as required. Since the metric on $T$ is isometric to a portion of $N$ in a collar of one component of $\partial T$, we can glue $T$ onto $N \smallsetminus \bigl(S^p \times D^q(\bar r)\bigr)$, getting a manifold $N'$ of positive scalar curvature with a single boundary component $S^p \times S^{q-1}(r_\infty)$, and with a metric that is a product metric in a collar neighborhood of the boundary.

Since $q-1 \geq 2$ and $r_\infty$ is very small, there is a homotopy of the metric on $S^p \times S^{q-1}(r_\infty)$ through metrics of positive scalar curvature to a Riemannian product of two standard spheres: $S^p(1)$ and $S^{q-1}(r_\infty)$. Even though $S^p(1)$ has zero curvature if $p \leq 1$, we have large positive scalar curvature since $S^{q-1}(r_\infty)$ has sectional curvature $r_\infty^{-2} \gg 0$. (See [GL2], Lemma 2.) This homotopy can be used to construct a metric of positive scalar curvature on a cylinder $S^p \times S^{q-1}(r_\infty) \times [0, a]$, which in a neighborhood of one boundary component matches the metric on a collar neighborhood of $\partial T$ in $T$, and which in a neighborhood of the other boundary component is a Riemannian product of standard spheres $S^p(1)$ and $S^{q-1}(r_\infty)$ with an interval. (See Proposition 3.3 below.) We glue this cylinder onto $N'$ to get $N''$, a manifold of positive scalar curvature with boundary $S^p(1) \times S^{q-1}(r_\infty)$, and with a product metric in a neighborhood of the boundary.

Finally, to finish off the proof, we glue onto $N''$ a Riemannian product $D^{p+1} \times S^{q-1}(r_\infty)$, where the disk $D^{p+1}$ has not the flat metric but a metric which is a Riemannian product $S^p(1) \times [0, b]$ in a neighborhood of the boundary. (Such metrics on the disk are easy to write down.) The endproduct of the construction is a metric of positive scalar curvature on $M$.

We're still left with the most delicate step, which is construction of a curve $\gamma$ with the properties listed on page 5 above. Obviously, there is no problem satisfying the first four conditions. To satisfy the last condition, we need to choose $\gamma$ so that $\kappa_T > 0$ in equation (3.1). Since $\kappa_N$ is bounded below by a positive constant, the constraint will be satisfied provided that

$$(1 + C' r^2)k \leq (q-2)\frac{\sin\theta}{r} + \kappa_0 \frac{r}{\sin\theta} - C r \sin\theta, \tag{3.2}$$

where $\kappa_0 > 0$ is $\frac{1}{q-1}$ times a lower bound for $\kappa_N$, and where the constants $C > 0$ and $C' > 0$ come from the $O(1)$ term and the $O(r)$ term in equation (3.1), respectively. (When $\theta = 0$, the right-hand side of inequality (3.2) is to be interpreted as $+\infty$.)

To satisfy this inequality, we begin by choosing

$$0 < \theta_0 < \arcsin\left(\sqrt{\frac{\kappa_0}{C}}\right).$$

Then for $0 \leq \theta \leq \theta_0$, the second term on the right in inequality (3.2) dominates the last term, and thus we can start at the point $(0, r_1)$ (where $\theta$ and $k$ are required to vanish) and find a small "bump function" of compact support for $k$ (as a function of arc length) satisfying (3.2), so that $\gamma$ bends in a small region around to a line segment with small positive $\theta$. Decreasing $\theta_0$ if necessary, we may assume this "first bend" ends at $\theta = \theta_0$. (So far the details are just as in [GL2], except that we have made the estimates more explicit.)

Next, we choose $r_0$ with

$$0 < r_0 < \min\left(\sqrt{\frac{1}{4C}}, \sqrt{\frac{1}{2C'}}\right).$$

This insures (since $q - 2 \geq 1$) that for $r \leq r_0$,

$$(q-2)\frac{\sin\theta}{r} - Cr\sin\theta \geq \frac{3\sin\theta}{4r}$$

and

$$1 + C'r^2 \leq \frac{3}{2},$$

so that $k$ can be as large as $\frac{2}{3} \cdot \frac{3\sin\theta}{4r} = \frac{\sin\theta}{2r}$. When $\gamma$ crosses the line $r = r_0$, we start the "second bend" by quickly bringing $k$ up to the allowed value of $\frac{\sin\theta}{2r}$ and thereafter following the solution of the differential equation $k = \frac{\sin\theta}{2r}$. If we write $r = f(t)$, then

$$\sin\theta = \frac{1}{\sqrt{1 + (f')^2}}, \qquad k = \frac{f''}{(1 + (f')^2)^{\frac{3}{2}}}.$$

So our differential equation can be rewritten

$$f'' = \frac{1 + (f')^2}{2f}.$$

This equation can be solved explicitly; the solution is

$$f(t) = \frac{1}{C_1} + \frac{C_1}{4}(t - C_2)^2,$$

for constants $C_1$ and $C_2$. Suppose we start following the differential equation at $t = t_1 \approx r_1 \arctan \theta_0$. Then we will need to take $f(t_1)$ very close to $r_0$ and $f'(t_1)$ very close to $-\cot \theta_0$. This can be accomplished by taking $C_2$ bigger than $t_1$, $C_1(C_2 - t_1)$ large, and $C_1$ huge. Then we follow the solution out until $t$ is very close to $C_2$, at which point $f(t)$ is approximately $\frac{1}{C_1}$, which is very small but positive, and $f'(t)$ is approximately 0, *i.e.*, $\theta$ is very close to $\frac{\pi}{2}$. Then we quickly bring $k$ back down to 0 and finish with a horizontal line, thereby satisfying all our requirements. $\square$

There is a slight strengthening of this due to Gajer, which provides information about manifolds with boundary.

**Theorem 3.2 (Improved Surgery Theorem, [Gaj1])** *Let $N$ be a closed manifold with a metric of positive scalar curvature $ds_N^2$, not necessarily connected, and let $M$ be obtained from $N$ by a surgery of codimension $\geq 3$. Let $W$ be the trace of this surgery (a cobordism from $N$ to $M$). Then $W$ can be given a metric of positive scalar curvature $ds_W^2$ which is a product metric $ds_N^2 + dt^2$ in a collar neighborhood of $N$ and a product metric $ds_M^2 + dt^2$ in a collar neighborhood of $M$.*

This indeed strengthens Theorem 3.1, since in a neighborhood of $M$, the scalar curvature of $ds_W^2$ is the same as that of $ds_M^2$, and thus we have given $M$ a metric of positive scalar curvature.

The study of metrics such as the one in Theorem 3.2, together with the obvious parallels in the theory of automorphisms of manifolds, motivates the following.

**Definition.** Let $ds_0^2$ and $ds_1^2$ be two Riemannian metrics on a compact manifold $M$, both with positive scalar curvature. (For the moment we take $M$ to be closed, though later we will also consider the case where $M$ has a boundary.) We say these metrics are *isotopic* if they lie in the same path component of the space of positive scalar curvature metrics on $M$, and *concordant* if there is a positive scalar curvature metric on a cylinder $W = M \times [0, a]$ which restricts to $ds_0^2 + dt^2$ in a collar neighborhood of $M \times \{0\}$ and to $ds_1^2 + dt^2$ in a collar neighborhood of $M \times \{a\}$. We denote by $\tilde{\pi}_0 \mathfrak{R}^+(M)$ the set of concordance classes of positive scalar curvature metrics on $M$.

There is one important and easy result relating isotopy and concordance of positive scalar curvature metrics.

**Proposition 3.3 ([GL2], Lemma 3; [Gaj1], pp. 184–185)** *Isotopic metrics of positive scalar curvature are concordant.*

*Sketch of Proof.* Suppose $ds_t^2$, $0 \leq t \leq 1$, is an isotopy between positive scalar curvature metrics on $M$. Consider the metric $ds_{t/a}^2 + dt^2$ on $W = M \times [0, a]$. This will have positive scalar curvature for $a \gg 0$, since a calculation shows that the scalar curvature $\kappa(x, t)$ at a point $(x, t)$ will be of the form $\kappa_{t/a}(x) + O(1/a)$, where $\kappa_{t/a}$ is the scalar curvature of $M$ for the metric $ds^2_{t/a}$. (In fact, if one is careful, the $O(1/a)$ can be improved to $O(1/a^2)$, though this doesn't matter to us.) Since $M$ is compact and all the metrics $ds^2_{t/a}$ have positive scalar curvature, we may choose $\kappa_0 > 0$ such that $\kappa_{t/a}(x) \geq \kappa_0 > 0$ for all $x$ and for all $t$. For $a$ large enough, the error terms will be less than $\kappa_0/2$, so $W$ also has positive scalar curvature. $\square$

It is still not known if the converse holds or not; indeed, there is no known methodology for approaching this question, as there is no known method for distinguishing between isotopy classes of positive scalar curvature metrics which is not based on distinguishing concordance classes. However, dimension 2 is special enough so that for the two closed 2-manifolds which admit positive scalar curvature metrics, $S^2$ and $\mathbb{RP}^2$, we can give a complete classification up to isotopy, and even say a bit more.

**Theorem 3.4** *Any two metrics of positive scalar curvature on $S^2$ or on $\mathbb{RP}^2$ are isotopic. In fact, the spaces $\mathfrak{R}^+(S^2)$ and $\mathfrak{R}^+(\mathbb{RP}^2)$ are contractible.*

*Proof.* We begin with a general observation. Let $M$ be any manifold, say for simplicity compact, and let $\mathrm{Diff}\, M$ be its diffeomorphism group, a topological group in the $C^\infty$ topology. (For $M$ compact, there is only one reasonable topology on $\mathrm{Diff}\, M$.) When $M$ is oriented, we denote the orientation-preserving subgroup of $\mathrm{Diff}\, M$ by $\mathrm{Diff}^+ M$. Let $C^\infty(M)$ be the smooth functions on $M$, viewed as a topological vector space (and, in particular, as a topological group under addition). Then one can form the semidirect product group $C^\infty(M) \rtimes \mathrm{Diff}\, M$, with $\mathrm{Diff}\, M$ acting on $C^\infty(M)$ by pre-composition. Note that $C^\infty(M) \rtimes \mathrm{Diff}\, M$ acts on Riemannian metrics on $M$ on the right by the formula

$$g \cdot (u, \varphi) = \varphi^*(e^u g), \qquad u \in C^\infty(M), \quad \varphi \in \mathrm{Diff}\, M,$$

and that this action is continuous for the $C^\infty$ topologies. Any two metrics in the same orbit for this action are said to be *conformal* to one another; any two metrics in the same orbit for the action of the subgroup $C^\infty(M)$ are said to be *pointwise conformal* to one another.

Now we need to recall the Uniformization Theorem for Riemann surfaces. When formulated in the language of differential geometry (rather than complex analysis), it says that if $M$ is an oriented connected closed 2-manifold, then $C^\infty(M) \rtimes \mathrm{Diff}^+ M$ acts transitively on the space of Riemannian metrics on $M$. Let's apply this to $S^2$. Then we get an identification of the (contractible) space of Riemannian metrics on $S^2$ with the quotient of $C^\infty(S^2) \rtimes \mathrm{Diff}^+ S^2$ by the subgroup fixing the standard metric $g_0$ of constant Gaussian curvature 1. This subgroup is identified with

$PSL(2, \mathbb{C})$, the group of Möbius transformations,[2] since a famous result of complex analysis says that all (orientation-preserving) pointwise conformal automorphisms for the standard spherical metric come from holomorphic automorphisms of $S^2 = \mathbb{CP}^1$. Since $PSL(2, \mathbb{C})$ has the homotopy type of its maximal compact subgroup $PSU(2) \cong SO(3)$, and since

$$\left(C^\infty(S^2) \rtimes \mathrm{Diff}^+ S^2\right) / PSL(2, \mathbb{C})$$

must be contractible, it follows that $\mathrm{Diff}^+ S^2$ has a deformation retraction down to its subgroup $SO(3)$, which in turn is the group of orientation-preserving isometries for the standard metric. Also observe that since $S^2$ is the double cover of $\mathbb{RP}^2$, taking the $\mathbb{Z}/2$-action into account shows that $C^\infty(\mathbb{RP}^2) \rtimes \mathrm{Diff}\,\mathbb{RP}^2$ acts transitively on the Riemannian metrics on $\mathbb{RP}^2$, and that the stabilizer of the standard metric is precisely $SO(3)$, the isometry group. So $\mathrm{Diff}\,\mathbb{RP}^2$ also has a deformation retraction down to $SO(3)$.

Let's come back to metrics of positive scalar curvature. If $g_0$ and $\bar{g}_0$ denote the standard metrics on $S^2$ or $\mathbb{RP}^2$ of constant Gaussian curvature 1, then a conformally related metric $g_0 \cdot (u, \varphi)$ (respectively, $\bar{g}_0 \cdot (u, \varphi)$) has positive scalar curvature if and only if $e^u g_0$ (resp., $e^u \bar{g}_0$) does (since positive scalar curvature is preserved under the action of Diff). Since $g_0$ has scalar curvature $\equiv 2$, the formula computing the change in scalar curvature under a conformal change in the metric (found in [KW1], for example) gives

$$\Delta(u) = 2 - e^u \kappa, \tag{3.3}$$

where $\Delta$ is the Laplace-Beltrami operator for the metric $g_0$ (with the sign convention making this a negative semi-definite operator) and $\kappa$ is the scalar curvature of the metric $e^u g_0$. We claim that the set

$$\mathcal{S} = \{u \in C^\infty : \kappa \text{ in (3.3) is strictly positive}\}$$

is star-shaped about the origin.

To prove this, suppose $u$ is such that $\kappa$ in (3.3) is strictly positive. Then if $\kappa_t$ denotes the scalar curvature of the metric $e^{tu} g_0$, replacing $u$ by $tu$ in (3.3) gives

$$\Delta(tu) = 2 - e^{tu} \kappa_t.$$

Since $\Delta$ is linear and $\kappa_0 \equiv 2$, we obtain:

$$2 - e^{tu} \kappa_t = t\Delta(u) = t\left(2 - e^u \kappa\right),$$

[2] Caution: While $PSL(2, \mathbb{C})$ embeds in $\mathrm{Diff}^+ S^2$, the identification of $PSL(2, \mathbb{C})$ with the stabilizer of $g_0$ is via a "diagonal embedding," since we need to take the "conformal factor" into account.

or

$$e^{tu}\kappa_t = te^u\kappa + 2(1-t).$$

Since, by assumption, $\kappa$ is everywhere positive and $0 \le t \le 1$, both terms on the right are non-negative. Furthermore, the first term on the right only vanishes when $t = 0$, and the second term only vanishes when $t = 1$. Thus $e^{tu}\kappa_t$ is everywhere positive, and so $\kappa_t$ is everywhere positive, proving that $\mathcal{S}$ is star-shaped (and thus contractible).

Finally, we see that $\mathfrak{R}^+(S^2)$ is identified with

$$\left(\mathcal{S}(S^2)\cdot \mathrm{Diff}^+(S^2)\right)/PSL(2,\mathbb{C}) \subset \left(C^\infty(S^2)\cdot \mathrm{Diff}^+(S^2)\right)/PSL(2,\mathbb{C}),$$

and similarly $\mathfrak{R}^+(\mathbb{RP}^2)$ is identified with

$$\left(\mathcal{S}(\mathbb{RP}^2)\cdot \mathrm{Diff}(\mathbb{RP}^2)\right)/SO(3) \subset \left(C^\infty(\mathbb{RP}^2)\cdot \mathrm{Diff}(\mathbb{RP}^2)\right)/SO(3).$$

As $\mathrm{Diff}^+(S^2)/PSL(2,\mathbb{C})$, $\mathrm{Diff}(\mathbb{RP}^2)/SO(3)$, $\mathcal{S}(S^2)$, and $\mathcal{S}(\mathbb{RP}^2)$ are all contractible, we see that $\mathfrak{R}^+(S^2)$ and $\mathfrak{R}^+(\mathbb{RP}^2)$ must be contractible. $\square$

## 4 The Gromov-Lawson Conjecture and its Variants

In the discussion so far, we have not explained (except in the case of dimension 2) why it is that there are closed manifolds which *cannot* admit a positive scalar curvature metric. Most of the known results of this sort, at least for manifolds of large dimension, stem from a fundamental discovery of Lichnerowicz [Li], which is that if $\not{D}$ is the Dirac operator on a spin manifold $M$ (a self-adjoint elliptic first-order differential operator, acting on sections of the spinor bundle), then

$$\not{D}^2 = \nabla^*\nabla + \frac{\kappa}{4}. \tag{4.1}$$

Here $\nabla$ is the covariant derivative on the spinor bundle induced by the Levi-Civita connection, and $\nabla^*$ is the adjoint of $\nabla$. Since the operator $\nabla^*\nabla$ is obviously self-adjoint and non-negative, it follows from equation (4.1) that the square of the Dirac operator for a metric of positive scalar curvature is bounded away from 0, and thus that the Dirac operator cannot have any kernel. It follows that any index-like invariant of $M$ which can be computed in terms of harmonic spinors (*i.e.*, the kernel of $\not{D}$) has to vanish. E.g., if $M$ is a spin manifold of dimension $n$, there is a version of the Dirac operator which commutes with the action of the Clifford algebra $C\ell_n$ (see

[LaM], § II.7). In particular, its kernel is a (graded) $C\ell_n$-module, which represents an element $\alpha(M)$ in the real $K$-theory group $KO_n = KO^{-n}(\mathrm{pt})$ (see [LaM], Def. II.7.4).

**Theorem 4.1 (Lichnerowicz [Li]; Hitchin [Hit])** *If $M^n$ is a closed spin manifold for which $\alpha(M) \neq 0$ in $KO_n$, then $M$ does not admit a metric of positive scalar curvature.*

We recall that $KO_n \cong \mathbb{Z}$ for $n \equiv 0 \mod 4$, that $KO_n \cong \mathbb{Z}/2$ for $n \equiv 1, 2 \mod 8$, and $KO_n = 0$ for all other values of $n$. Furthermore, for $n \equiv 0 \mod 4$, the invariant $\alpha(M)$ is essentially equal to Hirzebruch's $\widehat{A}$-genus $\widehat{A}(M)$, namely $\alpha(M) = \widehat{A}(M)$ for $n \equiv 0 \mod 8$, and $\alpha(M) = \widehat{A}(M)/2$ for $n \equiv 4 \mod 8$. So this result immediately shows that there are many manifolds, even simply connected ones, which do not lie in class (1) of the Kazdan-Warner trichotomy (see Theorem 2.2). E.g., the Kummer surface $K^4$, the hyperplane in the complex projective space $\mathbb{CP}^3$ given by the equation $z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0$, is spin and has $\widehat{A}(K) = 2$, and hence does not admit a metric of positive scalar curvature.

We observe that $\alpha(M)$ depends only on the spin bordism class $[M] \in \Omega_n^{\mathrm{spin}}$. In fact, we can interpret $\alpha(M)$ as the image of $[M]$ under a natural transformation of generalized homology theories as follows. Let $KO_*(X)$ and $ko_*(X)$ denote the periodic and connective real $K$-homology of a space $X$, respectively (so $KO_*(X)$ satisfies Bott periodicity, and $ko_* = ko_*(\mathrm{pt})$ is obtained from $KO_* = KO_*(\mathrm{pt})$ by killing the groups in negative degree). Then there are natural transformations

$$\Omega_*^{\mathrm{spin}}(X) \xrightarrow{D} ko_*(X) \xrightarrow{\mathrm{per}} KO_*(X),$$

the first of which sends the bordism class $[M, f]$ to $f_*([M]_{ko})$, where $[M]_{ko} \in ko_*(M)$ denotes the $ko$-fundamental class of $M$ determined by the spin structure. With this notation, $\alpha(M) = \mathrm{per} \circ D([M])$.

Next, we want to state an important consequence of Theorem 3.1, but first we need a relevant definition.

**Definition.** Let $B \to BO$ be a fibration. A $B$-*structure* on a manifold is defined to be a lifting of the (classifying map of the) stable normal bundle to a map into $B$. Then one has bordism groups $\Omega_n^B$ of manifolds with $B$-structures, defined in the usual way. (For instance, if $B = B\mathrm{Spin}$, mapping as usual to $BO$, then $\Omega_n^B = \Omega_n^{\mathrm{spin}}$.) We note that given a connected closed manifold $M$, there is a choice of such a $B$ [3] for which $M$ has a $B$-structure and the map $M \to B$ is a 2-equivalence. (**Example**: If $M$ is a spin manifold, choose $B = B\pi \times B\mathrm{Spin}$, where $\pi = \pi_1(M)$, and let $B \to BO$ be the projection onto the second factor composed with the map $B\mathrm{Spin} \to BO$

[3] We will see in Section 5 how to formalize this in a functorial way.

induced by Spin $\to O$. Map $M$ to the first factor by means of the classifying map for the universal cover, and to the second factor by means of the spin structure.)

The simply connected cases of the following theorem were proved in [GL2]; the general case, with this formulation, is in [RS1].

**Theorem 4.2 (Bordism Theorem)** *Let $M^n$ be a $B$-manifold with $n = \dim M \geq 5$, and assume that the map $M \to B$ is a 2-equivalence. Then $M$ admits a metric of positive scalar curvature if and only if there is some $B$-manifold of positive scalar curvature in the same $B$-bordism class.*

*Sketch of Proof.* Let $N$ be a $B$-manifold $B$-bordant to $M$. The hypotheses combine (via the method of proof of the $s$-Cobordism Theorem) to show that $M$ can be obtained from $N$ by surgeries in codimension $\geq 3$. Then if $N$ admits a metric of positive scalar curvature, one can apply Theorem 3.1 to conclude that the same is true for $M$. $\square$

**Remark.** Note that in the proof of Theorem 4.2, $M$ and $N$ do not quite play symmetrical roles. While $M$ can be obtained from $N$ by surgeries in codimension $\geq 3$, the converse may not be the case unless $N \to B$ is also a 2-equivalence. This is useful in applications, since often the "obvious" generators for $B$-bordism groups do not satisfy the 2-equivalence condition.

**Theorem 4.3 (Gromov-Lawson [GL2])** *If $M$ is a simply connected closed manifold of dimension $n \geq 5$, and if $w_2(M) \neq 0$, then $M$ admits a metric of positive scalar curvature.*

*Sketch of Proof.* If $M$ is simply connected with $w_2(M) \neq 0$, then the appropriate $B \to BO$ to use in Theorem 4.2 is just $BSO \to BO$, and the corresponding bordism theory is oriented bordism. Gromov-Lawson proceed to show that the generators of $\Omega_*$ constructed by Wall all admit positive scalar curvature metrics. $\square$

Of course, the restriction $w_2(M) \neq 0$ in Theorem 4.3 is important, because Theorem 4.1 shows that otherwise there can be obstructions to positive scalar curvature. It is also well-known that the maps $D_n : \Omega_n^{\mathrm{spin}} \to ko_n(\mathrm{pt})$ are all surjective, so all potential obstructions are in fact realized. In the simply connected spin case, Gromov and Lawson were not able to get as sharp a result as in the non-spin case, but at least they were able to prove:

**Theorem 4.4** *If $M$ is a simply connected closed manifold of dimension $n \geq 5$, and if $w_2(M) = 0$ (so that, once an orientation is fixed, $M$ defines a class $[M] \in \Omega_n^{\mathrm{spin}}$), then a finite connected sum of copies of $M$ admits a metric of positive scalar curvature if and only if $[M]$ maps to $0 \in KO_n(\mathrm{pt}) \otimes_{\mathbb{Z}} \mathbb{Q}$ under $\alpha$.*

For manifolds with a non-trivial fundamental group, the situation is more complicated, as can already be seen in the 2-dimensional case. (As we have already observed, no closed connected 2-dimensional with an infinite fundamental group admits a positive scalar curvature metric. Nevertheless, oriented surfaces map trivially to $KO_2(\mathrm{pt}) = \mathbb{Z}/2$, at least for the usual (bounding) choice of a spin structure.) It was shown in [GL1] and [SY] that tori never admit positive scalar curvature metrics (in any dimension), and that in general, there are extra obstructions to positive scalar curvature that come from the fundamental group. Extrapolating from Theorem 4.4 and from their results in [GL3], Gromov and Lawson arrived at:

**Conjecture 4.5 ("Gromov-Lawson Conjecture" [GL3])** *Suppose $M$ is a connected closed spin manifold of dimension $n \geq 5$ with "reasonable" fundamental group $\pi$ (in a sense to be discussed below). Let $f : M \to B\pi$ be the classifying map for the universal cover of $M$, so that $(M, f)$ defines a class $[M, f] \in \Omega_n^{\mathrm{spin}}(B\pi)$. Then $M$ admits a metric of positive scalar curvature if and only if* $\mathrm{per} \circ D([M, f]) = 0$ *in $KO_n(B\pi)$.*

The conjecture in the simply connected case was settled by:

**Theorem 4.6 (Stolz [St1])** *If $M$ is a simply connected closed manifold of dimension $n \geq 5$, and if $w_2(M) = 0$ (this means $M$ admits a spin structure, which since $M$ is simply connected is unique once we fix an orientation), then $M$ admits a metric of positive scalar curvature if and only if the Lichnerowicz-Hitchin obstruction $\alpha(M)$ vanishes in $KO_n(\mathrm{pt})$.*

*Sketch of Proof.* The first step in the proof is to reduce this to a 2-primary problem in homotopy theory. This reduction is primarily due to Miyazaki, who showed [Mi] by explicit construction of enough manifolds of positive scalar curvature that the subgroup of $\Omega_n^{\mathrm{spin}}$ generated by manifolds of positive scalar curvature is a subgroup of the kernel of $\alpha$ of index a power of 2. The main part of the proof is then based on the observation that the first non-trivial element in the kernel of $\alpha$ is the quaternionic projective space $\mathbb{HP}^2$. A careful transfer argument (relying on the mod 2 Adams spectral sequence) then shows that, after localizing at 2, the kernel of $\alpha$ in general is generated by the total spaces of fiber bundles over spin manifolds with fiber $\mathbb{HP}^2$ and structure group $PSp(3)$, the isometry group of $\mathbb{HP}^2$. It is not hard to show that all such fiber bundles admit positive scalar curvature metrics (since one can rescale the metric so that the positive scalar curvature on the projective space fibers dwarfs any contributions from the base). So the result follows from the simply connected case of Theorem 4.2. □

To explain progress regarding the conjecture in the non-simply connected case, we need one additional ingredient.

**Definition.** Let $\pi$ be any discrete group. Then the real group ring $\mathbb{R}\pi$ can be completed in two standard ways to get a $C^*$-algebra $C^*(\pi)$.[4] (Either one lets $\mathbb{R}\pi$ act on $\ell^2(\pi)$ on the left in the usual way, and takes the completion in the operator norm, obtaining what is usually called $C^*_r(\pi)$, or else one lets $\mathbb{R}\pi$ act on the Hilbert space direct sum of the spaces of *all* unitary representations of $\pi$ (suitably interpreted to avoid set-theoretic problems), and takes the completion in the operator norm, obtaining what is usually called $C^*_{\max}(\pi)$.) The two completions coincide if and only if $\pi$ is amenable, but for present purposes it will not matter which one we use, so we won't distinguish in the notation.

There is an *assembly map* $A : KO_n(B\pi) \to KO_n(C^*(\pi))$ defined as follows. Form the bundle $\mathcal{V}_{B\pi} = E\pi \times_\pi C^*(\pi)$ over $B\pi$ whose fibers are rank-one free (right) modules over $C^*(\pi)$. As a "$C^*(\pi)$-vector bundle" over $B\pi$, this has a stable class $[\mathcal{V}_{B\pi}]$ in a $K$-group $KO^0(B\pi; C^*(\pi))$, and $A$ is basically the "slant product" with $[\mathcal{V}_{B\pi}]$. The assembly map $A$ is functorial in $\pi$ (to the extent that this makes sense). Injectivity of $A$, often known as the Strong Novikov Conjecture, implies the Novikov Conjecture on homotopy invariance of higher signatures for manifolds with fundamental group $\pi$.

The results on one direction of the the Gromov-Lawson Conjecture all come from:

**Theorem 4.7 ([R2])** *Let $M$ be a closed connected spin manifold of positive scalar curvature, and let $f : M \to B\pi$ be the classifying map for the universal cover of $M$. Then $A \circ \mathrm{per} \circ D([M, f]) = 0$ in $KO_n(C^*(\pi))$. In particular, if the Strong Novikov Conjecture is true for $\pi$* (i.e., *$A$ is injective*), *then* $\mathrm{per} \circ D([M, f]) = 0$ *in $KO_n(B\pi)$.*

*Sketch of Proof.* This relies on an index theory, due to Mishchenko and Fomenko, for elliptic operators with coefficients in a $C^*(\pi)$-vector bundle. If $M$ is as in the theorem, then the (Clifford algebra linear) Dirac operator on $M$, with coefficients in the bundle $\mathcal{V}_{B\pi}$, has an index $\alpha(M, f) \in KO_n(C^*(\pi))$, which one can show by the Kasparov calculus is just $A \circ \mathrm{per} \circ D([M, f])$. Since $\mathcal{V}_{B\pi}$ is by construction a flat bundle, there are no correction terms due to curvature of the bundle, and formula (4.1) applies without change. Hence if $M$ has positive scalar curvature, the square of this Dirac operator is bounded away from 0, and the index vanishes. □

This result seems to be about the best one can do in (in the spin case) in attacking the Gromov-Lawson Conjecture 4.5 via index theory. It indicates that perhaps the "reasonable" groups for purposes of the Conjecture (which Gromov and Lawson did not make precise) should be a subset of the class

[4] A $C^*$-algebra is a Banach algebra with involution which is isometrically $*$-isomorphic to an algebra of operators on a Hilbert space which is closed under the adjoint operation and closed in the operator norm.

of those for which the assembly map $A$ is injective.[5] Many torsion-free groups are known to lie in this class, including for example all torsion-free amenable groups, all torsion-free subgroups of $GL(n, \mathbb{Q})$, and all torsion-free hyperbolic groups in the sense of Gromov.

For groups with torsion, even for finite cyclic groups, it is easy to find examples (see [R1]) where Conjecture 4.5 fails. The reason is simply that many classes in $KO_n(B\pi)$ can be represented by manifolds of positive scalar curvature, such as lens spaces. A first attempt at remedying this results in the following modified conjecture (which first appears in [R2], [R3]):

**Conjecture 4.8 ("Gromov-Lawson-Rosenberg Conjecture")** *Suppose $M$ is a connected closed spin manifold of dimension $n \geq 5$. Let $f : M \to B\pi$ be the classifying map for the universal cover of $M$, so that $(M, f)$ defines a class $[M, f] \in \Omega_n^{\mathrm{spin}}(B\pi)$. Then $M$ admits a metric of positive scalar curvature if and only if $\alpha(M, f)$, the generalized index of the Dirac operator, vanishes in $KO_n(C^*(\pi))$.*

There are analogues of this conjecture, involving indices of "twisted Dirac operators," for manifolds which are non-spin but which have spin universal covers. Rather than state them now, we will defer these cases to Section 5. However, it is worth pointing out that one way to rephrase Conjecture 4.8 is by saying that "the index of Dirac tells all." If this is the case even in the non-spin case, then it implies:

**Conjecture 4.9** *If $M$ is a connected closed manifold of dimension $n \geq 5$, and if the universal cover of $M$ does* not *admit a spin structure, then $M$ admits a metric of positive scalar curvature.*

Conjecture 4.9 is consistent with Theorem 4.3, but unfortunately it is known to fail for manifolds with large fundamental group. A counterexample suggested by [GL3], for which failure of the conjecture can be checked using the "minimal hypersurface technique" of [SY], is $T^6 \# (\mathbb{CP}^2 \times S^2)$. This suggests that Conjecture 4.8 should be false as well, though the following counterexample was only discovered recently.

**Counterexample 4.10 ([Sch])** *Let $M^5$ be the closed spin manifold obtained from $T^5$ by doing spin surgery to cut down the fundamental group to $\mathbb{Z}^4 \times \mathbb{Z}/3$, and let $f : M \to B(\mathbb{Z}^4 \times \mathbb{Z}/3)$ be the classifying map for its universal cover. Then $\alpha(M, f) = 0$ in $KO_n(C^*(\pi))$, but $M$ does not admit a metric of positive scalar curvature.*

What is most amazing about Conjectures 4.8 and 4.9 is not that there are cases where they fail, but that they indeed hold in a great number of cases.

[5] As far as we know at the moment, this class could include all torsion-free groups.

This should be viewed as a vindication of the intuition of Gromov and Lawson, since in many cases Conjecture 4.5 is true in its original formulation. Before stating some of these results, we should first explain how it is that one "narrows the gap" between the positive results of the Bordism Theorem, Theorem 4.2, and the results on obstructions in Theorem 4.7. While one could prove some of the results in greater generality, we will state them only in the spin and oriented non-spin cases.

**Theorem 4.11 (Stolz, Jung)** *Let $M^n$ be a connected closed manifold of dimension $n \geq 5$, and let $f : M \to B\pi$ be the classifying map for its universal cover. If $M$ is spin, then $M$ admits a metric of positive scalar curvature if and only if there is some spin manifold of positive scalar curvature representing the class $D([M, f])$ in $ko_n(B\pi)$. If $M$ is oriented and if the universal cover of $M$ does* not *admit a spin structure, then $M$ admits a metric of positive scalar curvature if and only if there is some oriented manifold of positive scalar curvature representing the class $f_*([M]) \in H_n(B\pi; \mathbb{Z})$.*

*Sketch of Proof.* This requires a number of techniques. The 2-primary calculation in the spin case is based on a generalization, found in [St2], of the $\mathbb{HP}^2$-bundle method of the proof of Theorem 4.6. The 2-primary calculation in the oriented non-spin case is easier, so we give it here. Localized at 2, the spectrum $MSO$ is known to be Eilenberg-MacLane (see [R4]), so $\Omega_n(B\pi)$, after localizing at 2, splits up as $\bigoplus_j H_{n-j}(B\pi; \Omega_j)$, with the summand $H_{n-j}(B\pi; \Omega_j)$ corresponding to bordism classes of the form

$$N^{n-j} \times P^j \xrightarrow{g} B\pi,$$

with $g$ collapsing $P$ to a point. But by the same calculation as in the proof of Theorem 4.3, each generator of $\Omega_j$ with $j > 0$ is represented by a manifold of positive scalar curvature. So by the Bordism Theorem, Theorem 4.2, we are reduced to looking at $H_n(B\pi; \mathbb{Z})$.

The proof at odd primes is based on the theory of homology theories derived from bordism, using "bordism with singularities." □

Using this result, it is easy to check certain cases of Conjectures 4.8 and 4.9. For example, one easily deduces:

**Theorem 4.12** *Conjecture 4.9 is true for orientable manifolds with finite cyclic fundamental group.*

*Proof.* The integral homology of a cyclic group is concentrated in odd degrees $n$, where (for $n \geq 3$) a generator is represented by a lens space (which has positive scalar curvature). □

Putting together Theorem 4.7 and Theorem 4.11, we obtain the following positive results on Conjecture 4.8:

**Theorem 4.13** *Suppose the discrete group $\pi$ has the following two properties:*

1. *The Strong Novikov Conjecture holds for $\pi$,* i.e., *the assembly map $A : KO_*(B\pi) \to KO_*(C^*(\pi))$ is injective.*
2. *The natural map* $\mathrm{per} : ko_*(B\pi) \to KO_*(B\pi)$ *is injective.*

*Then the Gromov-Lawson Conjecture, Conjecture 4.5, and the Gromov-Lawson-Rosenberg Conjecture, Conjecture 4.8, hold for spin manifolds with fundamental group $\pi$.*

*Proof.* Suppose $M^n$ is a spin manifold, with $n \geq 5$, and $f : M \to B\pi$ is the classifying map for its universal cover. If $\mathrm{per} \circ D([M, f]) = 0$ in $KO_n(B\pi)$, then $D([M, f]) = 0$ in $ko_n(B\pi)$ by Condition (2), and so $M$ admits a metric of positive scalar curvature by Theorem 4.11. But if $\mathrm{per} \circ D([M, f]) \neq 0$, condition (1) says that $\alpha(M, f) \neq 0$, and thus $M$ cannot admit a metric of positive scalar curvature, by Theorem 4.7. □

Theorem 4.13 applies to quite a number of torsion-free groups, for example, free groups and free abelian groups. It is not much help in studying finite groups, however. For finite groups, both of the conditions in Theorem 4.13 usually fail. Still, there are so far no counterexamples to the Gromov-Lawson-Rosenberg Conjecture in the case of finite fundamental groups. In fact, the Conjecture is true for the following class of finite groups. Recall that a finite group has periodic cohomology if and only if its Sylow subgroups are all cyclic or generalized quaternion.

**Theorem 4.14 ([BGS])** *The Gromov-Lawson-Rosenberg Conjecture, Conjecture 4.8, holds for any spin manifold with finite fundamental group with periodic cohomology.*

One might wonder whether the restriction to dimensions $n \geq 5$ in most of our results is truly necessary. In dimension 2, we already know the full story as far as positive scalar curvature is concerned, and in dimension 3, the Thurston Geometrization Conjecture would basically settle everything. Dimension 4 is different, however. Seiberg-Witten theory gives the following:

**Theorem 4.15 (primarily due to Taubes [Tau]; see also [LeB])** *Let $M^n$ be a closed, connected oriented 4-manifold with $b_2^+(M) > 1$. If $M$ admits a symplectic structure (in particular, if $M$ admits the structure of a Kähler manifold of complex dimension 2) then $M$ does not admit a positive scalar curvature metric (even one not well-behaved with respect to the symplectic structure).*

This dramatic result implies that the Gromov-Lawson-Rosenberg Conjecture fails badly in dimension 4, even in the simply connected case.

**Counterexample 4.16** *In dimension 4, there exist:*

1. *a simply connected spin manifold $M^4$ with $\widehat{A}(M) = 0$ but with no positive scalar curvature metric.*
2. *simply connected non-spin manifolds with no positive scalar curvature metric.*

The counterexamples we have listed to Conjectures 4.8 and 4.9, as well as the unusual behavior in dimension 4, suggest that it may be best to divide the Gromov-Lawson-Rosenberg Conjecture into two pieces: an "unstable" part, that may fail in some cases due to low-dimensional difficulties (or other factors), and a "stable" conjecture, which stands a better chance of being true in general. This, as well as the fact that the periodicity in $KO$-theory has no obvious geometric counterpart as far as positive scalar curvature is concerned, motivates:

**Conjecture 4.17 ("Stable Gromov-Lawson-Rosenberg Conjecture")** *Let $\mathrm{Bt}^8$ be the Bott manifold, a simply connected spin manifold of dimension 8 with $\widehat{A}(\mathrm{Bt}^8) = 1$. (This manifold is not unique, but any choice will do. What is essential here is that $\mathrm{Bt}^8$ geometrically represents Bott periodicity in $KO$-theory.) If $M^n$ is a spin manifold, and if $f : M \to B\pi$ is the classifying map for its universal cover, then $M \times \mathrm{Bt}^8 \times \cdots \times \mathrm{Bt}^8$ admits a metric of positive scalar curvature (for some sufficiently large number of $\mathrm{Bt}^8$ factors) if and only if $\alpha(M, f) = 0$ in $KO_n(C^*(\pi))$.*

The counterpart of Theorem 4.13 as far as the Stable Conjecture is concerned is simply:

**Theorem 4.18** *The Stable Gromov-Lawson-Rosenberg Conjecture, Conjecture 4.17, holds for spin manifolds with fundamental group $\pi$, provided that the assembly map $A : KO_*(B\pi) \to KO_*(C^*(\pi))$ is injective.*

At the other extreme of finite fundamental groups, we have:

**Theorem 4.19 ([RS2])** *The Stable Gromov-Lawson-Rosenberg Conjecture, Conjecture 4.17, holds for spin manifolds with finite fundamental group.*

For groups with torsion, the assembly map $A$ is not expected to be injective, so Baum, Connes, and Higson [BCH] suggested replacing it by the so-called Baum-Connes assembly map $KO_*^\pi(\underline{E\pi}) \to KO_*(C^*(\pi))$. Here $\underline{E\pi}$ is the universal *proper* $\pi$-space and $KO_*^\pi(\underline{E\pi})$ is its $\pi$-equivariant $KO$-homology. The space $\underline{E\pi}$ coincides with $E\pi$, the universal free $\pi$-space, exactly when $\pi$ is torsion-free, and in this case one recovers the usual assembly map. For a finite group, $\underline{E\pi}$ is a point and the Baum-Connes assembly map is an isomorphism. The following result generalizes Theorems 4.18 and 4.19.

**Theorem 4.20 ([St5])** *The Stable Gromov-Lawson-Rosenberg Conjecture, Conjecture 4.17, holds for spin manifolds with fundamental group* $\pi$, *provided that the Baum-Connes assembly map* $KO_*^\pi(\underline{E\pi}) \to KO_*(C^*(\pi))$ *is injective.*

The hypothesis of this theorem is known to be satisfied in a great many cases, for example, whenever $\pi$ can be embedded discretely in a Lie group with finitely many connected components.

# 5 Parallels with Wall's Surgery Theory

Surgery theory is the main tool in the study of smoothings of Poincaré complexes. As we have seen, it is also the main tool in the study of metrics of positive scalar curvature. In this section we want to discuss similarities and differences between the resulting theories.

A central role in our understanding of smoothings of a Poincaré complex $X$ is played by Wall's surgery obstruction groups $L_i(\pi, w)$; these are abelian groups, which depend on the fundamental group $\pi = \pi_1(X)$, the first Stiefel-Whitney class $w = w_1(X)$, and an integer $i$. The group relevant for the *existence* of a smoothing of $X$ is $L_n(\pi, w)$, $n = \dim X$, while $L_{n+1}(\pi, w)$ plays a role in the *classification* of smoothings of $X$.

The analog of the Wall group in the study of positive scalar curvature metrics on a manifold $M$ is an abelian group $R_i(\pi, w, \widehat{\pi})$, which depends on the fundamental group $\pi = \pi_1(M)$ and the first Stiefel-Whitney class $w\colon \pi \to \mathbb{Z}/2$, as well as an extension $\widehat{\pi}$ of $\pi$. Geometrically, the extension $\widehat{\pi} \twoheadrightarrow \pi$ is given by applying the fundamental group functor to the fiber bundle $O(M)/\mathbb{Z}/2 \to M$, where $O(M)$ is the frame bundle of $M$ and $\mathbb{Z}/2$ acts on $O(M)$ by mapping an isometry $f\colon \mathbb{R}^n \to T_xM$ to the composition $f \circ r$, where $r\colon \mathbb{R}^n \to \mathbb{R}^n$ is the reflection in the hyperplane perpendicular to $(1, 0, \ldots, 0)$.

Up to isomorphism, the extension $\widehat{\pi} \twoheadrightarrow \pi$ is determined by the second Stiefel-Whitney class $w_2(M)$ as follows. If the universal cover of $M$ is spin, then $w_2(M) = u^*(e)$ for a unique $e \in H^2(B\pi; \mathbb{Z}/2)$ where $u\colon M \to B\pi$ is the classifying map of the universal covering of $M$; in this case $\widehat{\pi} \to \pi$ is the central $\mathbb{Z}/2$-extension classified by $e$. Otherwise $\widehat{\pi} \to \pi$ is an isomorphism.

Before defining the groups $R_i(\pi, w, \widehat{\pi})$, we want to state and discuss the following result which shows the central role of these groups for the study of positive scalar curvature metrics.

**Theorem 5.1 ([St4])** *Let* $M$ *be a smooth, connected, compact manifold of dimension* $n \geq 5$, *possibly with boundary. Let* $\pi = \pi_1(M)$ *be the fundamental group,* $w\colon \pi \to \mathbb{Z}/2$ *the first Stiefel-Whitney class, and let* $\widehat{\pi} \to \pi$ *be the extension described above.*

**Existence.** *A positive scalar curvature metric $h$ on $\partial M$ extends to a positive scalar curvature metric on $M$ which is a product metric near the boundary if and only if an obstruction $\sigma(M,h) \in R_n(\pi, w, \widehat{\pi})$ vanishes.*

**Concordance Classification.** *If $h$ extends to a positive scalar curvature metric on $M$, then the group $R_{n+1}(\pi, w, \widehat{\pi})$ acts freely and transitively on the concordance classes of such metrics.*

The groups $R_i(\gamma)$ for $\gamma = (\pi, 0, \pi \times \mathbb{Z}/2)$ (corresponding to spin manifolds) were first introduced by Hajduk [Haj]; he also proved the existence statement in that case.

We wish to compare Theorem 5.1 with the corresponding statements concerning smoothings of a Poincaré complex $X$. We recall that a *smoothing* of $X$ is a (simple) homotopy equivalence $f\colon N \to X$ between a closed manifold $N$ and $X$; two such pairs $(N, f)$, $(N', f')$ are identified if there is a diffeomorphism $g\colon N \to N'$ such that $f$ is homotopic to $f' \circ g$. A necessary condition for the existence of a smoothing is that the Spivak normal bundle of $X$ is stably fiber homotopy equivalent to the sphere bundle of a vector bundle. In homotopy theoretic terms this condition means that the map $X \to BG$ classifying the Spivak normal bundle factors through the canonical map $BO \to BG$. Since this map fits into a homotopy fibration $BO \to BG \to B(G/O)$, the condition is equivalent to the composition $X \to BG \to B(G/O)$ being homotopic to the constant map.

A fiber homotopy equivalence $\Phi$ between the Spivak normal bundle of $X$ and the sphere bundle of a vector bundle determines via the Pontryagin-Thom construction a degree one normal map $f\colon N \to X$ up to bordism. The pair $(N, f)$ is bordant to a smoothing if and only if its "surgery obstruction" $\sigma(N, f) \in L_n(\pi, w)$ vanishes. In particular, if the group $[X, B(G/O)]$ of pointed homotopy classes of maps from $X$ to $B(G/O)$ is trivial, then the vanishing of $\sigma(N, f)$ is sufficient for the existence of a smoothing of $X$; if in addition the group $[X, G/O]$ is trivial, then the fiber homotopy equivalence $\Phi$ is unique up to homotopy. It follows that the bordism class of the degree one normal map $f\colon N \to X$ and hence the surgery obstruction $\sigma(N, f)$ is independent of the choices made in the construction of $(N, f)$. Thus in this case, the vanishing of $\sigma(N, f)$ is also a necessary condition for the existence of a smoothing of $X$.

Concerning classification, the group $L_{n+1}(\pi, w)$ acts on the set $\mathfrak{S}(X)$ of smoothings of $X$. The "surgery exact sequence" describes the orbits as well as the isotropy groups of this action. The orbits are the fibers of a map $\mathfrak{S}(X) \to [X, G/O]$, and the isotropy subgroups are the images of homomorphisms $[\Sigma X, G/O] \to L_{n+1}(\pi, w)$. In particular, if the groups $[X, G/O]$ and $[\Sigma X, G/O]$ are trivial, then $L_{n+1}(\pi, w)$ acts freely and transitively on $\mathfrak{S}(X)$.

The upshot of this discussion is that if the groups $[X, B(G/O)]$, $[X, G/O]$, and $[\Sigma X, G/O]$ vanish, then the main result of surgery theory takes precisely the form of Theorem 5.1, with concordance classes of positive scalar curvature metrics replaced by smoothings and $R_i(\pi, w, \widehat{\pi})$ replaced by $L_i(\pi, w)$.

We recall that Wall's $L_i$-groups have an algebraic description as well as a description as bordism groups. So far, there is only a bordism description of $R_i$.

**Definition 5.2** Let $\gamma$ be a triple $(\pi, w, \widehat{\pi})$, where $w\colon \pi \to \mathbb{Z}/2$ is a group homomorphism and $\widehat{\pi} \to \pi$ is an extension of $\pi$ such that $\ker(\widehat{\pi} \to \pi)$ is either $\mathbb{Z}/2$ or the trivial group. Let $\sigma\colon \mathrm{Spin}(n) \to SO(n)$ be the non-trivial double covering of the special orthogonal group $SO(n)$. We note that the conjugation action of $O(n)$ on $SO(n)$ lifts to an action on $\mathrm{Spin}(n)$. Let $\widehat{\pi} \ltimes \mathrm{Spin}(n)$ be the semi direct product, where $\widehat{g} \in \widehat{\pi}$ acts on the normal subgroup $\mathrm{Spin}(n)$ by conjugation by $r^{w(\widehat{g})}$. Here $r \in O(n)$ is the reflection in the hyperplane perpendicular to $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$. Abusing notation, we also use the notation $w$ for the composition $\widehat{\pi} \to \pi \to \mathbb{Z}/2$. We define $G(\gamma, n)$ to be the quotient of $\widehat{\pi} \ltimes \mathrm{Spin}(n)$ by the central subgroup generated by $(k, -1)$, where $k \in \widehat{\pi}$ is the (possibly trivial) generator of $\ker(\widehat{\pi} \to \pi)$. Sending $[a, b] \in G(\gamma, n)$ to $r^{w(a)}\sigma(b)$ defines a homomorphism $\rho(\gamma, n)\colon G(\gamma, n) \to O(n)$.

A *$\gamma$-structure* on an $n$-dimensional Riemannian manifold $M$ is a principal $G(\gamma, n)$-bundle $P \to M$ together with a $G(\gamma, n)$-equivariant map $\rho\colon P \to O(M)$. Here $O(M)$ is the orthogonal frame bundle of $M$, a principal bundle for the orthogonal group $O(n)$, and $G(\gamma, n)$ acts on $O(M)$ via the homomorphism $\rho(\gamma, n)$.

**Remark 5.3**

1. If $\pi$ is the trivial group, then $G(\gamma, n) = SO(n)$ (resp. $\mathrm{Spin}(n)$) if $\ker(\widehat{\pi} \to \pi)$ is trivial (resp. non-trivial). In this case a $\gamma$-structure on $M$ amounts to an orientation (resp. spin structure) on $M$ (cf. [LaM], Def. II.1.3).

2. More generally, if $w = 0$ and $\widehat{\pi} = \pi$ (resp. $\widehat{\pi} = \pi \times \mathbb{Z}/2$), then $G(\gamma, n) = \pi \times SO(n)$ (resp. $G(\gamma, n) = \pi \times \mathrm{Spin}(n)$); in this case, a $\gamma$-structure amounts to an orientation (resp. spin structure) on $M$, together with a principal $\pi$-bundle $\widetilde{M} \to M$.

3. A $\gamma$-structure determines a principal $\pi$-bundle $\widetilde{M} \stackrel{\text{def}}{=} P/G_1 \to M$, where $G_1$ is the identity component of $G(\gamma, n)$. We note that $G_1 = SO(n)$ if $\ker(\widehat{\pi} \to \pi)$ is trivial, and $G_1 = \mathrm{Spin}(n)$ otherwise. Hence the principal $G_1$-bundle $P \to \widetilde{M}$ can be identified with the oriented frame bundle of $\widetilde{M}$ or a double cover thereof.

**Definition 5.4** Given a triple $\gamma$ as above, $R_n(\gamma)$ is the bordism group of pairs $(N, h)$, where $N$ is a $n$-dimensional manifold with $\gamma$-structure and $h$ is a positive scalar curvature metric on the boundary $\partial N$ (possibly empty). The obstruction $\sigma(M, h) \in R_n(\gamma(M))$ to extending the positive scalar curvature metric $h$ on $\partial M$ to a positive scalar curvature metric on $M$ is just the bordism class $[M, h]$ (every manifold $M$ has a *canonical* $\gamma(M)$-structure).

*Sketch of Proof of Theorem 5.1.* Both the existence and the classification statement are fairly direct consequences of the surgery results discussed in Section 3. Concerning existence, it is easy to see that if $h$ extends to a positive scalar curvature metric on $M$, then $(M, h)$ represents zero in the bordism group $R_n(\gamma)$, $\gamma = \gamma(M) = (\pi, w, \widehat{\pi})$. (The manifold $M \times [0, 1]$ with some corners suitably rounded represents a zero bordism.) Conversely, a zero bordism for $(M, h)$ provides us with a manifold $M'$ with boundary $\partial M' = \partial M$ over which $h$ extends to a positive scalar curvature metric (which is a product metric near the boundary), and a manifold $W$ of dimension $n + 1$ whose boundary is $\partial W = M \cup_{\partial M} M'$. Moreover, the $\gamma$-structure on $M$ extends to a $\gamma$-structure on $W$. Doing some surgery on $W$ if necessary, we may assume that the map $W \to BG(n + 1, \gamma)$ provided by the $\gamma$-structure on $W$ is a 3-equivalence (i.e., it induces an isomorphism on homotopy groups $\pi_i$ for $i < 3$, and a surjection for $i = 3$). The restriction of this map to $M$ is a 2-equivalence (this is a property of the "canonical" $\gamma(M)$-structure of $M$). It follows that the inclusion $M \subset W$ is a 2-equivalence; this implies that $W$ can be built by attaching handles of dimension $\geq 3$ to $M \times [0, 1]$. Reversing the roles of $M$ and $M'$, it follows that $W$ can be constructed from $M'$ by attaching handles of codimension $\geq 3$; in particular, $M$ is obtained from $M'$ by a sequence of surgeries in the interior of $M'$ of codimension $\geq 3$. Hence the Surgery Theorem 3.1 shows that $h$ extends to a positive scalar curvature metric on $M$.

We turn to the classification up to concordance. Our first goal is to define the action of $R_{n+1}(\gamma)$ on $\widetilde{\pi}_0\mathfrak{R}^+(M \operatorname{rel} h)$. We do so by describing for each $[g] \in \widetilde{\pi}_0\mathfrak{R}^+(M \operatorname{rel} h)$ the map

$$m_{[g]} : R_{n+1}(\gamma) \to \widetilde{\pi}_0\mathfrak{R}^+(M \text{ rel } h) \qquad r \mapsto r \cdot [g].$$

We note that our claim that the action is free and transitive translates into the statement that for each $[g] \in \widetilde{\pi}_0\mathfrak{R}^+(M \operatorname{rel} h)$ the map $m_{[g]}$ is bijective. It seems difficult to describe the map $m_{[g]}$ directly. Instead we construct a map

$$i_{[g]} : \widetilde{\pi}_0\mathfrak{R}^+(M \text{ rel } h) \to R_{n+1}(\gamma),$$

show that it is a bijection, and define $m_{[g]}$ to be the inverse of $i_{[g]}$. To define $i_{[g]}([g'])$, consider the positive scalar curvature metric

$$g \cup (h \times s) \cup g' \quad \text{on} \quad \partial(M \times I) = (M \times \{0\}) \cup (\partial M \times I) \cup (M \times \{1\}),$$

where $s$ is the standard metric on $I$, and $h \times s$ is the product metric on $\partial M \times I$. We define $i_{[g]}([g'])$ to be the bordism class of $M \times I$ (furnished with its canonical $\gamma$-structure) together with the metric $g \cup (h \times s) \cup g'$ on its boundary.

Injectivity of $i_{[g]}$ follows immediately from the existence statement proved above. Surjectivity of $i_{[g]}$ is proved in two steps. First we show that every element of $R_{n+1}(\gamma)$ has a representative of the form $(T, q)$ with $q \in \mathfrak{R}^+(\partial T)$, where $T$ is an $(n+1)$-thickening of the 2-skeleton of $M$ (i.e., $T \subset M \times I$ is a codimension zero submanifold with boundary simply homotopy equivalent to a 2-skeleton of $M$). To prove this, let $(N, k)$ be a representative of a given element of $R_{n+1}(\gamma)$. After modifying $N$ if necessary by surgeries in the interior, we may assume that the map $N \to BG(\gamma, n+1)$ given by the $\gamma$-structure on $N$ is a 3-equivalence. Then using Wall's classification of thickenings in the stable range [Wa], Prop. 5.1, it can be shown that $T$ embeds into the interior of $N$. Another application of the Improved Surgery Theorem 3.2 then shows that $k$ extends to a positive scalar curvature metric $K$ on $N \setminus \operatorname{int} T$, which implies $[N, k] = [T, K_{|\partial T}]$.

In a second step, the Improved Surgery Theorem 3.2 is used again to argue that the positive scalar curvature metric $g \cup (h \times s) \cup H_{|\partial T}$ which lives on a part of the boundary of $(M \times I) \setminus T$ can always be extended to a positive scalar curvature metric $G$ on $(M \times I) \setminus T$. This shows that $i_{[g]}$ maps $[G_{|M\times\{1\}}] \in \widetilde{\pi}_0\mathfrak{R}^+(M \text{ rel } h)$ to $[T, H_{|\partial T}]$. $\square$

As mentioned above, there is so far no *algebraic* description of the $R_n$-groups. Worse yet, there is no pair $(n, \gamma)$, with $n \geq 5$, for which $R_n(\gamma)$ is known. However, in many cases, we can obtain a lower bound for the size of $R_n(\gamma)$ by means of an "index homomorphism"

$$\theta\colon R_n(\gamma) \to KO_n(C_r^*\gamma).$$

Here $C^*\gamma$ is a $\mathbb{Z}/2$-graded $C^*$-algebra associated to $\gamma = (\pi, w, \widehat{\pi})$.[6] It is defined as an ideal in the group $C^*$-algebra $C^*\widehat{\pi}$; namely multiplication by the generator $k$ of $\ker(\widehat{\pi} \to \pi)$ is an involution on $C^*\widehat{\pi}$ whose $-1$-eigenspace is $C^*\gamma$. The $\mathbb{Z}/2$-grading is given by the $\{\pm 1\}$-eigenspaces of the involution $C^*\gamma \to C^*\gamma$ which is the restriction of the involution $C^*\widehat{\pi} \to C^*\widehat{\pi}$ given by $\hat{g} \mapsto (-1)^{\hat{w}(\hat{g})}$ for $\hat{g} \in \widehat{\pi} \subset C^*\widehat{\pi}$, where $\hat{w}$ is the composition of the projection map $\widehat{\pi} \to \pi$ and $w\colon \pi \to \mathbb{Z}/2$. In particular, $C^*\gamma = 0$ if $\widehat{\pi} = \pi$ and $C^*\gamma = C^*\pi$ if $w = 0$ and $\widehat{\pi} = \pi \times \mathbb{Z}/2$.

**Remark 5.5** The index homomorphism $\theta$ is a generalization of the index $\alpha(N, f) \in KO_n(C^*\pi)$ for $n$-dimensional closed spin manifolds $N$ equipped with a map $f\colon M \to B\pi$. By remark 5.3, the spin structure and the map

[6] For the meaning of the subscript $_r$, which we henceforth suppress, see the discussion on page 15.

$f$ amount to a $\gamma$-structure on $N$, $\gamma = (\pi, 0, \pi \times \mathbb{Z}/2)$, and hence the closed manifold $N$ represents an element $[N]$ in the bordism group $R_n(\gamma)$. Then

$$\alpha(N, f) = \theta([N]) \in KO_n(C^*\pi) = KO_n(C^*\gamma).$$

In particular, $\theta$ generalizes $\alpha$ to non-spin manifolds, and to manifolds with boundary (whose boundary is equipped with a positive scalar curvature metric).

**Definition 5.6** To define the index homomorphism $\theta$, it is convenient to describe its range $KO_n(C^*\gamma)$ as equivalence classes of "Kasparov modules" $(H, F)$. Here $H$ is a Hilbert module over the real $C^*$-algebra $A = C^*\gamma \otimes C\ell_n$ [Bla], §13; i.e., $H$ is a right $A$-module equipped with a compatible $A$-valued inner product, which is complete with respect to a norm derived from this inner product. (When $A = \mathbb{R}$ or $\mathbb{C}$, a Hilbert $A$-module is just a real or complex Hilbert space.) Here $F$ is an $A$-linear bounded operator on $H$ satisfying certain properties generalizing the main features of elliptic pseudodifferential operators of order 0. (If $A = \mathbb{R}$ or $\mathbb{C}$, these properties imply in particular that $F$ is Fredholm.)

Hence to define $\theta$, we need to describe the pair $(H, F)$ that represents $\theta([N, h])$, where $N$ is manifold with $\gamma$-structure and $h$ is a positive scalar curvature metric on $\partial N$. The Hilbert module $H$ is the space of $L^2$-sections of a bundle $S$ over the complete manifold without boundary $\widehat{N} = N \cup_{\partial N} \partial N \times [0, \infty)$ obtained by attaching a cylindrical end to $N$.

The key fact for the construction of $S$ is the existence of a homomorphism from $G(\gamma, n)$ to $O^{ev}(A)$, the group of even orthogonal elements of the $C^*$-algebra $A = C^*\gamma \otimes C\ell_n$. (An element $x$ of a real $C^*$-algebra is orthogonal if $x^*x = xx^* = 1$.) This homomorphism is given by

$$\rho\colon G(\gamma, n) = \widehat{\pi} \ltimes_{\mathbb{Z}/2} \mathrm{Spin}(n) \to O^{ev}(A) \qquad [a, b] \mapsto ea \otimes e_1^{w(a)} b.$$

Here $e = (1 - k)/2 \in C^*\widehat{\pi}$ is the unit of the ideal $C^*\gamma \subset C^*\widehat{\pi}$, and $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$. We remark that $e_1$ has order four in $C\ell_n$ (its square is $-1$); to make the above map well-defined, we decree $w(b) \in \{0, 1\} \subset \mathbb{Z}$ (this gives in fact a homomorphism!).

If $P \to \widehat{N}$ is the principal $G(\gamma, n)$-bundle given by the $\gamma$-structure on $N$ extended to $\widehat{N}$, then we define the "spinor" bundle $S_{\widehat{N}}$ by

$$S_{\widehat{N}} \stackrel{\text{def}}{=} P \times_{G(\gamma,n)} A,$$

where $g \in G(\gamma, n)$ acts on $A$ by left multiplication by $\rho(g)$.

We note that the fibers of $S_{\widehat{N}}$ are right $A$-modules and are furnished with an $A$-valued inner product $\langle \quad , \quad \rangle$ given by $\langle [p, a], [p, b] \rangle = a^*b \in A$ (we note that two elements in the same fiber of $S_{\widehat{N}}$ can be written in the form $[p,a],[p,b]$ with $p \in P$, $a,b \in A$). Upon integration over $\widehat{N}$, this gives the space $L^2(S_{\widehat{N}})$ of $L^2$-sections of $S_{\widehat{N}}$ the structure of a Hilbert $A$-module.

To construct a "Dirac operator" $D_{\widehat{N}}\colon L^2(S_{\widehat{N}}) \to L^2(S_{\widehat{N}})$ it suffices to note that the Levi-Civita connection on $\widehat{N}$ induces a connection on $S_{\widehat{N}}$, and that the $\gamma$-structure can be used to make the fiber of $S_{\widehat{N}}$ over a point $x \in \widehat{N}$ a left-module over the Clifford algebra generated by the tangent space $T_xM$. Then $D_{\widehat{N}}$ is defined by the usual formula (cf. [LaM], Ch. II, formula 5.0).

The operator $D_{\widehat{N}}$ is $A$-linear, but it is *not* a bounded operator on the Hilbert $A$-module $H = L^2(S_{\widehat{N}})$ (not even in the classical case $A = \mathbb{C}$). One needs to replace $D_{\widehat{N}}$ by a bounded operator $f(D_{\widehat{N}})$, where $f$ is a suitable real valued function on $\mathbb{R}$, and $f(D_{\widehat{N}})$ is defined by "functional calculus" [Lan]. On a compact manifold the usual choice is $f(x) = x(x^2+1)^{-1/2}$. This doesn't work on the non-compact manifold $\widehat{N}$, since $f(D_{\widehat{N}})^2 - 1$ is not compact, which is one of the requirements for a Kasparov module. However, it is shown in [St4] that if $4c^2$ is a lower bound for the scalar curvature of the metric $f$ on $\partial N$ (and hence a lower bound for the scalar curvature of $\widehat{N}$ outside a compact set), and if $f\colon \mathbb{R} \to \mathbb{R}$ is an odd function with $f(x) = 1$ for $x \geq c$ and $f(x) = -1$ for $x \leq -c$, then $(L^2(S_{\widehat{N}}), f(D_{\widehat{N}}))$ is in fact a Kasparov module. Moreover, its $K$-theory class $[L^2(S_{\widehat{N}}), f(D_{\widehat{N}})] \in KO(A) = KO_n(C^*\gamma)$ is independent of the choice of $f$ and the Riemannian metric on $N$ extending $h \in \mathfrak{R}^+(\partial N)$.

Bunke's relative index theorem for $K$-valued indices [Bun], Theorem 1.2, shows furthermore that the $K$-theory class $[L^2(S_{\widehat{N}}), f(D_{\widehat{N}})]$ depends only on the bordism class of $(N,h)$ in $R_n(\gamma)$; this shows that

$$\theta\colon R_n(\gamma) \to KO_n(C^*\gamma) \qquad [N,h] \mapsto [L^2(S_{\widehat{N}}), f(D_{\widehat{N}})]$$

is a well-defined homomorphism.

We have seen in Section 4 that there are closed spin manifolds with trivial $\alpha$-invariant, which do not admit a metric of positive scalar curvature. In view of Theorem 5.1 and Remark 5.5 this implies that

$$\theta\colon R_n(\gamma) \to KO_n(C^*\gamma)$$

is not in general injective.

We observe that the target of $\theta$ is 8-periodic and that the isomorphism $KO_n(C^*\gamma) \cong KO_{n+8}(C^*\gamma)$ is given by multiplication with the Bott element, the generator of $KO_8(\mathbb{R}) \cong \mathbb{Z}$. Under $\theta$, this correponds to the map $R_n(\gamma) \to R_{n+8}(\gamma)$ given by Cartesian product with the Bott manifold $\mathrm{Bt}^8$. However, this map is *not* an isomorphism in general; in fact, the above examples represent non-trivial elements of $R_n(\gamma)$, whose product with a suitable power of Bt is trivial.

We note that the groups $R_n(\gamma)$ can be made 8-periodic by "inverting" the Bott manifold; i.e., by defining a "periodic" or "stable" version of the $R_n$-groups by

$$R_n(\gamma)[\mathrm{Bt}^{-1}] \stackrel{\mathrm{def}}{=} \varinjlim \left( R_n(\gamma) \xrightarrow{\times \mathrm{Bt}} R_{n+8}(\gamma) \xrightarrow{\times \mathrm{Bt}} \dots \right).$$

Then $\theta$ factors through a "stable" homomorphism

$$\theta_{st} \colon R_n(\gamma)[\mathrm{Bt}^{-1}] \to KO_n(C^*\gamma).$$

**Conjecture 5.7 ([St4])** *The homomorphism $\theta_{st}$ is an isomorphism.*

The rest of this section is devoted to discussing the status of this conjecture. First, we look at the case $\ker(\widehat{\pi} \to \pi) = 0$, which corresponds to manifolds whose universal covering is not spin. In this case $C^*\gamma$ and hence also $KO_n(C^*\gamma)$ are trivial. It is a simple observation that also $R_n(\gamma)[\mathrm{Bt}^{-1}]$ vanishes. The argument is the following: Cartesian product gives $R_*(\gamma)$ the structure of a module over the spin bordism ring $\Omega_*^{\mathrm{spin}}$; if $\ker(\widehat{\pi} \to \pi)$ is trivial, it is in fact a module over the oriented bordism ring $\Omega_*^{SO}$. In the latter, the Bott manifold is bordant to a linear combination of the quaternionic plane $\mathbb{HP}^2$ and the complex projective space $\mathbb{CP}^4$, which generate $\Omega_8^{SO} \cong \mathbb{Z} \oplus \mathbb{Z}$. Both of these manifolds admit metrics of positive scalar curvature, and hence the product of any element in $R_n(\gamma)$ with $\mathrm{Bt}^8$ is the trivial element in $R_{n+8}(\gamma)$.

Injectivity of $\theta_{st}$ is closely related to the Stable Conjecture 4.17. In fact, having the index homomorphism $\theta$ at our disposal, we can formulate the following more general conjecture, which agrees with Conjecture 4.17 for spin manifolds.

**Conjecture 5.8** *A closed manifold $M$ admits stably a positive scalar curvature metric if and only if $\theta([M])$ vanishes in $KO_n(\gamma(M))$ (here $M$ is equipped with its canonical $\gamma(M)$-structure).*

We note that injectivity of the homomorphism $\theta_{st}$ implies Conjecture 5.8, but not vice versa; in fact, Conjecture 5.8 is equivalent to the statement that $\theta_{st}$ is injective when restricted to the image of $\Omega_n(\gamma) \to R_n(\gamma)[\mathrm{Bt}^{-1}]$, where $\Omega_n(\gamma)$ is the bordism group of $n$-dimensional *closed* manifolds with $\gamma$-structure. We note that this map factors in the form

$$\Omega_n(\gamma) \to KO_n(\gamma) \stackrel{\mathrm{def}}{=} \left(\Omega_n(\gamma)/T_n(\gamma)\right)[\mathrm{Bt}^{-1}] \xrightarrow{F} R_n(\gamma)[\mathrm{Bt}^{-1}], \tag{5.1}$$

where $T_n(\gamma) \subset \Omega_n(\gamma)$ consists of the bordism classes represented by total spaces of $\mathbb{HP}^2$-bundles. In the spin case $\gamma = (\pi, 0, \pi \times \mathbb{Z}/2)$, a (homotopy theoretic) result of Kreck and the second author [KS], Theorem C, implies that $KO_n(\gamma)$ can be identified with the $KO$-homology of $B\pi$. Composing the forgetful map $F$ and the index map $\theta$ we obtain a homomorphism

$$A\colon KO_n(\gamma) \xrightarrow{F} R_n(\gamma)[\mathrm{Bt}^{-1}] \xrightarrow{\theta_{st}} KO_n(C^*\gamma)$$

which agrees with the assembly map in the spin case $\gamma = (\pi, 0, \pi \times \mathbb{Z}/2)$.

In the authors' opinion, Conjecture 5.8 (assuming as in Theorem 4.20 that a Baum-Connes type map is injective) seems to be within reach; an important ingredient in the proof will be a homotopy theoretic interpretation of $KO_n(\gamma)$ as a 'twisted' $KO$-homology group of $B\pi$. This is work in progress by Michael Joachim based on his thesis [Joa].

Proving injectivity of $\theta_{st}$ seems hard due to an apparent lack of tools; proving injectivity in the simplest case $\gamma = (0, 0, \mathbb{Z}/2)$ is equivalent to giving an affirmative solution to Problem 6.1 discussed in the next section.

Surjectivity of $\theta_{st}$ is closely related to the Baum-Connes Conjecture of [BCH]. We recall that for torsion-free groups $\pi$ this Conjecture claims that the assembly map $A\colon KO_n(B\pi) \to KO_n(C_r^*\pi)$ is an isomorphism. The factorization (5.1) of $A$ shows that surjectivity of $A$ implies that $\theta_{st}$ is surjective.

If $\pi$ is a finite group, then $A$ is in general far from being surjective. Still, Laszlo Feher shows in his thesis [Feh] that $\theta_{st}$ is surjective in the "spin case" $\gamma = (\pi, 0, \pi \times \mathbb{Z}/2)$, provided $\pi$ is a finite $p$-group (i.e., a finite group whose order is a power of $p$ for some prime $p$).

# 6 Future Directions

In this final section, we mention just a few of the most important open problems concerning positive scalar curvature metrics. These problems appear to be quite hard, but they play such fundamental roles that it seems we will never fully understand the subject of positive scalar curvature until some progress is made on them.

**Problem 6.1** *Suppose $g$ is a positive scalar curvature metric on $S^n$. Then there is an index theoretic obstruction with values in $KO_{n+1}$, studied in* [Hit], [GL3]*, and in Section 5 above, to extending $g$ to a positive scalar curvature metric on $D^{n+1}$ which is a product metric on a neighborhood of the boundary. Is this the only obstruction? In other words, if the index obstruction vanishes in $KO_{n+1}$, does $g$ extend to a positive scalar curvature metric on $D^{n+1}$? If not, is this at least true "stably" (after taking a Riemannian product with enough copies of the Bott manifold* $\mathrm{Bt}^8$*,*[7] *or*

[7] It is worth noting here that it is now known that there is a model for the Bott manifold which admits a Ricci-flat metric [J]. If we use this particular choice, then taking a Riemannian product with $\mathrm{Bt}^8$ does not change the scalar curvature.

*after taking a Riemannian product with a flat torus of sufficiently high dimension)?*

*Discussion.* This problem is absolutely fundamental, since without its solution, there is no hope for computing the $R$-groups described in Section 5 above, and thus no hope for a complete concordance classification of positive scalar curvature metrics, even on the very simplest manifolds. At the moment, we know the answer to this question only in the case $n = 2$, where it is easy to see from Theorem 3.4 that every positive scalar curvature metric extends (and the index obstruction always vanishes).

A case which may be exceptional (because of the peculiarities of 4-dimensional smooth topology) is $n = 3$. For this case, Seiberg-Witten theory could conceivably be of use; though it is more likely that Seiberg-Witten theory is only useful in studying the extension problem for more complicated pairs $(M^4, \partial M)$ where $b_2^+(M) > 0$. At the moment, we also do not know anything about the image of the index obstruction in $KO_4 \cong \mathbb{Z}$ when $n = 3$. However, it is proved in [GL3], pp. 130–131, that the obstruction takes all values in $KO_8 \cong \mathbb{Z}$ when $n = 7$.

One possible method of attack in this problem (which could potentially be used in any dimension $> 2$) is the following. We may as well assume that the scalar curvature of $g$ is a positive constant, say 1. If we extend $g$ any way we like to a metric $\overline{g}$ on $D^{n+1}$ which is a product metric in a neighborhood of $S^n = \partial(D^{n+1})$, then we can try to make a pointwise conformal change in the metric $\overline{g}$, supported away from the boundary, to a metric of positive scalar curvature of the special form $e^f\overline{g}$, $f$ supported on the interior of $D$. This leads to the study of the "Yamabe equation with Dirichlet boundary conditions." Rewriting the conformal factor $e^f$ as $v^{4/(n-2)}$, we obtain the boundary value problem

$$\begin{aligned} &-\Delta v + \tfrac{n-2}{n-1}\tfrac{\kappa}{4} v = \tfrac{n-2}{n-1}\tfrac{\kappa_1}{4} v^{\frac{n+2}{n-2}} \qquad \text{in} \quad \operatorname{int} D^{n+1}, \\ &v > 0 \quad \text{in} \quad \operatorname{int} D^{n+1}, \quad v \equiv 1 \quad \text{near} \quad \partial(D^{n+1}). \end{aligned} \tag{6.1}$$

Here $\kappa$ is the scalar curvature of the original metric $\overline{g}$, which is 1 on a neighborhood of $\partial(D^{n+1})$ and has unknown behavior in the interior, $\Delta$ is the Laplace-Beltrami operator with respect to $\overline{g}$ (with the sign convention for which this operator is non-positive), and $\kappa_1$ is the scalar curvature for the new metric (which we want to be everywhere positive).

Note from equation (6.1) that if the "conformal Laplacian," the linear operator

$$L_0 = -\Delta + \frac{n-2}{n-1}\frac{\kappa}{4},$$

has positive spectrum (with Dirichlet boundary conditions, in other words on functions vanishing at the boundary), then it follows that the metric

$g$ has an extension with positive scalar curvature. The reasoning, copied in part from [KW1] and [KW2], is as follows. We may assume that the minimum value of $\kappa$ is $-\kappa_0$, some non-positive number. (Otherwise we're already done.) The eigenfunction $\varphi$ of $L_0$ corresponding to the lowest eigenvalue $\lambda$ cannot change sign, by an application of the maximum principle, so we may assume $\varphi \geq 0$ in $\operatorname{int} D^{n+1}$, and clearly there must be some $\varepsilon > 0$ such that $\varphi > \varepsilon$ on the compact set where $\kappa \leq 0$. Then if $v = 1 + \mu\varphi$, $v > 0$ on $D^{n+1}$, $v \equiv 1$ on $\partial(D^{n+1})$, and

$$L_0 v = \frac{n-2}{n-1}\frac{\kappa}{4} + \lambda\mu\varphi,$$

which we can arrange to be everywhere positive by taking $\mu$ large enough to have $\lambda\mu\varepsilon > \frac{n-2}{n-1}\frac{\kappa_0}{4}$. So $v$ satisfies equation (6.1) except for the condition that $v$ be constant near the boundary. We can achieve this by making a small perturbation in $\varphi$ near the boundary. (This destroys its being an eigenfunction for $L_0$, but doesn't change the condition we really need, which is that $L_0(1 + \mu\varphi)$ should be everywhere positive.)

A curious feature of equation (6.1), which suggests that the answer to our "stable" question is "yes," is that the operator $L_0$ bears a remarkable similarity to equation (4.1) for the square of the Dirac operator. (In fact, the lower-order terms $\frac{n-2}{n-1}\frac{\kappa}{4}$ and $\frac{\kappa}{4}$ become the same in the stable limit as $n \to \infty$.) A challenge before us is therefore to figure out how to apply information about the Dirac operator, which acts on spinors, to the study of the scalar equation (6.1). $\square$

**Problem 6.2** *Are we missing additional "unstable" obstructions to positive scalar curvature (in the closed manifold case, and in dimensions other than* 4*) which do not come from the theory of minimal hypersurfaces?*

*Discussion.* The existence of counterexamples to Conjectures 4.8 and 4.9, as well as the fact that there are many classes in $H_n(B\pi)$ or $ko_n(B\pi)$ for finite groups $\pi$ (see Theorem 4.11) which no one has been able to represent by manifolds of positive scalar curvature, suggests that this may be the case. (The minimal hypersurface method of [SY] can only be applied to manifolds which have a covering space with positive first Betti number, clearly a very restrictive condition not applying when the fundamental group is finite.) Conceivably, additional obstructions to positive scalar curvature might come from the study of certain non-linear partial differential equations, for example, from higher-dimensional analogues of Seiberg-Witten theory, that involve coupling of the Dirac operator to something else, or from the study of moduli spaces of solutions to variants of the Yamabe problem. $\square$

**Problem 6.3** *Are concordant positive scalar curvature metrics necessarily isotopic?*

*Discussion.* This question is still wide open. See the comments following Proposition 3.3. In the analogous problem for automorphisms of manifolds, it is known that invariants from algebraic $K$-theory (especially $K_2$ and Waldhausen's $K$-theory of spaces) play a role here. It would be very interesting to see if any similar phenomena occur in the theory of positive scalar curvature metrics. $\square$

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