Equivariant Euler characteristics and K-homology Euler classes for proper cocompact G-manifolds

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Abstract

Let G be a countable discrete group and let M be a smooth proper cocompact G-manifold without boundary. The Euler operator defines via Kasparov theory an element, called the equivariant Euler class, in the equivariant KO-homology of M. The universal equivariant Euler characteristic of M, which lives in a group $U^G(M)$, counts the equivariant cells of M, taking the component structure of the various fixed point sets into account. We construct a natural homomorphism from $U^G(M)$ to the equivariant KO-homology of M. The main result of this paper says that this map sends the universal equivariant Euler characteristic to the equivariant Euler class. In particular this shows that there are no "higher" equivariant Euler characteristics. We show that, rationally, the equivariant Euler class carries the same information as the collection of the orbifold Euler characteristics of the components of the L-fixed point sets M^L, where L runs through the finite cyclic subgroups of G. However, we give an example of an action of the symmetric group S_3 on the 3-sphere for which the equivariant Euler class has order 2, so there is also some torsion information.

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0 Background and statements of results

Given a countable discrete group G and a cocompact proper smooth G-manifold M without boundary and with G-invariant Riemannian metric, the Euler characteristic operator defines via Kasparov theory an element, the *equivariant Euler class*, in the equivariant real K-homology group of M

$$\mathrm{Eul}^G(M) \quad \in \quad KO_0^G(M). \tag{0.1}$$

The *Euler characteristic operator* is the minimal closure, or equivalently, the maximal closure, of the densely defined operator

$$(d + d^*)\colon\ \Omega^*(M) \subseteq L^2\Omega^*(M) \to L^2\Omega^*(M),$$

with the $\mathbb{Z}/2$-grading coming from the degree of a differential p-form. The *equivariant signature operator* is the same underlying operator, but with a different grading coming from the Hodge star operator. The signature operator also defines an element

$$\mathrm{Sign}^G(M) \quad \in \quad K_0^G(M),$$

which carries a lot of geometric information about the action of G on M. (Rationally, when $G = \{1\}$, $\mathrm{Sign}(M)$ is the Poincaré dual of the total $\mathcal{L}$-class, the Atiyah-Singer L-class, which differs from the Hirzebruch L-class only by certain well-understood powers of 2, but in addition, it also carries quite interesting integral information [11], [22], [27]. A partial analysis of the class $\mathrm{Sign}^G(M)$ for G finite may be found in [26] and [24].)

We want to study how much information $\mathrm{Eul}^G(M)$ carries. This has already been done by the second author [23] in the non-equivariant case. Namely, given a closed Riemannian manifold M, not necessarily connected, let

$$e\colon\ \bigoplus_{\pi_0(M)} \mathbb{Z} = \bigoplus_{\pi_0(M)} KO_0(\{*\}) \ \to\ KO_0(M)$$

be the map induced by the various inclusions $\{*\} \to M$. This map is split injective; a splitting is given by the various projections $C \to \{*\}$ for $C \in \pi_0(M)$, and sends $\{\chi(C) \mid C \in \pi_0(M)\}$ to $\mathrm{Eul}(M)$. Hence $\mathrm{Eul}(M)$ carries precisely the same information as the Euler characteristics of the various components of M, and there are no "higher" Euler classes. Thus the situation is totally different from what happens with the signature operator.

We will see that in the equivariant case there are again no "higher" Euler characteristics and that $\mathrm{Eul}^G(M)$ is determined by the *universal equivariant*

Euler characteristic (see Definition 2.5)

$$\chi^G(M) \quad \in \quad U^G(M) = \bigoplus_{(H)\in\mathrm{consub}(G)} \; \bigoplus_{WH\backslash\pi_0(M^H)} \mathbb{Z}.$$

Here and elsewhere $\mathrm{consub}(G)$ is the set of conjugacy classes of subgroups of G and $NH = \{g \in G \mid g^{-1}Hg = H\}$ is the *normalizer* of the subgroup $H \subseteq G$ and $WH := NH/H$ is its *Weyl group*. The component of $\chi^G(M)$ associated to $(H) \in \mathrm{consub}(G)$ and $WH \cdot C \in WH\backslash\pi_0(M^H)$ is the (ordinary) Euler characteristic $\chi(WH_C\backslash(C, C \cap M^{>H}))$, where WH_C is the isotropy group of $C \in \pi_0(M^H)$ under the WH-action. There is a natural homomorphism

$$e^G(M)\colon U^G(M) \quad \to \quad KO_0^G(M). \tag{0.2}$$

It sends the basis element associated to $(H) \subseteq \mathrm{consub}(G)$ and $WH \cdot C \in WH\backslash\pi_0(M^H)$ to the image of the class of the trivial H-representation $\mathbb{R}$ under the composition

$$R_{\mathbb{R}}(H) = KO_0^H(\{*\}) \xrightarrow{(\alpha)_*} KO_0^G(G/H) \xrightarrow{KO_0^G(x)} KO_0^G(M),$$

where $(\alpha)_*$ is the isomorphism coming from induction via the inclusion $\alpha\colon H \to G$ and $x\colon G/H \to M$ is any G-map with $x(1H) \in C$. The main result of this paper is

Theorem 0.3 (Equivariant Euler class and Euler characteristic) *Let G be a countable discrete group and let M be a cocompact proper smooth G-manifold without boundary. Then*

$$e^G(M)(\chi^G(M)) = \mathrm{Eul}^G(M).$$

The proof of Theorem 0.3 involves two independent steps. Let Ξ be an equivariant vector field on M which is transverse to the zero-section. Let $\mathrm{Zero}(\Xi)$ be the set of points $x \in M$ with $\Xi(x) = 0$. Then $G\backslash\mathrm{Zero}(\Xi)$ is finite. The zero-section $i\colon M \to TM$ and the inclusion $j_x\colon T_xM \to TM$ induce an isomorphism of G_x-representations

$$T_xi \oplus T_0j_x\colon T_xM \oplus T_xM \xrightarrow{\cong} T_{i(x)}(TM)$$

if we identify $T_0(T_xM) = T_xM$ in the obvious way. If pr_i denotes the projection onto the i-th factor for $i = 1, 2$ we obtain a linear G_x-equivariant isomorphism

$$d_x\Xi\colon T_xM \xrightarrow{T_x\Xi} T_{i(x)}(TM) \xrightarrow{(T_xi\oplus T_xj_x)^{-1}} T_xM \oplus T_xM \xrightarrow{\mathrm{pr}_2} T_xM. \tag{0.4}$$

Notice that we obtain the identity if we replace pr_2 by pr_1 in the expression (0.4) above. One can even achieve that Ξ is *canonically transverse to the zero-section*, i.e., it is transverse to the zero-section and $d_x\Xi$ induces the identity on $T_xM/(T_xM)^{G_x}$ for G_x the isotropy group of x under the G-action. This is proved in [29, Theorem 1A on page 133] in the case of a finite group and the argument directly carries over to the proper cocompact case. Define the *index* of Ξ at a zero x by

$$s(\Xi, x) \;=\; \frac{\det\left((d_x\Xi)^{G_x}\colon\ (T_xM)^{G_x} \to (T_xM)^{G_x}\right)}{|\det\left((d_x\Xi)^{G_x}\colon\ (T_xM)^{G_x} \to (T_xM)^{G_x}\right)|} \quad \in \{\pm 1\}.$$

For $x \in M$ let $\alpha_x\colon G_x \to G$ be the inclusion, $(\alpha_x)_*\colon R_{\mathbb{R}}(G_x) = KO_0^{G_x}(\{*\}) \to KO_0^G(G/G_x)$ be the map induced by induction via α_x and let $x\colon G/G_x \to M$ be the G-map sending g to $g \cdot x$. By perturbing the equivariant Euler operator using the vector field Ξ we will show :

Theorem 0.5 (Equivariant Euler class and vector fields) *Let G be a countable discrete group and let M be a cocompact proper smooth G-manifold without boundary. Let Ξ be an equivariant vector field which is canonically transverse to the zero-section. Then*

$$\mathrm{Eul}^G(M) \;=\; \sum_{Gx \in G\backslash \operatorname{Zero}(\Xi)} s(\Xi, x) \cdot KO_0^G(x) \circ (\alpha_x)_*([\mathbb{R}]),$$

where $[\mathbb{R}] \in R_{\mathbb{R}}(G_x) = K_0^{G_x}(\{*\})$ is the class of the trivial G_x-representation $\mathbb{R}$, we consider x as a G-map $G/G_x \to M$ and $\alpha_x\colon G_x \to G$ is the inclusion.

In the second step one has to prove

$$e^G(M)(\chi^G(M)) \;=\; \sum_{Gx \in G\backslash \operatorname{Zero}(\Xi)} s(\Xi, x) \cdot KO_0^G(x) \circ (\alpha_x)_*([\mathbb{R}]). \tag{0.6}$$

This is a direct conclusion of the equivariant Poincaré-Hopf theorem proved in [20, Theorem 6.6] (in turn a consequence of the equivariant Lefschetz fixed point theorem proved in [20, Theorem 0.2]), which says

$$\chi^G(M) \;=\; i^G(\Xi). \tag{0.7}$$

where $i^G(\Xi)$ is the equivariant index of the vector field Ξ defined in [20, (6.5)]. Since we get directly from the definitions

$$e^G(M)(i^G(\Xi)) \;=\; \sum_{Gx \in G\backslash \operatorname{Zero}(\Xi)} s(\Xi, x) \cdot KO_0^G(x) \circ (\alpha_x)_*([\mathbb{R}]), \tag{0.8}$$

equation (0.6) follows from (0.7) and (0.8). Hence Theorem 0.3 is true if we can prove Theorem 0.5, which will be done in Section 1.

We will factorize $e^G(M)$ as

$$e^G(M)\colon\ U^G(M) \xrightarrow{e_1^G(M)} H_0^{\mathsf{Or}(G)}(M;\underline{R_{\mathbb{Q}}}) \xrightarrow{e_2^G(M)} H_0^{\mathsf{Or}(G)}(M;\underline{R_{\mathbb{R}}}) \xrightarrow{e_3^G(M)} KO_0^G(M),$$

where $H_0^{\mathsf{Or}(G)}(M;\underline{R_F})$ is the Bredon homology of M with coefficients in the coefficient system which sends G/H to the representation ring $R_F(H)$ for the field $F=\mathbb{Q},\mathbb{R}$. We will show that $e_2^G(M)$ and $e_3^G(M)$ are rationally injective (see Theorem 3.6). We will analyze the map $e_1^G(M)$, which is not rationally injective in general, in Theorem 3.21.

The rational information carried by $\mathrm{Eul}^G(M)$ can be expressed in terms of orbifold Euler characteristics of the various components of the L-fixed point sets for all finite cyclic subgroups $L\subseteq G$. For a component $C\in\pi_0(M^H)$ denote by WH_C its isotropy group under the WH-action on $\pi_0(M^H)$. For $H\subseteq G$ finite WH_C acts properly and cocompactly on C and its *orbifold Euler characteristic* (see Definition 2.5), which agrees with the more general notion of *L^2-Euler characteristic*,

$$\chi^{\mathbb{Q}WH_C}(C)\in\mathbb{Q},$$

is defined. Notice that for finite WH_C the orbifold Euler characteristic is given in terms of the ordinary Euler characteristic by

$$\chi^{\mathbb{Q}WH_C}(C)=\frac{\chi(C)}{|WH_C|}.$$

There is a character map (see (2.6))

$$\mathrm{ch}^G(M)\colon\ U^G(M)\to\bigoplus_{(H)\in\mathrm{consub}(G)}\ \bigoplus_{WH\backslash\pi_0(M^H)}\mathbb{Q}$$

which sends $\chi^G(M)$ to the various L^2-Euler characteristics $\chi^{\mathbb{Q}WH_C}(C)$ for $(H)\in\mathrm{consub}(G)$ and $WH\cdot C\in WH\backslash\pi_0(M^L)$. Recall that rationally $\mathrm{Eul}^G(M)$ carries the same information as $e_1^G(M)(\chi^G(M))$ since the rationally injective map $e_3^G(M)\circ e_2^G(M)$ sends $e_1^G(M)(\chi^G(M))$ to $\mathrm{Eul}^G(M)$. Rationally $e_1^G(M)(\chi^G(M))$ is the same as the collection of all these orbifold Euler characteristics $\chi^{\mathbb{Q}WH_C}(C)$ if one restricts to finite cyclic subgroups H. Namely, we will prove (see Theorem 3.21):

Theorem 0.9 *There is a bijective natural map*

$$\gamma_{\mathbb{Q}}^G\colon\ \bigoplus_{\substack{(L)\in\mathrm{consub}(G)\\ L\text{ finite cyclic}}}\ \bigoplus_{WL\backslash\pi_0(M^L)}\mathbb{Q}\ \xrightarrow{\cong}\ \mathbb{Q}\otimes_{\mathbb{Z}}H_0^{\mathsf{Or}(G)}(X;\underline{R_{\mathbb{Q}}})$$

which maps

$$\{\chi^{\mathbb{Q}WL_C)}(C) \mid (L) \in \operatorname{consub}(G), L \text{ finite cyclic}, WL \cdot C \in WL\backslash\pi_0(M^L)\}$$

to $1 \otimes_{\mathbb{Z}} e_1^G(M)(\chi^G(M))$.

However, we will show that $\mathrm{Eul}^G(M)$ does carry some torsion information. Namely, we will prove:

Theorem 0.10 *There exists an action of the symmetric group* S_3 *of order* $3!$ *on the* 3*-sphere* S^3 *such that* $\mathrm{Eul}^{S_3}(S^3) \in KO_0^{S_3}(S^3)$ *has order* 2.

The relationship between $\mathrm{Eul}^G(M)$ and the various notions of equivariant Euler characteristic is clarified in sections 2 and 4.3.

The paper is organized as follows:

This paper subsumes and replaces the preprint [25], which gave a much weaker version of Theorem 0.3.

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1 Perturbing the equivariant Euler operator by a vector field

Let M^n be a complete Riemannian manifold without boundary, equipped with an isometric action of a discrete group G. Recall that the de Rham operator $D = d + d^*$, acting on differential forms on M (of all possible degrees) is a formally self-adjoint elliptic operator, and that on the Hilbert space of L^2

forms, it is essentially self-adjoint [8]. With a certain grading on the form bundle (coming from the Hodge $*$-operator), D becomes the *signature operator*; with the more obvious grading of forms by parity of the degree, D becomes the *Euler characteristic operator* or simply the *Euler operator*. When M is compact and G is finite, the kernel of D, the space of harmonic forms, is naturally identified with the real or complex[1] cohomology of M by the Hodge Theorem, and in this way one observes that the (equivariant) index of D (with respect to the parity grading) in the real representation ring of G is simply the (equivariant homological) Euler characteristic of M, whereas the index with respect to the other grading is the G-signature [2].

Now by Kasparov theory (good general references are [4] and [9]; for the detailed original papers, see [12] and [13]), an elliptic operator such as D gives rise to an equivariant K-homology class. In the case of a compact manifold, the equivariant index of the operator is recovered by looking at the image of this class under the map collapsing M to a point. However, the K-homology class usually carries far more information than the index alone; for example, it determines the G-index of the operator with coefficients in any G-vector bundle, and even determines the families index in $K^*_G(Y)$ of a family of twists of the operator, as determined by a G-vector bundle on $M \times Y$. (Y here is an auxiliary parameter space.) When M is non-compact, things are similar, except that usually there is no index, and the class lives in an appropriate Kasparov group $K_G^{-*}(C_0(M))$, which is *locally finite* K^G-homology, *i.e.*, the relative group $K^G_*(\overline{M}, \{\infty\})$, where $\overline{M}$ is the one-point compactification of M.[2] We will be restricting attention to the case where the action of G is proper and cocompact, in which case $K_G^{-*}(C_0(M))$ may be viewed as a kind of orbifold K-homology for the compact orbifold $G\backslash M$ (see [4, Theorem 20.2.7].)

We will work throughout with real scalars and real K-theory, and use a variant of the strategy found in [23] to prove Theorem 0.5.

Proof of Theorem 0.5 Recall that since Ξ is transverse to the zero-section, its zero set $\mathrm{Zero}(\Xi)$ is discrete, and since M is assumed G-cocompact, $\mathrm{Zero}(\Xi)$ consists of only finitely many G-orbits. Write $\mathrm{Zero}(\Xi) = \mathrm{Zero}(\Xi)^+ \amalg \mathrm{Zero}(\Xi)^-$,

[1]depending on what scalars one is using

[2]Here $C_0(M)$ denotes continuous real- or complex-valued functions on M vanishing at infinity, depending on whether one is using real or complex scalars. This algebra is *contra*variant in M, so a contravariant functor of $C_0(M)$ is *co*variant in M. Excision in Kasparov theory identifies $K_G^{-*}(C_0(M))$ with $K_G^{-*}(C(\overline{M}), C(\mathrm{pt}))$, which is identified with relative K^G-homology. When $\overline{M}$ does not have finite G-homotopy type, K^G-homology here means Steenrod K^G-homology, as explained in [10].

according to the signs of the indices $s(\Xi, x)$ of the zeros $x \in \mathrm{Zero}(\Xi)$. We fix a G-invariant Riemannian metric on M and use it to identify the form bundle of M with the Clifford algebra bundle $\mathrm{Cliff}(TM)$ of the tangent bundle, with its standard grading in which vector fields are sections of $\mathrm{Cliff}(TM)^-$, and D with the Dirac operator on $\mathrm{Cliff}(TM)$.[3] (This is legitimate by [15, II, Theorem 5.12].) Let $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ be the $\mathbb{Z}/2$-graded Hilbert space of L^2 sections of $\mathrm{Cliff}(TM)$. Let A be the operator on $\mathcal{H}$ defined by *right* Clifford multiplication by Ξ on $\mathrm{Cliff}(TM)^+$ (the even part of $\mathrm{Cliff}(TM)$) and by right Clifford multiplication by $-\Xi$ on $\mathrm{Cliff}(TM)^-$ (the odd part). We use *right* Clifford multiplication since it commutes with the symbol of D. Observe that A is self-adjoint, with square equal to multiplication by the non-negative function $|\Xi(x)|^2$. Furthermore, A is odd with respect to the grading and commutes with multiplication by scalar-valued functions.

For $\lambda \geq 0$, let $D_\lambda = D + \lambda A$. As in [23], each D_λ defines an unbounded G-equivariant Kasparov module in the same Kasparov class as D. In the "bounded picture" of Kasparov theory, the corresponding operator is

$$B_\lambda = D_\lambda \big(1 + D_\lambda^2\big)^{-\frac{1}{2}} = \frac{1}{\lambda} D_\lambda \left(\frac{1}{\lambda^2} + \frac{1}{\lambda^2} D_\lambda^2 \right)^{-\frac{1}{2}}. \tag{1.1}$$

The axioms satisfied by this operator that insure that it defines a Kasparov K^G-homology class (in the "bounded picture") are the following:

(B1) It is self-adjoint, of norm ≤ 1, and commutes with the action of G.

(B2) It is odd with respect to the grading of $\mathrm{Cliff}(TM)$.

(B3) For $f \in C_0(M)$, $fB_\lambda \sim B_\lambda f$ and $fB_\lambda^2 \sim f$, where $\sim$ denotes equality modulo compact operators.

We should point out that (B1) is somewhat stronger than it needs to be when G is infinite. In that case, we can replace invariance of B_λ under G by "G-continuity," the requirement (see [13] and [4, §20.2.1]) that

(B1′) $f(g \cdot B_\lambda - B_\lambda) \sim 0$ for $f \in C_0(M)$, $g \in G$.

In order to simplify the calculations that are coming next, we may assume without loss of generality that we've chosen the G-invariant Riemannian metric on M so that for each $z \in \mathrm{Zero}(\Xi)$, in some small open G_z-invariant neighborhood U_z of z, M is G_z-equivariantly isometric to a ball, say of radius 1, about the origin in Euclidean space $\mathbb{R}^n$ with an orthogonal G_z-action, with z corresponding to the origin. This can be arranged since the exponential map induces a

[3] Since sign conventions differ, we emphasize that for us, unit tangent vectors on M have square -1 in the Clifford algebra.

G_z-diffeomorphism of a small G_z-invariant neighborhood of $0 \in T_zM$ onto a G_z-invariant neighborhood of z such that 0 is mapped to z and its differential at 0 is the identity on T_zM under the standard identification $T_0(T_zM) = T_zM$. Thus the usual coordinates $x_1, x_2, \ldots, x_n$ in Euclidean space give local coordinates in M for $|x| < 1$, and $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}$ define a local orthonormal frame in TM near z. We can arrange that $(\mathbb{R}^n)^{G_z}$ contains the points with $x_2 = \ldots = x_n = 0$ if $(\mathbb{R}^n)^{G_z}$ is different from $\{0\}$. In these exponential local coordinates, the point $x_1 = x_2 = \cdots = x_n = 0$ corresponds to z. We may assume we have chosen the vector field Ξ so that in these local coordinates, Ξ is given by the radial vector field

$$x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + \cdots + x_n \frac{\partial}{\partial x_n} \tag{1.2}$$

if $z \in \mathrm{Zero}(\Xi)^+$, or by the vector field

$$-x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + \cdots + x_n \frac{\partial}{\partial x_n} \tag{1.3}$$

if $z \in \mathrm{Zero}(\Xi)^-$. Thus $|\Xi(x)| = 1$ on ∂U_z for each z, and we can assume (rescaling Ξ if necessary) that $|\Xi| \geq 1$ on the complement of $\bigcup_{z \in \mathrm{Zero}(\Xi)} U_z$. Recall that $D_\lambda = D + \lambda A$.

Lemma 1.4 *Fix a small number $\varepsilon > 0$, and let P_λ denote the spectral projection of D_λ^2 corresponding to $[0, \varepsilon]$. Then for λ sufficiently large,* range P_λ *is G-isomorphic to $L^2(\mathrm{Zero}(\Xi))$ (a Hilbert space with* $\mathrm{Zero}(\Xi)$ *as orthonormal basis, with the obvious unitary action of G coming from the action of G on* $\mathrm{Zero}(\Xi)$*), and there is a constant $C > 0$ such that $(1 - P_\lambda)D_\lambda^2 \geq C\lambda$. (In other words, $(\varepsilon, C\lambda) \cap (\operatorname{spec} D_\lambda^2) = \emptyset$.) Furthermore, the functions in* range P_λ *become increasingly concentrated near* $\mathrm{Zero}(\Xi)$ *as $\lambda \to \infty$.*

Proof First observe that in Euclidean space $\mathbb{R}^n$, if Ξ is defined by (1.2) or (1.3) and A and D_λ are defined from Ξ as on M, then $S_\lambda = D_\lambda^2$ is basically a Schrödinger operator for a harmonic oscillator, so one can compute its spectral decomposition explicitly. (For example, if $n = 1$, then $S_\lambda = -\frac{d^2}{dx^2} + \lambda^2 x^2 \pm \lambda$, the sign depending on whether $z \in \mathrm{Zero}(\Xi)^+$ or $z \in \mathrm{Zero}(\Xi)^-$ and whether one considers the action on $\mathcal{H}^+$ or $\mathcal{H}^-$.) When $z \in \mathrm{Zero}(\Xi)^+$, the kernel of S_λ in L^2 sections of $\mathrm{Cliff}(T\mathbb{R}^n)$ is spanned by the Gaussian function

$$(x_1, x_2, \ldots, x_n) \mapsto e^{-\lambda |x|^2/2},$$

and if $z \in \mathrm{Zero}(\Xi)^-$, the L^2 kernel is spanned by a similar section of $\mathrm{Cliff}(TM)^-$, $e^{-\lambda|x|^2/2}\frac{\partial}{\partial x_1}$. Also, in both cases, S_λ has discrete spectrum lying on an arithmetic progression, with one-dimensional kernel (in L^2) and first non-zero eigenvalue given by $2n\lambda$.

Now let's go back to the operator on M. Just as in [23, Lemma 2], we have the estimate

$$-K\lambda \leq D_\lambda^2 - (D^2 + \lambda^2 A^2) \leq K\lambda, \tag{1.5}$$

where $K > 0$ is some constant (depending on the size of the covariant derivatives of Ξ).[4] But $D_\lambda^2 \geq 0$, and also, from (1.5),

$$\frac{1}{\lambda^2} D_\lambda^2 \geq A^2 + \frac{1}{\lambda^2} D^2 - \frac{K}{\lambda}, \tag{1.6}$$

which implies that

$$\frac{1}{\lambda^2} D_\lambda^2 \geq \text{multiplication by } |\Xi(x)|^2 - \frac{K}{\lambda}. \tag{1.7}$$

So if ξ_λ is a unit vector in range P_λ, we have

$$\frac{\varepsilon}{\lambda^2} \geq \left\langle \frac{1}{\lambda^2} D_\lambda^2 \xi_\lambda,\, \xi_\lambda \right\rangle \geq \int_M |\Xi(x)|^2 \, |\xi_\lambda(x)|^2 \, d\text{vol} - \frac{K}{\lambda} \left\| \xi_\lambda \right\|^2. \tag{1.8}$$

Now $\|\xi_\lambda\| = 1$, and if we fix $\eta > 0$, we only make the integral smaller by replacing $|\Xi(x)|^2$ by η on the set $E_\eta = \{x : |\Xi(x)|^2 \geq \eta\}$ and by 0 elsewhere. So

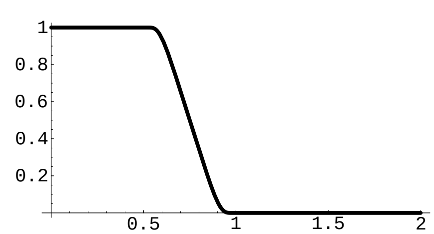
$$\frac{\varepsilon}{\lambda^2} \geq -\frac{K}{\lambda} + \eta \int_{E_\eta} |\xi_\lambda(x)|^2 \, d\text{vol}$$

or

$$\left\| \xi_\lambda \chi_{E_\eta} \right\|^2 \leq \frac{K}{\eta\lambda} + \frac{\varepsilon}{\eta\lambda^2}. \tag{1.9}$$

This being true for any η, we have verified that as $\lambda \to \infty$, ξ_λ becomes increasingly concentrated near the zeros of Ξ, in the sense that the L^2 norm of its restriction to the complement of any neighborhood of $\text{Zero}(\Xi)$ goes to 0.

It remains to compute range P_λ (as a unitary representation space of G) and to prove that D_λ^2 has the desired spectral gap. Define a C^2 cut-off function $\varphi(t)$, $0 \leq t < \infty$, so that $0 \leq \varphi(t) \leq 1$, $\varphi(t) = 1$ for $0 \leq t \leq \frac{1}{2}$, $\varphi(t) = 0$ for $t \geq 1$, and φ is decreasing on the interval $\left[\frac{1}{2}, 1\right]$. In other words, φ is supposed to have a graph like this:



[4]We are using the cocompactness of the G-action to obtain a uniform estimate.

We can arrange that $|\varphi'(t)| \leq 3$ and that $|\varphi''(t)| \leq 20$. For each element z of $\mathrm{Zero}(\Xi)$, recall that we have a G_z-invariant neighborhood U_z that can be identified with the unit ball in $\mathbb{R}^n$ equipped with an orthogonal G_z-action. So the function $\psi_{z,\lambda}(x) = \varphi(r)e^{-\lambda r^2/2}$, where $r = |x|$ is the radial coordinate in $\mathbb{R}^n$, makes sense as a function in $C^2(M)$, with support in U_z. For simplicity suppose $z \in \mathrm{Zero}(\Xi)^+$; the other case is exactly analogous except that we need a 1-form instead of a function. Then D_λ^2, acting on radial functions, becomes

$$-\Delta + \lambda^2|x|^2 - n\lambda = -\frac{\partial^2}{\partial r^2} - (n-1)\frac{1}{r}\frac{\partial}{\partial r} + \lambda^2 r^2 - n\lambda.$$

As we mentioned before, this operator on $\mathbb{R}^n$ annihilates $x \mapsto e^{-\lambda r^2/2}$, so we have

$$\begin{aligned}
\frac{\|D_\lambda \psi_{z,\lambda}\|^2}{\|\psi_{z,\lambda}\|^2} &= \frac{\left\langle D_\lambda^2 \psi_{z,\lambda},\, \psi_{z,\lambda}\right\rangle}{\langle \psi_{z,\lambda},\, \psi_{z,\lambda}\rangle} \\
&= \frac{\int_0^1 \varphi(r)\left(-r\varphi''(r) + (1-n+2r^2\lambda)\varphi'(r)\right)e^{-\lambda r^2} r^{n-2}\,dr}{\int_0^1 \varphi(r)^2 e^{-\lambda r^2} r^{n-1}\,dr} \\
&\leq \frac{\int_{1/2}^1 (20r + 6\lambda)e^{-\lambda r^2} r^{n-2}\,dr}{\int_0^{1/2} e^{-\lambda r^2} r^{n-1}\,dr}.
\end{aligned} \tag{1.10}$$

The expression (1.10) goes to 0 faster than λ^{-k} for any $k \geq 1$, since the numerator dies rapidly and the denominator behaves like a constant times $\lambda^{-n/2}$ for large λ, so $P_\lambda \psi_{z,\lambda}$ is non-zero and very close to $\psi_{z,\lambda}$. Rescaling constructs a unit vector in $\mathrm{range}\, P_\lambda$ concentrated near z, regardless of the value of ε, provided λ is sufficiently large (depending on ε). And the action of $g \in G$ sends this unit vector to the corresponding unit vector concentrated near $g \cdot z$. In particular, $\mathrm{range}\, P_\lambda$ contains a Hilbert space G-isomorphic to $L^2(\mathrm{Zero}(\Xi))$.

To complete the proof of the Lemma, it will suffice to show that if ξ is a unit vector in the domain of D which is orthogonal to each $\psi_{z,\lambda}$, then $\|D_\lambda \xi\|^2 \geq C\lambda$ for some constant $C > 0$, provided λ is sufficiently large Let $E = \bigcup_{z \in \mathrm{Zero}(\Xi)} V_z$, where V_z corresponds to the ball about the origin of radius $\frac{1}{2}$ when we identify U_z with the ball about the origin in $\mathbb{R}^n$ of radius 1. Let χ_E be the characteristic function of E. Then

$$1 = \|\xi\|^2 = \|\chi_E \xi\|^2 + \|(1-\chi_E)\xi\|^2.$$

Hence we must be in one of the following two cases:

(a) $\|(1-\chi_E)\xi\|^2 \geq \frac{1}{2}$.

(b) $\|\chi_E \xi\|^2 \geq \frac{1}{2}$.

In case (1), we can argue just as in the inequalities (1.8) and (1.9) with $\eta = \frac{1}{4}$, since E is precisely the set where $|\Xi(x)|^2 < \frac{1}{4}$. So we obtain

$$\frac{1}{\lambda^2}\|D_\lambda\xi\|^2 = \left\langle \frac{1}{\lambda^2} D_\lambda^2 \xi,\, \xi \right\rangle \geq -\frac{K}{\lambda} + \frac{1}{4}\|(1-\chi_E)\xi\|^2 \geq \frac{1}{8} - \frac{K}{\lambda},$$

which gives $\|D_\lambda\xi\|^2 \succeq \text{const}\cdot\lambda^2$ once λ is sufficiently large. So now consider case (2). Then for some z, we must have $\|\chi_{G\cdot V_z}\xi\|^2 \geq \frac{1}{2|G\backslash \mathrm{Zero}(\Xi)|}$. But by assumption, $\xi \perp \psi_{g\cdot z,\lambda}$ (for this same z and all $g \in G$). Assume for simplicity that $\xi \in \mathcal{H}^+$ and $z \in \mathrm{Zero}(\Xi)^+$. If $\xi \in \mathcal{H}^+$ and $z \in \mathrm{Zero}(\Xi)^-$, there is no essential difference, and if $\xi \in \mathcal{H}^-$, the calculations are similar, but we need 1-forms in place of functions. Anyway, if we let ξ_g denote $\xi|_{U_g\cdot z}$ transported to $\mathbb{R}^n$, we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \varphi(|x|)\xi_g(x)e^{-\lambda|x|^2/2}\,dx \qquad (\forall g),\\ 1 &\geq \sum_g \int_{|x|\leq\frac{1}{2}} \varphi(|x|)^2\big|\xi_g(x)\big|^2\,dx \geq \frac{1}{2|G\backslash \mathrm{Zero}(\Xi)|}. \end{aligned}$$

Now we use the fact that the Schrödinger operator S_λ on $\mathbb{R}^n$ has one-dimensional kernel in L^2 spanned by $x \mapsto e^{-\lambda|x|^2/2}$ (if $z \in \mathrm{Zero}(\Xi)^+$), and spectrum bounded below by $2n\lambda$ on the orthogonal complement of this kernel. (If $z \in \mathrm{Zero}(\Xi)^-$, the entire spectrum of S_λ on $\mathcal{H}^+$ is bounded below by $2n\lambda$.) So compute as follows:

$$\begin{aligned} \big\|D_\lambda\big(\varphi(|x|)\xi_g\big)\big\|^2 &= \big\langle D_\lambda^2\big(\varphi(|x|)\xi_g\big),\, \varphi(|x|)\xi_g\big\rangle\\ &\geq 2n\lambda\,\langle\varphi(|x|)\xi_g,\, \varphi(|x|)\xi_g\rangle\,. \end{aligned} \tag{1.11}$$

Let ω be the function on M which is 0 on the complement of $\bigcup_g U_{g\cdot z}$ and given by $\varphi(|x|)$ on $U_{g\cdot z}$ (when we use the local coordinate system there centered at $g\cdot z$). Then:

$$\begin{aligned} \|D_\lambda\xi\|^2 = \big\|D_\lambda\big(\omega\xi\big)\big\|^2 &+ \big\|D_\lambda\big((1-\omega)\xi\big)\big\|^2\\ &+ 2\,\big\langle D_\lambda^2\big((1-\omega)\xi\big),\, \omega\xi\big\rangle\,. \end{aligned} \tag{1.12}$$

Since D_λ is local and ω is supported on the $U_{g\cdot z}$, $g \in G$, the first term on the right is simply

$$\big\|D_\lambda\big(\omega\xi\big)\big\|^2 = \sum_g \big\|D_\lambda\big(\varphi(|x|)\xi_g\big)\big\|^2 \geq \frac{2n\lambda}{2|G\backslash \mathrm{Zero}(\Xi)|} \tag{1.13}$$

by (1.11). In the inner product term in (1.12), since $\omega\xi$ is a sum of pieces with disjoint supports $U_{g\cdot z}$, we can split this as a sum over terms we can transfer to

$\mathbb{R}^n$, getting

$$2\sum_g \left\langle S_\lambda\big((1-\varphi(|x|))\xi_g\big),\, \varphi(|x|)\xi_g\right\rangle = 2\sum_g \left\langle D^2\big((1-\varphi(|x|))\xi_g\big),\, \varphi(|x|)\xi_g\right\rangle$$

$$+\,2\sum_g \int_{\frac{1}{2}\le|x|\le 1} (\lambda^2|x|^2+T\lambda)\,\varphi(|x|)\big(1-\varphi(|x|)\big)|\xi_g(x)|^2\,dx,$$

where $-K \le T \le K$. Since $\lambda^2|x|^2+T\lambda > 0$ on $\frac{1}{2} \le |x| \le 1$ for large enough λ, the integral here is nonnegative, and the only possible negative contributions to $\|D_\lambda\xi\|^2$ are the terms $2\left\langle D^2\big((1-\varphi(|x|))\xi_g\big),\, \varphi(|x|)\xi_g\right\rangle$, which do not grow with λ. So from (1.12), (1.13), and (1.11), $\|D_\lambda\xi\|^2 \ge \text{const}\cdot\lambda$ for large enough λ, which completes the proof. □

Proof of Theorem 0.5, continued We begin by defining a continuous field $\mathcal{E}$ of $\mathbb{Z}/2$-graded Hilbert spaces over the closed interval $[0, +\infty]$. Over the open interval $[0, +\infty)$, the field is just the trivial one, with fiber $\mathcal{E}_\lambda = \mathcal{H}$, the L^2 sections of $\mathrm{Cliff}(TM)$. But the fiber $\mathcal{E}_\infty$ over $+\infty$ will be the direct sum of $\mathcal{H}\oplus V$, where $V = L^2(\mathrm{Zero}(\Xi))$ is a Hilbert space with orthonormal basis v_z, $z \in \mathrm{Zero}(\Xi)$. We put a $\mathbb{Z}/2$-grading on V by letting $V^+ = L^2(\mathrm{Zero}(\Xi)^+)$, $V^- = L^2(\mathrm{Zero}(\Xi)^-)$. To define the continuous field structure, it is enough by [7, Proposition 10.2.3] to define a suitable total set of continuous sections near the exceptional point $\lambda = \infty$. We will declare ordinary continuous functions $[0,\infty] \to \mathcal{H}$ to be continuous, but will also allow additional continuous fields that become increasingly concentrated near the points of $\mathrm{Zero}(\Xi)$. Namely, suppose $z \in \mathrm{Zero}(\Xi)$. By Lemma 1.4, for λ large, D_λ has an element $\psi_{z,\lambda}$ in its "approximate kernel" increasingly supported close to z, and we have a formula for it. So we declare $(\xi(\lambda))_{\lambda<\infty}$ to define a continuous field converging to cv_z at $\lambda = \infty$ if for any neighborhood U of z,

$$\int_{M\smallsetminus U} |\xi(\lambda)(m)|^2\, d\mathrm{vol}(m) \to 0 \qquad \text{as } \lambda\to\infty,$$

and if (assuming $z \in \mathrm{Zero}(\Xi)^+$) $\xi(\lambda) \in \mathcal{H}^+$ and

$$\left\| \xi(\lambda) - c\left(\frac{\lambda}{\pi}\right)^{\frac{n}{4}} \psi_{z,\lambda} \right\| \to 0 \qquad \text{as } \lambda\to\infty. \tag{1.14}$$

The constant reflects the fact that the L^2-norm of $e^{-\lambda|x|^2/2}$ is $\left(\frac{\pi}{\lambda}\right)^{\frac{n}{4}}$. If $z \in \mathrm{Zero}(\Xi)^-$, we use the same definition, but require $\xi(\lambda) \in \mathcal{H}^-$.

This concludes the definition of the continuous field of Hilbert spaces $\mathcal{E}$, which we can think of as a Hilbert C^*-module over $C(I)$, I the interval $[0, +\infty]$. We will use this to define a Kasparov $(C_0(M), C(I))$-bimodule, or in other words, a homotopy of Kasparov $(C_0(M), \mathbb{R})$-modules. The action of $C_0(M)$ on $\mathcal{E}$ is the obvious one: $C_0(M)$ acts on $\mathcal{H}$ the usual way, and it acts on V (the other summand of $\mathcal{E}_\infty$) by evaluation of functions at the points of $\mathrm{Zero}(\Xi)$:

$$f \cdot v_z = f(z)v_z, \qquad z \in \mathrm{Zero}(\Xi),\ f \in C_0(M).$$

We define a field T of operators on $\mathcal{E}$ as follows. For $\lambda < \infty$, $T_\lambda \in \mathcal{L}(\mathcal{E}_\lambda) = \mathcal{L}(\mathcal{H})$ is simply B_λ as defined in (1.1), where recall that $D_\lambda = D + \lambda A$. For $\lambda = \infty$, $\mathcal{E}_\infty = \mathcal{H} \oplus V$ and T_∞ is 0 on V and is given on $\mathcal{H}$ by the operator $B_\infty = A/|A|$ which is right Clifford multiplication by $\frac{\Xi(x)}{|\Xi(x)|}$ (an L^∞, but possibly discontinuous, vector field) on $\mathcal{H}^+$ and by $\frac{-\Xi(x)}{|\Xi(x)|}$ on $\mathcal{H}^-$. Note that T_∞^2 is 0 on V and the identity on $\mathcal{H}$. While 1_V is not compact on V if $\mathrm{Zero}(\Xi)$ is infinite, this is not a problem since for $f \in C_c(M)$, the action of f on V has finite rank (since f annihilates v_z for $z \notin \operatorname{supp} f$).

Now we check the axioms for $(\mathcal{E}, T)$ to define a homotopy of Kasparov modules from $[D]$ to the class of

$$\big(C_0(M), \mathcal{E}_\infty, T_\infty\big) = \big(C_0(M), \mathcal{H}, B_\infty\big) \oplus \big(C_0(M), V, 0\big).$$

But $\big(C_0(M), \mathcal{H}, B_\infty\big)$ is a *degenerate* Kasparov module, since B_∞ commutes with multiplication by functions and has square 1. So the class of $\big(C_0(M), \mathcal{E}_\infty, T_\infty\big)$ is just the class of $\big(C_0(M), V, 0\big)$, which (essentially by definition) is the image under the inclusion $\mathrm{Zero}(\Xi) \hookrightarrow M$ of the sum (over $G\backslash\mathrm{Zero}(\Xi)$) of $+1$ times the canonical class $KO_0^G(z) \circ (\alpha_z)_*([\mathbb{R}])$ for $G \cdot z \subseteq \mathrm{Zero}(\Xi)^+$ and of -1 times this class if $G \cdot z \subseteq \mathrm{Zero}(\Xi)^-$. This will establish Theorem 0.5, assuming we can verify that we have a homotopy of Kasparov modules.

The first thing to check is that the action of $C_0(M)$ on $\mathcal{E}$ is continuous, i.e., given by a $*$-homomorphism $C_0(M) \to \mathcal{L}(\mathcal{E})$. The only issue is continuity at $\lambda = \infty$. In other words, since the action on $\mathcal{H}$ is constant, we just need to know that if ξ is a continuous field converging as $\lambda \to \infty$ to a vector v in V, then for $f \in C_0(M)$, $f \cdot \xi(\lambda) \to f \cdot v$. But it's enough to consider the special kinds of continuous fields discussed above, since they generate the structure, and if $\xi(\lambda) \to cv_z$, then $\xi(\lambda)$ becomes increasingly concentrated at z (in the sense of L^2 norm), and hence $f \cdot \xi(\lambda) \to cf(z)v_z$, as required.

Next, we need to check that $T \in \mathcal{L}(\mathcal{E})$. Again, the only issue is (strong operator) continuity at $\lambda = \infty$. Because of the way continuous fields are defined at $\lambda = \infty$, there are basically two cases to check. First, if $\xi \in \mathcal{H}$, we need to check that $B_\lambda \xi \to B_\infty \xi$ as $\lambda \to \infty$. Since the B_λ's all have norm ≤ 1, we also only need to check this on a dense set of ξ's. First, fix $\varepsilon > 0$ small and suppose ξ

is smooth and supported on the open set where $|\Xi(x)|^2 > \varepsilon$. Then for λ large, Lemma 1.4 implies that there is a constant $C > 0$ (depending on ε but not on λ) such that $\langle D_\lambda^2 \xi, \xi\rangle > C\lambda \|\xi\|^2$. In fact, if P_λ is the spectral projection of D_λ^2 for the interval $[0, C\lambda]$, Lemma 1.4 implies that $\|P_\lambda \xi\| \le \varepsilon \|\xi\|$ for λ sufficiently large. (This is because the condition on the support of ξ forces ξ to be almost orthogonal to the spectral subspace where $D_\lambda^2 \le \varepsilon$.) Now let E_λ^+ and E_λ^- be the spectral projections for D_λ corresponding to the intervals $(0, \infty)$ and $(-\infty, 0)$, and let F^+ and F^- be the spectral projections for A corresponding to the same intervals. Since the vector field Ξ vanishes only on a discrete set, the operator A has no kernel, and hence $F^+ + F^- = 1$. Now we appeal to two results in Chapter VIII of [14]: Corollary 1.6 in §1, and Theorem 1.15 in §2. The former shows that the operators $A + \frac{1}{\lambda} D$, all defined on $\operatorname{dom} D$, "converge strongly in the generalized sense" to A. Since the positive and negative spectral subspaces for $A + \frac{1}{\lambda} D$ are the same as for D_λ (since the operators only differ by a homothety), [14, Chapter VIII, §2, Theorem 1.15] then shows that $E_\lambda^+ \to F^+$ and $E_\lambda^- \to F^+$ in the strong operator topology. Note that the fact that A has no kernel is needed in these results.

Now since $\|P_\lambda \xi\| \le \varepsilon \|\xi\|$ for λ sufficiently large, we also have

$$B_\infty \xi = F^+ \xi - F^- \xi, \quad \text{and} \quad \|B_\lambda \xi - (E_\lambda^+ \xi - E_\lambda^- \xi)\| \le 2\varepsilon$$

for λ sufficiently large. Hence

$$\|B_\lambda \xi - B_\infty \xi\| \le 2\varepsilon + \|(E_\lambda^+ \xi - F^+ \xi) - (E_\lambda^- \xi - F^- \xi)\| \to 2\varepsilon.$$

Now let $\varepsilon \to 0$. Since, with ε tending to zero, ξ's satisfying our support condition are dense, we have the required strong convergence.

There is one other case to check, that where $\xi(\lambda) \to c v_z$ in the sense of the continuous field structure of $\mathcal{E}$. In this case, we need to show that $B_\lambda \xi(\lambda) \to 0$. This case is much easier: $\xi(\lambda) \to c v_z$ means

$$\left\| \xi(\lambda) - c \left(\frac{\lambda}{\pi}\right)^{\frac{n}{4}} \psi_{z,\lambda} \right\| \to 0 \qquad \text{by (1.14),}$$

while $\|B_\lambda\| \le 1$ and

$$\left\| D_\lambda \left(\left(\frac{\lambda}{\pi}\right)^{\frac{n}{4}} \psi_{z,\lambda} \right) \right\| \to 0 \qquad \text{by (1.10),}$$

so $B_\lambda \xi(\lambda) \to 0$ in norm.

Thus $T \in \mathcal{L}(\mathcal{E})$. Obviously, T satisfies (B1) and (B2) of page 9, so we need to check the analogues of (B3), which are that $f(1 - T^2)$ and $[T, f]$ lie in $\mathcal{K}(\mathcal{E})$

for $f \in C_0(M)$. First consider $1-T^2$. $1-T_\lambda^2$ is locally compact (i.e., compact after multiplying by $f \in C_c(M)$) for each λ, since

$$1-T_\lambda^2 = 1-B_\lambda^2 = (1+D_\lambda^2)^{-1} = \left(1+(D+\lambda A)^2\right)^{-1}$$

is locally compact for $\lambda < \infty$, and $1-T_\infty^2$ is just projection onto V, where functions f of compact support act by finite-rank operators. So we just need to check that $1-T_\lambda^2$ is a norm-continuous field of operators on $\mathcal{E}$. Continuity for $\lambda < \infty$ is routine, and implicit in [3, Remarques 2.5]. To check continuity at $\lambda = \infty$, we use Lemma 1.4, which shows that $(1+D_\lambda^2)^{-1} = P_\lambda + O\left(\frac{1}{\lambda}\right)$, and also that P_λ is increasingly concentrated near Zero(Ξ). So near $\lambda = \infty$, we can write the field of operators $(1+D_\lambda^2)^{-1}$ as a sum of rank-one projections onto vector fields converging to the various v_z's (in the sense of our continuous field structure) and another locally compact operator converging in norm to 0.

This leaves just one more thing to check, that for $f \in C_0(M)$, $[f,\, T_\lambda]$ lies in $\mathcal{K}(\mathcal{E})$. We already know that $[f,\, B_\lambda] \in \mathcal{K}(\mathcal{H})$ for fixed λ and is norm-continuous in λ for $\lambda < \infty$, so since T_∞ commutes with multiplication operators, it suffices to show that $[f,\, B_\lambda]$ converges to 0 in norm as $\lambda \to 0$. We follow the method of proof in [23, p. 3473], pointing out the changes needed because of the zeros of the vector field Ξ.

We can take $f \in C_c^\infty(M)$ with critical points at all of the points of the set Zero(Ξ), since such functions are dense in $C_0(M)$. Then estimate as follows:

$$\begin{aligned}[f,\, B_\lambda] &= \left[f,\, D_\lambda(1+D_\lambda^2)^{-1/2}\right] \\ &= [f,\, D_\lambda](1+D_\lambda^2)^{-1/2} + D_\lambda\left[f,\, (1+D_\lambda^2)^{-1/2}\right]. \end{aligned} \tag{1.15}$$

We have $[f,\, D_\lambda] = [f,\, D]$, which is a 0'th order operator determined by the derivatives of f, of compact support since f has compact support, and we've seen that $(1+D_\lambda^2)^{-1/2}$ converges as $\lambda \to \infty$ (in the norm of our continuous field) to projection onto the space $V = L^2(\text{Zero}(\Xi))$. Since the derivatives of f vanish on Zero(Ξ), the product $[f,\, D_\lambda](1+D_\lambda^2)^{-1/2}$, which is the first term in (1.15), goes to 0 in norm. As for the second term, we have (following [4, p. 199])

$$D_\lambda\left[f,\, (1+D_\lambda^2)^{-1/2}\right] = \frac{1}{\pi}\int_0^\infty \mu^{-\frac{1}{2}} D_\lambda\left[f,\, (1+D_\lambda^2+\mu)^{-1}\right]\, d\mu, \tag{1.16}$$

and

$$D_\lambda\left[f,\, (1+D_\lambda^2+\mu)^{-1}\right] = D_\lambda(1+D_\lambda^2+\mu)^{-1}\left[1+D_\lambda^2+\mu,\, f\right](1+D_\lambda^2+\mu)^{-1}.$$

Now use the fact that

$$\left[1+D_\lambda^2+\mu,\, f\right] = \left[D_\lambda^2,\, f\right] = D_\lambda[D_\lambda,\, f] + [D_\lambda,\, f]D_\lambda = D_\lambda[D,\, f] + [D,\, f]D_\lambda.$$

We obtain that

$$
\begin{aligned}
&D_\lambda \left[f,\, (1+D_\lambda^2+\mu)^{-1}\right] \\
&\quad = \frac{D_\lambda^2}{1+D_\lambda^2+\mu}[D,\, f]\frac{1}{1+D_\lambda^2+\mu} + \frac{D_\lambda}{1+D_\lambda^2+\mu}[D,\, f]\frac{D_\lambda}{1+D_\lambda^2+\mu}. \qquad (1.17)
\end{aligned}
$$

Again a slight modification of the argument in [23, p. 3473] is needed, since D_λ has an “approximate kernel” concentrated near the points of Zero(Ξ). So we estimate the norm of the right side of (1.17) as follows:

$$
\begin{aligned}
\left\|D_\lambda \left[f,\, (1+D_\lambda^2+\mu)^{-1}\right]\right\| &\leq \left\|\frac{D_\lambda^2}{1+D_\lambda^2+\mu}[D,\, f]\frac{1}{1+D_\lambda^2+\mu}\right\| && (1.18)\\
&+ \left\|\frac{D_\lambda}{1+D_\lambda^2+\mu}[D,\, f]\frac{D_\lambda}{1+D_\lambda^2+\mu}\right\|. && (1.19)
\end{aligned}
$$

The first term, (1.18), is bounded by the second, (1.19), plus an additional commutator term:

$$
\left\|\frac{D_\lambda}{1+D_\lambda^2+\mu}[D,[D,f]]\frac{1}{1+D_\lambda^2+\mu}\right\|. \qquad (1.20)
$$

Now the contribution of the term (1.19) is estimated by observing that the function

$$
\frac{x}{1+x^2+\mu}, \qquad -\infty < x < \infty
$$

has maximum value $\frac{1}{2\sqrt{1+\mu}}$ at $x = \sqrt{1+\mu}$, is increasing for $0 < x < \sqrt{1+\mu}$, and is decreasing to 0 for $x > \sqrt{1+\mu}$. Fix $\varepsilon > 0$ small. Since, by Lemma 1.4, $|D_\lambda|$ has spectrum contained in $[0,\, \sqrt{\varepsilon}] \cup [\sqrt{C\lambda},\, \infty)$, we find that

$$
\left\|\frac{D_\lambda}{1+D_\lambda^2+\mu}\right\| \leq \begin{cases} \frac{1}{2\sqrt{1+\mu}}, \\ \qquad \sqrt{C\lambda} \leq \sqrt{1+\mu}, \text{ or } \mu \geq C\lambda - 1, \\ \\ \max\left(\frac{\sqrt{\varepsilon}}{1+\varepsilon+\mu}, \frac{\sqrt{C\lambda}}{1+\mu+C\lambda}\right), \\ \qquad \sqrt{C\lambda} \geq \sqrt{1+\mu}, \text{ or } 0 \leq \mu \leq C\lambda - 1. \end{cases} \qquad (1.21)
$$

Thus the contribution of the term (1.19) to the integral in (1.16) is bounded by

$$
\begin{aligned}
&\frac{\|[D,f]\|}{\pi}\int_0^\infty \left\|\frac{D_\lambda}{1+D_\lambda^2+\mu}\right\|^2 \frac{1}{\sqrt{\mu}}\,d\mu \\
&\le \frac{\|[D,f]\|}{\pi}\Big(\int_{C\lambda-1}^\infty \frac{1}{\sqrt{\mu}}\frac{1}{4(1+\mu)}\,d\mu \qquad (1.22)\\
&\qquad + \int_0^{C\lambda-1}\frac{1}{\sqrt{\mu}}\max\left(\frac{C\lambda}{(1+\mu+C\lambda)^2},\frac{\varepsilon}{(1+\varepsilon+\mu)^2}\right)d\mu\Big) \\
&\le \frac{\|[D,f]\|}{\pi}\left(\frac{1}{4}\int_{C\lambda-1}^\infty \mu^{-\frac{3}{2}}\,d\mu + \int_0^{C\lambda}\frac{1}{\sqrt{\mu}}\frac{C\lambda}{(C\lambda)^2}\,d\mu + \int_0^\infty \frac{\varepsilon}{\sqrt{\mu}(1+\mu)^2}\,d\mu\right) \\
&= \frac{\|[D,f]\|}{\pi}\left(\frac{1}{2\sqrt{C\lambda-1}}+\frac{2}{\sqrt{C\lambda}}+\frac{\pi\varepsilon}{2}\right)\to\frac{\|[D,f]\|}{2}\varepsilon. \qquad (1.23)
\end{aligned}
$$

We can make this as small as we like by taking ε small enough. Similarly, the contribution of term (1.20) to the integral in (1.16) is bounded by

$$
\begin{aligned}
&\frac{\|[D,[D,f]]\|}{\pi}\int_0^\infty \left\|\frac{D_\lambda}{1+D_\lambda^2+\mu}\right\| \frac{1}{1+\mu}\frac{1}{\sqrt{\mu}}\,d\mu \\
&\le \frac{\|[D,[D,f]]\|}{\pi}\Big(\int_{C\lambda-1}^\infty \frac{1}{2\sqrt{1+\mu}}\frac{1}{1+\mu}\frac{1}{\sqrt{\mu}}\,d\mu \\
&\qquad + \int_0^{C\lambda-1}\frac{\sqrt{C\lambda}}{(1+\mu+C\lambda)}\frac{1}{1+\mu}\frac{1}{\sqrt{\mu}}\,d\mu + \int_0^\infty \frac{\varepsilon}{\sqrt{\mu}(1+\mu)^2}\,d\mu\Big) \\
&\le \frac{\|[D,[D,f]]\|}{\pi}\Big(\int_{C\lambda-1}^\infty \frac{1}{2\mu^2}\,d\mu + \int_0^\infty \frac{1}{\sqrt{C\lambda}}\frac{1}{\sqrt{\mu}(1+\mu)}\,d\mu \\
&\qquad + \int_0^\infty \frac{\varepsilon}{\sqrt{\mu}(1+\mu)^2}\,d\mu\Big) \\
&\le \frac{\|[D,[D,f]]\|}{\pi}\left(\frac{1}{2(C\lambda-1)}+\frac{\pi}{\sqrt{C\lambda}}+\frac{\pi\varepsilon}{2}\right)\to\frac{\|[D,[D,f]]\|}{2}\varepsilon, \qquad (1.24)
\end{aligned}
$$

which again can be taken as small as we like. This completes the proof. □

2 Review of notions of equivariant Euler characteristic

Next we briefly review the universal equivariant Euler characteristic, as well as some other notions of equivariant Euler characteristic, so we can see exactly how they are related to the KO^G-Euler class $\mathrm{Eul}^G(M)$. We will use the following notation in the sequel.

Notation 2.1 Let G be a discrete group and $H \subseteq G$ be a subgroup. Let $NH = \{g \in G \mid gHg^{-1} = H\}$ be its *normalizer* and let $WH := NH/H$ be its *Weyl group.*

Denote by $\operatorname{consub}(G)$ the set of conjugacy classes (H) of subgroups $H \subseteq G$.

Let X be a G-CW-complex. Put

$$\begin{aligned} X^H &:= \{x \in X \mid H \subseteq G_x\}; \\ X^{>H} &:= \{x \in X \mid H \subsetneq G_x\}, \end{aligned}$$

where G_x is the isotropy group of x under the G-action.

Let $x\colon G/H \to X$ be a G-map. Let $X^H(x)$ be the component of X^H containing $x(1H)$. Put

$$X^{>H}(x) = X^H(x) \cap X^{>H}.$$

Let WH_x be the isotropy group of $X^H(x) \in \pi_0(X^H)$ under the WH-action.

Next we define the group $U^G(X)$, in which the universal equivariant Euler characteristic takes its values. Let $\Pi_0(G, X)$ be the *component category* of the G-space X in the sense of tom Dieck [6, I.10.3]. Objects are G-maps $x\colon G/H \to X$. A morphism σ from $x\colon G/H \to X$ to $y\colon G/K \to X$ is a G-map $\sigma\colon G/H \to G/K$ such that $y \circ \sigma$ and x are G-homotopic. A G-map $f\colon X \to Y$ induces a functor $\Pi_0(G, f)\colon \Pi_0(G, X) \to \Pi_0(G, Y)$ by composition with f. Denote by $\operatorname{Is}\Pi_0(G, X)$ the set of isomorphism classes $[x]$ of objects $x\colon G/H \to X$ in $\Pi_0(G, X)$. Define

$$U^G(X) \quad := \quad \mathbb{Z}[\operatorname{Is}\Pi_0(G, X)], \tag{2.2}$$

where for a set S we denote by $\mathbb{Z}[S]$ the free abelian group with basis S. Thus we obtain a covariant functor from the category of G-spaces to the category of abelian groups. Obviously $U^G(f) = U^G(g)$ if $f, g\colon X \to Y$ are G-homotopic.

There is a natural bijection

$$\operatorname{Is}\Pi_0(G, X) \quad \xrightarrow{\cong} \quad \coprod_{(H) \in \operatorname{consub}(G)} WH\backslash\pi_0(X^H), \tag{2.3}$$

which sends $x\colon G/H \to X$ to the orbit under the WH-action on $\pi_0(X^H)$ of the component $X^H(x)$ of X^H which contains the point $x(1H)$. It induces a natural isomorphism

$$U^G(X) \quad \xrightarrow{\cong} \quad \bigoplus_{(H) \in \operatorname{consub}(G)} \quad \bigoplus_{WH\backslash\pi_0(X^H)} \mathbb{Z}. \tag{2.4}$$

Definition 2.5 Let X be a finite G-CW-complex X. We define the *universal equivariant Euler characteristic* of X

$$\chi^G(X) \quad \in \quad U^G(X)$$

by assigning to $[x\colon G/H \to X] \in \operatorname{Is}\Pi_0(G,X)$ the (ordinary) Euler characteristic of the pair of finite CW-complexes $(WH_x\backslash X^H(x), WH_x\backslash X^{>H}(x))$.

If the action of G on X is proper (so that the isotropy group of any open cell in X is finite), we define the *orbifold Euler characteristic* of X by:

$$\chi^{\mathbb{Q}G}(X) \quad := \quad \sum_{p\geq 0} \quad \sum_{G\cdot e\in G\backslash I_p(X)} |G_e|^{-1} \qquad \in \mathbb{Q},$$

where $I_p(X)$ is the set of open cells of X (after forgetting the group action).

The orbifold Euler characteristic $\chi^{\mathbb{Q}G}(X)$ can be identified with the more general notion of the L^2-Euler characteristic $\chi^{(2)}(X;\mathcal{N}(G))$, where $\mathcal{N}(G)$ is the group von Neumann algebra of G. One can compute $\chi^{(2)}(X;\mathcal{N}(G))$ in terms of L^2-homology

$$\chi^{(2)}(X;\mathcal{N}(G)) \;=\; \sum_{p\geq 0}(-1)^p \cdot \dim_{\mathcal{N}(G)}\left(H_p^{(2)}(X;\mathcal{N}(G))\right),$$

where $\dim_{\mathcal{N}(G)}$ denotes the von Neumann dimension (see for instance [17, Section 6.6]).

Next we define for a proper G-CW-complex X the *character map*

$$\operatorname{ch}^G(X)\colon U^G(X) \quad \to \quad \bigoplus_{\operatorname{Is}\Pi_0(G,X)} \mathbb{Q} \;=\; \bigoplus_{(H)\in\operatorname{consub}(G)} \quad \bigoplus_{WH\backslash\pi_0(X^H)} \mathbb{Q}. \qquad (2.6)$$

We have to define for an isomorphism class $[x]$ of objects $x\colon G/H \to X$ in $\Pi_0(G,X)$ the component $\operatorname{ch}^G(X)([x])_{[y]}$ of $\operatorname{ch}^G(X)([x])$ which belongs to an isomorphism class $[y]$ of objects $y\colon G/K \to X$ in $\Pi_0(G,X)$, and check that $\chi^G(X)([x])_{[y]}$ is different from zero for at most finitely many $[y]$. Denote by $\operatorname{mor}(y,x)$ the set of morphisms from y to x in $\Pi_0(G,X)$. We have the left operation

$$\operatorname{aut}(y,y) \times \operatorname{mor}(y,x) \to \operatorname{mor}(y,x), \qquad (\sigma,\tau) \mapsto \tau\circ\sigma^{-1}.$$

There is an isomorphism of groups

$$WK_y \xrightarrow{\cong} \operatorname{aut}(y,y)$$

which sends $gK \in WK_y$ to the automorphism of y given by the G-map

$$R_{g^{-1}}\colon G/K \to G/K, \qquad g'K \mapsto g'g^{-1}K.$$

Thus $\operatorname{mor}(y,x)$ becomes a left WK_y-set.

The WK_y-set $\operatorname{mor}(y,x)$ can be rewritten as

$$\operatorname{mor}(y,x) \;=\; \{g \in G/H^K \mid g \cdot x(1H) \in X^K(y)\},$$

where the left operation of WK_y on $\{g \in G/H^K \mid g \cdot x(1H) \in Y^K(y)\}$ comes from the canonical left action of G on G/H. Since H is finite and hence contains only finitely many subgroups, the set $WK\backslash(G/H^K)$ is finite for each $K \subseteq G$ and is non-empty for only finitely many conjugacy classes (K) of subgroups $K \subseteq G$. This shows that $\operatorname{mor}(y,x) \neq \emptyset$ for at most finitely many isomorphism classes $[y]$ of objects $y \in \Pi_0(G,X)$ and that the WK_y-set $\operatorname{mor}(y,x)$ decomposes into finitely many WK_y orbits with finite isotropy groups for each object $y \in \Pi_0(G,X)$. We define

$$\operatorname{ch}^G(X)([x])_{[y]} \quad := \sum_{\substack{WK_y\cdot\sigma\in\\ WK_y\backslash\operatorname{mor}(y,x)}} |(WK_y)_\sigma|^{-1}, \tag{2.7}$$

where $(WK_y)_\sigma$ is the isotropy group of $\sigma \in \operatorname{mor}(y,x)$ under the WK_y-action.

Lemma 2.8 *Let X be a finite proper G-CW-complex. Then the map $\operatorname{ch}^G(X)$ of (2.6) is injective and satisfies*

$$\operatorname{ch}^G(X)(\chi^G(X))_{[y]} \;=\; \chi^{\mathbb{Q}WK_y}(X^K(y)).$$

The induced map

$$\operatorname{id}_{\mathbb{Q}} \otimes_{\mathbb{Z}} \operatorname{ch}^G(X)\colon\; \mathbb{Q} \otimes_{\mathbb{Z}} U^G(X) \;\xrightarrow{\cong}\; \bigoplus_{(H)\in\operatorname{consub}(G)} \;\bigoplus_{WH\backslash\pi_0(X^H)} \mathbb{Q}$$

is bijective.

Proof Injectivity of $\chi^G(X)$ and $\operatorname{ch}^G(X)(\chi^G(X))_{[y]} \;=\; \chi^{\mathbb{Q}WK_y}(X^K(y))$. is proved in [20, Lemma 5.3]. The bijectivity of $\operatorname{id}_{\mathbb{Q}} \otimes_{\mathbb{Z}} \operatorname{ch}^G(X)$ follows since its source and its target are $\mathbb{Q}$-vector spaces of the same finite $\mathbb{Q}$-dimension. □

Now let us briefly summarize the various notions of equivariant Euler characteristic and the relations among them. Since some of these are only defined when M is compact and G is finite, we temporarily make these assumptions *for the rest of this section only.*

Definition 2.9 If G is a finite group, the *Burnside ring* $A(G)$ of G is the Grothendieck group of the (additive) monoid of finite G-sets, where the addition comes from disjoint union. This becomes a ring under the obvious multiplication coming from the Cartesian product of G sets. There is a natural map of rings $j_1\colon A(G) \to R_{\mathbb{R}}(G) = KO_0^G(\mathrm{pt})$ that comes from sending a finite G-set X to the orthogonal representation of G on the finite-dimensional real Hilbert space $L^2_{\mathbb{R}}(X)$. This map can fail to be injective or fail to be surjective, even rationally.

The rank of $A(G)$ is the number of conjugacy classes of subgroups of G, while the rank of $R_{\mathbb{R}}(G)$ is the number of $\mathbb{R}$-conjugacy classes in G, where x and y are called $\mathbb{R}$-conjugate if they are conjugate or if x^{-1} and y are conjugate [28, §13.2]. Thus $\operatorname{rank} A\big((\mathbb{Z}/2)^3\big) = 16 > \operatorname{rank} R_{\mathbb{R}}\big((\mathbb{Z}/2)^3\big) = 8$; on the other hand, $\operatorname{rank} A(\mathbb{Z}/5) = 2 < \operatorname{rank} R_{\mathbb{R}}(\mathbb{Z}/5) = 3$.

Definition 2.10 Now let G be a finite group, M a compact G-manifold (without boundary). We define three more equivariant Euler characteristics for G:

(a) the analytic equivariant Euler characteristic $\chi_a^G(M) \in KO_0^G$, the equivariant index of the Euler operator on M. Since the index of an operator is computed by pushing its K-homology class forward to K-homology of a point, $\chi_a^G(M) = c_*(\mathrm{Eul}^G(M))$, where $c\colon M \to \mathrm{pt}$ and c_* is the induced map on KO_0^G.

(b) the stable homotopy-theoretic equivariant Euler characteristic $\chi_s^G(M) \in A(G)$. This is discussed, say, in [6], Chapter IV, §2.

(c) a certain unstable homotopy-theoretic equivariant Euler characteristic, which we will denote here $\chi_u^G(M)$ to distinguish it from $\chi_s^G(M)$. This invariant is defined in [30], and shown to be the obstruction to existence of an everywhere non-vanishing G-invariant vector field on M. The invariant $\chi_u^G(M)$ lives in a group $A_u^G(M)$ (Waner and Wu call it $A_M(G)$, but the notation $A_u^G(M)$ is more consistent with our notation for $U^G(M)$) defined as follows: $A_u^G(M)$ is the free abelian group on finite G-sets *embedded in* M, modulo isotopy (if s_t is a 1-parameter family of finite G-sets embedded in M, all isomorphic to one another as G-sets, then $s_0 \sim s_1$) and the relation $[s \amalg t] = [s] + [t]$. Waner and Wu define a map $d\colon A_u^G(M) \to A(G)$ (defined by forgetting that a G-set s is embedded in M, and just viewing it abstractly) which maps $\chi_u^G(M)$ to $\chi_s^G(M)$. Both $\chi_u^G(M)$ and $\chi_s^G(M)$ may be computed from the virtual finite G-set given by the singularities of a G-invariant canonically transverse vector field, where the signs are given by the the indices at the singularities.

Proposition 2.11 *Let G be a finite group, and let M be a compact G-manifold (without boundary). The following diagram commutes:*

$$\begin{tikzcd}
A_u^G(M) \arrow[r] \arrow[dr, "d"'] & U^G(M) \arrow[rr, "e^G(M)"] \arrow[d, "c_*"] & & KO_0^G(M) \arrow[d, "c_*"] \\
& A(G) = U^G(\mathrm{pt}) \arrow[rr, "j_1"] & & R_{\mathbb{R}}(G) = KO_0^G(\mathrm{pt}).
\end{tikzcd} \tag{2.12}$$

The map $A_u^G(M) \to U^G(M)$ in the upper left is an isomorphism if (M, G) satisfies the weak gap hypothesis, *that is, if whenever $H \subsetneq K$ are subgroups of G, each component of G^K has codimension at least 2 in the component of G^H that contains it* [30]*. Furthermore, under the maps of this diagram,*

$$\begin{aligned}
\chi_u^G(M) &\mapsto \chi^G(M), & e^G(M)\colon\ \chi^G(M) &\mapsto \mathrm{Eul}^G(M),\\
c_*\colon\ \chi^G(M) &\mapsto \chi_s^G(M), & j_1\colon\ \chi_s^G(M) &\mapsto \chi_a^G(M),\\
d\colon\ \chi_u^G(M) &\mapsto \chi_s^G(M), & c_*\colon\ \mathrm{Eul}^G(M) &\mapsto \chi_a^G(M).
\end{aligned}$$

Proof This is just a matter of assembling known information. The facts about the map $A_u^G(M) \to U^G(M)$ are in [30, §2] and in [20]. That $e^G(M)$ sends $\chi^G(M)$ to $\mathrm{Eul}^G(M)$ is Theorem 0.3. Commutativity of the square follows immediately from the definition of $e^G(M)$, since $c_* \circ e^G(M)$ sends the basis element associated to $(H) \subseteq \mathrm{consub}(G)$ and $WH \cdot C \in WH\backslash\pi_0(M^H)$ to the class of the orthogonal representation of G on $L^2(G/H)$. But under c_*, this same basis element maps to the G-set G/H in $A(G)$, which also maps to the orthogonal representation of G on $L^2(G/H)$ under j_1. □

3 The transformation $e^G(X)$

Next we factorize the transformation $e^G(M)$ defined in (0.2) as

$$e^G(M)\colon\ U^G(M) \xrightarrow{e_1^G(M)} H_0^{\mathsf{Or}(G)}(M; \underline{R_{\mathbb{Q}}}) \xrightarrow{e_2^G(M)} H_0^{\mathsf{Or}(G)}(M; \underline{R_{\mathbb{R}}}) \\ \xrightarrow{e_3^G(M)} KO_0^G(M),$$

where $e_2^G(X)$ and $e_3^G(X)$ are rationally injective. Rationally we will identify $H_0^{\mathsf{Or}(G)}(M; \underline{R_{\mathbb{Q}}})$ and the element $e_1^G(M)(\chi^G(M))$ in terms of the orbifold Euler characteristics $\chi^{WL_C}(C)$, where (L) runs through the conjugacy classes of finite cyclic subgroups L of G and $WL \cdot C$ runs through the orbits in $WL\backslash\pi_0(X^L)$.

Here WL_C is the isotropy group of $C \in \pi_0(X^L)$ under the $WL = NL/L$-action. Notice that $e_1^G(M)(\chi^G(M))$ carries rationally the same information as $\mathrm{Eul}^G(M) \in KO_0^G(M)$.

Here and elsewhere $H_0^{\mathsf{Or}(G)}(M; \underline{R_F})$ is the Bredon homology of M with coefficients the covariant functor

$$\underline{R_F} \colon \mathsf{Or}(G; \mathcal{F}\mathsf{in}) \to \mathbb{Z}\text{-}\mathsf{Mod}.$$

The orbit category $\mathsf{Or}(G; \mathcal{F}\mathsf{in})$ has as objects homogeneous spaces G/H with finite H and as morphisms G-maps (Since M is proper, it suffices to consider coefficient systems over $\mathsf{Or}(G; \mathcal{F}\mathsf{in})$ instead over the full orbit category $\mathsf{Or}(G)$.) The functor $\underline{R_F}$ into the category $\mathbb{Z}\text{-}\mathsf{Mod}$ of $\mathbb{Z}$-modules sends G/H to the representation ring $R_F(H)$ of the group H over the field $F = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. It sends a morphism $G/H \to G/K$ given by $g'H \mapsto g'gK$ for some $g \in G$ with $g^{-1}Hg \subseteq K$ to the induction homomorphism $R_F(H) \to R_F(K)$ associated with the group homomorphism $H \to K, h \mapsto g^{-1}hg$. This is independent of the choice of g since an inner automorphism of K induces the identity on $R_{\mathbb{R}}(K)$. Given a covariant functor $V \colon \mathsf{Or}(G) \to \mathbb{Z}\text{-}\mathsf{Mod}$, the *Bredon homology* of a G-CW-complex X with coefficients in V is defined as follows. Consider the cellular (contravariant) $\mathbb{Z}\mathsf{Or}(G)$-chain complex $C_*(X^-) \colon \mathsf{Or}(G) \to \mathbb{Z}\text{-}\mathsf{Chain}$ which assigns to G/H the cellular chain complex of the CW-complex $X^H = \mathrm{map}_G(G/H, X)$. One can form the tensor product over the orbit category (see for instance [16, 9.12 on page 166]) $C_*(X^-) \otimes_{\mathbb{Z}\mathsf{Or}(G;\mathcal{F}\mathsf{in})} V$ which is a $\mathbb{Z}$-chain complex and whose homology groups are defined to be $H_p^{\mathsf{Or}(G)}(X; V)$.

The zero-th Bredon homology can be made more explicit. Let

$$Q \colon \Pi_0(G; X) \to \mathsf{Or}(G; \mathcal{F}\mathsf{in}) \tag{3.1}$$

be the forgetful functor sending an object $x \colon G/H \to X$ to G/H. Any covariant functor $V \colon \mathsf{Or}(G; \mathcal{F}\mathsf{in}) \to \mathbb{Z}\text{-}\mathsf{Mod}$ induces a functor $Q^*V \colon \Pi_0(G; X) \to \mathbb{Z}\text{-}\mathsf{Mod}$ by composition with Q. The colimit (= direct limit) of the functor $Q^*\underline{R_F}$ is naturally isomorphic to the Bredon homology

$$\beta_F^G(X) \colon \varinjlim_{\Pi_0(G,X)} Q^* R_F(H) \xrightarrow{\cong} H_0^{\mathsf{Or}(G)}(X; \underline{R_F}). \tag{3.2}$$

The isomorphism $\beta_F^G(X)$ above is induced by the various maps

$$R_F(H) = H_0^{\mathsf{Or}(G)}(G/H; \underline{R_F}) \xrightarrow{H_0^{\mathsf{Or}(G)}(x; \underline{R_F})} H_0^{\mathsf{Or}(G)}(X; \underline{R_F}),$$

where x runs through all G-maps $x\colon G/H \to X$. We define natural maps

$$\begin{aligned} e_1^G(X)\colon U^G(X) &\to H_0^{\mathrm{Or}(G)}(X;\underline{R_{\mathbb{Q}}}); & (3.3)\\ e_2^G(X)\colon H_0^{\mathrm{Or}(G)}(X;\underline{R_{\mathbb{Q}}}) &\to H_0^{\mathrm{Or}(G)}(X;\underline{R_{\mathbb{R}}}); & (3.4)\\ e_3^G(X)\colon H_0^{\mathrm{Or}(G)}(X;\underline{R_{\mathbb{R}}}) &\to KO_0^G(X) & (3.5)\end{aligned}$$

as follows. The map $\beta_{\mathbb{Q}}^G(X)^{-1}\circ e_1^G(X)$ sends the basis element $[x\colon G/H \to X]$ to the image of the trivial representation $[\mathbb{Q}] \in R_{\mathbb{Q}}(H)$ under the canonical map associated to x

$$R_{\mathbb{Q}}(H) \to \underrightarrow{\lim}_{\Pi_0(G,X)} Q^*R_{\mathbb{Q}}(H).$$

The map $e_2^G(X)$ is induced by the change of fields homomorphisms $R_{\mathbb{Q}}(H) \to R_{\mathbb{R}}(H)$ for $H \subseteq G$ finite. The map $e_3^G(X)\circ \beta^G(X)$ is the colimit over the system of maps

$$R_{\mathbb{R}}(H) = KO_0^H(\{*\}) \xrightarrow{(\alpha_H)_*} KO_0^G(G/H) \xrightarrow{KO_0^G(x)} KO_0^G(X)$$

for the various G-maps $x\colon G/H \to X$, where $\alpha_H\colon H \to G$ is the inclusion.

Theorem 3.6 *Let X be a proper G-CW-complex. Then*

(a) *The map $e^G(X)$ defined in (0.2) factorizes as*

$$e^G(X)\colon U^G(X) \xrightarrow{e_1^G(X)} H_0^{\mathrm{Or}(G)}(X;\underline{R_{\mathbb{Q}}}) \xrightarrow{e_2^G(X)} H_0^{\mathrm{Or}(G)}(X;\underline{R_{\mathbb{R}}}) \xrightarrow{e_3^G(X)} KO_0^G(X);$$

(b) *The map*

$$\mathbb{Q}\otimes_{\mathbb{Z}} e_2^G(X)\colon \mathbb{Q}\otimes_{\mathbb{Z}} H_0^{\mathrm{Or}(G)}(X;\underline{R_{\mathbb{Q}}}) \to \mathbb{Q}\otimes_{\mathbb{Z}} H_0^{\mathrm{Or}(G)}(X;\underline{R_{\mathbb{R}}})$$

is injective;

(c) *For each $n \in \mathbb{Z}$ there is an isomorphism, natural in X,*

$$\mathrm{chern}_n^G(X)\colon \bigoplus_{\substack{p,q\in\mathbb{Z},\\ p\geq 0, p+q=n}} \mathbb{Q}\otimes_{\mathbb{Z}} H_p^{\mathrm{Or}(G)}(X;\underline{KO_q^G}) \xrightarrow{\cong} \mathbb{Q}\otimes_{\mathbb{Z}} KO_n^G(X),$$

where $\underline{KO_q^G}$ is the covariant functor from $\mathsf{Or}(G;\mathcal{F}\mathsf{in})$ to $\mathbb{Z}$-Mod sending G/H to $KO_q^G(G/H)$. The map

$$\mathrm{id}_{\mathbb{Q}}\otimes_{\mathbb{Z}} e_3^G(X)\colon \mathbb{Q}\otimes_{\mathbb{Z}} H_0^{\mathrm{Or}(G)}(M;\underline{R_{\mathbb{R}}}) \to \mathbb{Q}\otimes_{\mathbb{Z}} KO_0^G(M)$$

is the restriction of $\mathrm{chern}_n^G(X)$ to the summand for $p = q = 0$ and is hence injective;

(d) *Suppose* $\dim(X) \leq 4$ *and one of the following conditions is satisfied:*

   (a) *either* $\dim(X^H) \leq 2$ *for* $H \subseteq G$, $H \neq \{1\}$, *or*

   (b) *no subgroup* H *of* G *has irreducible representations of complex or quaternionic type.*

   Then

$$e_3^G(X)\colon\ H_0^{\mathrm{Or}(G)}(M; \underline{R_{\mathbb{R}}}) \to KO_0^G(M)$$

   is injective.

Proof (a) follows directly from the definitions.

(b) will be proved later.

(c) An equivariant Chern character chern_*^G for equivariant homology theories such as equivariant K-homology is constructed in [18, Theorem 0.1] (see also [19, Theorem 0.7]). The restriction of $\mathrm{chern}_0^G(M)$ to $H_0^{\mathrm{Or}(G)}(X; \underline{R_{\mathbb{R}}})$ is just $e_3^G(X)$ under the identification $\underline{R_{\mathbb{R}}} = \underline{KO_0^G}$.

(d) Consider the equivariant Atiyah-Hirzebruch spectral sequence which converges to $KO_{p+q}^G(X)$ (see for instance [5, Theorem 4.7 (1)]). Its E^2-term is $E_{p,q}^2 = H_p^{\mathrm{Or}(G)}(X; \underline{KO_q^G})$. The abelian group $KO_q^G(G/H)$ is isomorphic to the real topological K-theory $KO_q(\mathbb{R}H)$ of the real C^*-algebra $\mathbb{R}H$. The real C^*-algebra $\mathbb{R}H$ splits as a product of matrix algebras over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, with as many summands of a given type as there are irreducible real representations of H of that type (see [28, §13.2]). By Morita invariance of topological real K-theory, we conclude that $KO_q^G(G/H)$ is a direct sum of copies of the non-equivariant K-homologies $KO_q(*) = KO_q(\mathbb{R})$, $KU_q(*) = KO_q(\mathbb{C})$, and $KSp_q(*) = KO_q(\mathbb{H})$. In particular, we conclude that $KO_q^G(G/H) = 0$ for $q \equiv -1 \pmod 8$. As a consequence, $\underline{KO_{-1}^G} = 0$ and $E_{p,-1}^2 = H_p^{\mathrm{Or}(G)}(X; \underline{KO_{-1}^G}) = 0$. If no subgroup H of G has irreducible representations of complex or quaternionic type, then similarly $KO_q^G(G/H) = 0$ for all subgroups H of G and $q = -2, -3$, and $E_{p,-2}^2 = 0$, $E_{p,-3}^2 = 0$, as well.

Let $X^{>1} \subseteq X$ be the subset $\{x \in X \mid G_x \neq 1\}$. There is a short exact sequence

$$H_p^{\mathrm{Or}(G)}(X^{>1}; \underline{KO_q^G}) \to H_p^{\mathrm{Or}(G)}(X; \underline{KO_q^G}) \to H_p^{\mathrm{Or}(G)}(X, X^{>1}; \underline{KO_q^G}).$$

Since the isotropy group of any point in $X - X^{>1}$ is trivial, we get an isomorphism

$$H_p^{\mathrm{Or}(G)}(X, X^{>1}; \underline{KO_q^G}) = H_p(C_*(X, X^{>1}) \otimes_{\mathbb{Z}G} KO_q^G(G/1))$$

Since $KO_q^G(G/1)) = KO_q(\mathbb{R})$ vanishes for $q \in \{-2,-3\}$, we get for $p \in \mathbb{Z}$ and $q \in \{-2,-3\}$

$$H_p^{\mathrm{Or}(G)}(X, X^{>1}; \underline{KO_q^G}) = 0.$$

So if $\dim(X^H) \leq 2$ for $H \subseteq G$, $H \neq \{1\}$, we have $\dim(X^{>1}) \leq 2$. This implies for $p \geq 3$ and $q \in \mathbb{Z}$

$$H_p^{\mathrm{Or}(G)}(X^{>1}; \underline{KO_q^G}) = 0.$$

We conclude that for $p \geq 3$ and $q \in \{-2,-3\}$

$$E^2_{p,q} = H_p^{\mathrm{Or}(G)}(X; \underline{KO_q^G}) = 0,$$

just as in the previous case.

Now there are no non-trivial differentials out of $E^r_{0,0}$, and since $\dim X \leq 4$, $d^r_{r,1-r}\colon E^r_{r,1-r} \to E^r_{0,0}$ must be zero for $r > 4$. But we have just seen that $E^2_{2,-1} = 0$, $E^2_{3,-2} = 0$, and $E^2_{4,-3} = 0$. Hence each differential $d^r_{p,q}$ which has $E^r_{0,0}$ as source or target is trivial. Hence the edge homomorphism restricted to $E^2_{0,0}$ is injective. But this map is $e_3^G(X)$. □

Remark 3.7 We conclude from Theorem 0.3 and Theorem 3.6 that $\mathrm{Eul}^G(M)$ carries rationally the same information as the image of the equivariant Euler characteristic $\chi^G(X)$ under the map $e_1^G(X)\colon U^G(X) \to H_0^{\mathrm{Or}(G)}(M; \underline{R_{\mathbb{R}}})$. Moreover, in contrast to the class of the signature operator, the class $\mathrm{Eul}^G(M) \in KO_0^G(M)$ of the Euler operator does not carry "higher" information because its preimage under the equivariant Chern character is concentrated in the summand corresponding to $p = q = 0$.

Next we recall the definition of the Hattori-Stallings rank of a finitely generated projective RG-module P, for some commutative ring R and a group G. Let $R[\mathrm{con}(G)]$ be the R-module with the set of conjugacy classes $\mathrm{con}(G)$ of elements in G as basis. Define the universal RG-trace

$$\mathrm{tr}^u_{RG}\colon RG \to R[\mathrm{con}(G)], \quad \sum_{g \in G} r_g \cdot g \;\mapsto\; \sum_{g \in G} r_g \cdot (g).$$

Choose a matrix $A = (a_{i,j}) \in M_n(RG)$ such that $A^2 = A$ and the image of the map $r_A\colon RG^n \to RG^n$ sending x to xA is RG-isomorphic to P. Define the *Hattori-Stallings rank*

$$\mathrm{HS}_{RG}(P) \quad := \textstyle\sum_{i=1}^n \mathrm{tr}^u_{RG}(a_{i,i}) \quad \in R[\mathrm{con}(G)]. \tag{3.8}$$

Let $\alpha\colon H_1 \to H_2$ be a group homomorphism. It induces a map

$$\mathrm{con}(H_1) \to \mathrm{con}(H_2), \quad (h) \mapsto (\alpha(h))$$

and thus an R-linear map $\alpha_*\colon R[\mathrm{con}(H_1)] \to R[\mathrm{con}(H_2)]$. If $\alpha_* P$ is the $R[H_2]$-module obtained by induction from the finitely generated projective $R[H_1]$-module P, then

$$\mathrm{HS}_{RH_2}(\alpha_* P) \quad = \quad \alpha_*(\mathrm{HS}_{RH_1}(P)). \tag{3.9}$$

Next we compute $F \otimes_{\mathbb{Z}} \left(\varinjlim_{\Pi_0(G,X)} Q^* R_F(H)\right)$ for $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. Two elements g_1 and g_2 of a group G are called $\mathbb{Q}$-*conjugate* if and only if the cyclic subgroup $\langle g_1 \rangle$ generated by g_1 and the cyclic subgroup $\langle g_2 \rangle$ generated by g_2 are conjugate in G. Two elements g_1 and g_2 of a group G are called $\mathbb{R}$-*conjugate* if and only if g_1 and g_2 or g_1^{-1} and g_2 are conjugate in G. Two elements g_1 and g_2 of a group G are called $\mathbb{C}$-*conjugate* if and only if they are conjugate in G (in the usual sense). We denote by $(g)_F$ the set of elements of G which are F-conjugate to g. Denote by $\mathrm{con}_F(G)$ the set of F-conjugacy classes $(g)_F$ of elements of finite order $g \in G$. Let $\mathrm{class}_F(G)$ be the F-vector space generated by the set $\mathrm{con}_F(G)$. This is the same as the F-vector space of functions $\mathrm{con}_F(G) \to \mathbb{R}$ whose support is finite.

Let $\mathrm{pr}_F\colon \mathrm{con}(H) \to \mathrm{con}_F(H)$ be the canonical epimorphism for a finite group H. It extends to an F-linear epimorphism $F[\mathrm{pr}_F]\colon F[\mathrm{con}(H)] \to \mathrm{class}_F(H)$. Define for a finite-dimensional H-representation V over F for a finite group H

$$\mathrm{HS}_{F,H}(V) \quad := \quad F[\mathrm{pr}_F](\mathrm{HS}_{FH}(V)) \in \mathrm{class}_F(H). \tag{3.10}$$

Let $\alpha\colon H_1 \to H_2$ be a homomorphism of finite groups. It induces a map $\mathrm{con}_F(H_1) \to \mathrm{con}_F(H_2), (h)_F \mapsto (\alpha(h))_F$ and thus an F-linear map

$$\alpha_*\colon \ \mathrm{class}_F(H_1) \to \mathrm{class}_F(H_2).$$

If V is a finite-dimensional H_1-representation over F, we conclude from (3.9):

$$\mathrm{HS}_{F,H_2}(\alpha_* V) \quad = \quad \alpha_*(\mathrm{HS}_{F,H_1}(V)). \tag{3.11}$$

Lemma 3.12 *Let H be a finite group and $F = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. Then the Hattori-Stallings rank defines an isomorphism*

$$\mathrm{HS}_{F,H}\colon \ F \otimes_{\mathbb{Z}} R_F(H) \xrightarrow{\cong} \mathrm{class}_F(H)$$

which is natural with respect to induction with respect to group homomorphism $\alpha\colon H_1 \to H_2$ of finite groups.

Proof One easily checks for a finite-dimensional H-representation over F of a finite group H

$$\mathrm{HS}_{FH}(V) \ = \sum_{(h) \in \mathrm{con}(H)} \frac{|(h)|}{|H|} \cdot \mathrm{tr}_F(l_h).$$

This explains the relation between the Hattori-Stallings rank and the character of a representation — they contain equivalent information. (We prefer the Hattori-Stallings rank because it behaves better under induction.) We conclude from [28, page 96] that $\mathrm{HS}_{FH}(V)$ as a function $\mathrm{con}(H) \to F$ is constant on the F-conjugacy classes of elements in H and that $\mathrm{HS}_{F,H}$ is bijective. Naturality follows from (3.11). $\square$

Let $\underline{\mathrm{class}_F}$ be the covariant functor $\mathsf{Or}(G;\mathcal{F}\mathsf{in}) \to F\text{-}\mathsf{Mod}$ which sends an object G/H to $\mathrm{class}_F(H)$. The isomorphisms $\mathrm{HS}_{F,H}\colon F \otimes_{\mathbb{Z}} R_F(H) \xrightarrow{\cong} \mathrm{class}_F(H)$ yield a natural equivalence of covariant functors from $\pi_0(G,X)$ to $F\text{-}\mathsf{Mod}$. Thus we obtain an isomorphism

$$\begin{aligned}\mathrm{HS}_F^G(X)\colon\ & F \otimes_{\mathbb{Z}} \varinjlim_{\Pi_0(G,X)} Q^*\underline{R_F} \\ & \xrightarrow{\cong} \varinjlim_{\Pi_0(G,X)} Q^* F \otimes_{\mathbb{Z}} \underline{R_F} \xrightarrow{\cong} \varinjlim_{\Pi_0(G,X)} Q^*\underline{\mathrm{class}_F}. \quad (3.13)\end{aligned}$$

Let $f\colon S_0 \to S_1$ be a map of sets. It extends to an F-linear map $F[f]\colon F[S_0] \to F[S_1]$. Suppose that the preimage of any element in S_1 is finite. Then we obtain an F-linear map

$$f^*\colon F[S_1] \to F[S_0], \qquad s_1 \mapsto \sum_{s_0 \in f^{-1}(s_1)} s_0. \qquad (3.14)$$

If we view elements in $F[S_i]$ as functions $S_i \to F$, then f^* is given by composing with f. One easily checks that $F[f] \circ f^*$ is bijective and that for a second map $g\colon S_1 \to S_2$, for which the preimages of any element in S_2 is finite, we have $f^* \circ g^* = (g \circ f)^*$.

Now we can finish the proof of Theorem 3.6 by explaining how assertion (b) is proved.

Proof Let H be a finite group. Let $p_H\colon \mathrm{con}_{\mathbb{R}}(H) \to \mathrm{con}_{\mathbb{Q}}(H)$ be the projection. If V is a finite-dimensional H-representation over $\mathbb{Q}$, then $\mathbb{R} \otimes_{\mathbb{Q}} V$ is a finite-dimensional H-representation over $\mathbb{R}$ and $\mathrm{HS}^{\mathbb{R}H}(\mathbb{R} \otimes_{\mathbb{Q}} V)$ is the image of $\mathrm{HS}^{\mathbb{Q}H}(V)$ under the obvious map $\mathbb{Q}[\mathrm{con}(H)] \to \mathbb{R}[\mathrm{con}(H)]$. Recall that $\mathrm{HS}_{F,H}(V)$ is the image of $\mathrm{HS}^{FH}(V)$ under $F[\mathrm{pr}_F]$ for $\mathrm{pr}_F\colon \mathrm{con}(H) \to \mathrm{con}_F(H)$ the canonical projection; $\mathrm{HS}^{FH}(V)$ is constant on the F-conjugacy classes of elements in H. This implies that the following diagram commutes

$$\begin{array}{ccc} \mathbb{R} \otimes_{\mathbb{Z}} R_{\mathbb{Q}}(H) & \xrightarrow[\cong]{\mathrm{id}_{\mathbb{R}} \otimes_{\mathbb{Z}} \mathrm{HS}_{\mathbb{Q},H}} & \mathbb{R} \otimes_{\mathbb{Q}} \mathrm{class}_{\mathbb{Q}}(H) \\ {\scriptstyle \mathrm{ind}(H)}\big\downarrow & & {\scriptstyle \mathrm{ind}(H)}\big\downarrow \\ \mathbb{R} \otimes_{\mathbb{Z}} R_{\mathbb{R}}(H) & \xrightarrow[\cong]{\mathrm{HS}_{\mathbb{R},H}} & \mathrm{class}_{\mathbb{R}}(H), \end{array}$$

where the left vertical arrow comes from inducing a $\mathbb{Q}$-representation to a $\mathbb{R}$-representation and the right vertical arrow is

$$\mathbb{R}\otimes_{\mathbb{Q}}\operatorname{class}_{\mathbb{Q}}(H)=\mathbb{R}\otimes_{\mathbb{Q}}\mathbb{Q}[\operatorname{con}_{\mathbb{Q}}(H)] \xrightarrow{\operatorname{id}_{\mathbb{R}}\otimes_{\mathbb{Q}}(p_H)^*} \mathbb{R}\otimes_{\mathbb{Q}}\mathbb{Q}[\operatorname{con}_{\mathbb{R}}(H)]=\operatorname{class}_{\mathbb{R}}(H).$$

Let

$$q(H)\colon\ \operatorname{class}_{\mathbb{R}}(H)\to\mathbb{R}\otimes_{\mathbb{Q}}\operatorname{class}_{\mathbb{Q}}(H)$$

be the map

$$\mathbb{R}[p_H]\colon\ \mathbb{R}[\operatorname{con}_{\mathbb{R}}(H)]\to\mathbb{R}[\operatorname{con}_{\mathbb{Q}}(H)]=\mathbb{R}\otimes_{\mathbb{Q}}\mathbb{Q}[\operatorname{con}_{\mathbb{Q}}(H)]$$

The $q(H)\circ\operatorname{ind}(H)$ is bijective.

We get natural transformations of functors from $\mathsf{Or}(G;\mathcal{F}\mathsf{in})$ to $\mathbb{R}$-Mod:

$$\begin{array}{rcl}\operatorname{ind}\colon\ \underline{\mathbb{R}\otimes_{\mathbb{Z}}R_{\mathbb{Q}}} & \to & \underline{\mathbb{R}\otimes_{\mathbb{Z}}R_{\mathbb{R}}},\\ \operatorname{ind}\colon\ \underline{\mathbb{R}\otimes_{\mathbb{Q}}\operatorname{class}_{\mathbb{Q}}} & \to & \underline{\operatorname{class}_{\mathbb{R}}},\\ q\colon\ \underline{\operatorname{class}_{\mathbb{R}}} & \to & \underline{\mathbb{R}\otimes_{\mathbb{Q}}\operatorname{class}_{\mathbb{Q}}}.\end{array}$$

Also $q\circ\operatorname{ind}$ is a natural equivalence. This implies that ind induces a split injection on the colimits

$$\varinjlim_{\Pi_0(G;X)}Q^*\operatorname{ind}\colon\ \varinjlim_{\Pi_0(G;X)}Q^*\underline{\mathbb{R}\otimes_{\mathbb{Q}}\operatorname{class}_{\mathbb{Q}}}\to\varinjlim_{\Pi_0(G;X)}Q^*\underline{\operatorname{class}_{\mathbb{R}}}.$$

Since the following diagram commutes and has isomorphisms as vertical arrows

$$\begin{tikzcd}
\mathbb{R}\otimes_{\mathbb{Z}}H_0^{\mathsf{Or}(G)}(X;\underline{R_{\mathbb{Q}}}) \arrow[r, "\mathbb{R}\otimes_{\mathbb{Z}}e_2^G(X)"] & \mathbb{R}\otimes_{\mathbb{Z}}H_0^{\mathsf{Or}(G)}(X;\underline{R_{\mathbb{R}}}) \\
\varinjlim_{\Pi_0(G;X)}Q^*\underline{\mathbb{R}\otimes_{\mathbb{Z}}R_{\mathbb{Q}}} \arrow[u, "\mathbb{R}\otimes_{\mathbb{Z}}\beta^G_{\mathbb{Q}}(X)", "\cong"'] \arrow[r, "\varinjlim_{\Pi_0(G;X)}Q^*\operatorname{ind}"] \arrow[d, "\mathbb{R}\otimes_{\mathbb{Q}}\operatorname{HS}^G_{\mathbb{Q}}(X)"', "\cong"] & \varinjlim_{\Pi_0(G;X)}Q^*\underline{\mathbb{R}\otimes_{\mathbb{Z}}R_{\mathbb{R}}} \arrow[u, "\mathbb{R}\otimes_{\mathbb{Z}}\beta^G_{\mathbb{R}}(X)", "\cong"'] \arrow[d, "\mathbb{R}\otimes_{\mathbb{Z}}\operatorname{HS}^G_{\mathbb{R}}(X)"', "\cong"] \\
\varinjlim_{\Pi_0(G;X)}Q^*\underline{\mathbb{R}\otimes_{\mathbb{Q}}\operatorname{class}_{\mathbb{Q}}} \arrow[r, "\varinjlim_{\Pi_0(G;X)}Q^*\operatorname{ind}"] & \varinjlim_{\Pi_0(G;X)}Q^*\underline{\operatorname{class}_{\mathbb{R}}}\,,
\end{tikzcd}$$

the top horizontal arrow is split injective. Hence $e_2^G(X)$ is rationally split injective. This finishes the proof of Theorem 3.6 (b). $\square$

Notation 3.15 Consider $g\in G$ of finite order. Denote by $\langle g\rangle$ the finite cyclic subgroup generated by g. Let $y\colon G/\langle g\rangle\to X$ be a G-map. Let F be one of the fields $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$. Define

$$\begin{array}{rcl}C_{\mathbb{Q}}(g) & = & \{g'\in G,(g')^{-1}gg'\in\langle g\rangle\}\};\\ C_{\mathbb{R}}(g) & = & \{g'\in G,(g')^{-1}gg'\in\{g,g^{-1}\}\};\\ C_{\mathbb{C}}(g) & = & \{g'\in G,(g')^{-1}gg'=g\}.\end{array}$$

Since $C_F(g)$ is a subgroup of the normalizer $N\langle g\rangle$ of $\langle g\rangle$ in G and contains $\langle g\rangle$, we can define a subgroup $Z_F(g) \subseteq W\langle g\rangle$ by

$$Z_F(g) \;:=\; C_F(g)/\langle g\rangle.$$

Let $Z_F(g)_y$ be the intersection of $W\langle g\rangle_y$ (see Notation 2.1) with $Z_F(g)$, or, equivalently, the subgroup of $Z_F(g)$ represented by elements $g \in C_F(g)$ for which $g \cdot y(1\langle g\rangle)$ and $y(1\langle g\rangle)$ lie in the same component of $X^{\langle g\rangle}$.

It is useful to interpret the group $C_F(g)$ as follows. Let G act on the set $G_f := \{g \mid g \in G, |g| < \infty\}$ by conjugation. Then the projection $G_f \to \mathrm{con}_{\mathbb{C}}(G)$ induces a bijection $G\backslash G_f \xrightarrow{\cong} \mathrm{con}_{\mathbb{C}}(G)$ and the isotropy group of $g \in G_f$ is $C_{\mathbb{C}}(g)$. Let $G_f/\{\pm 1\}$ be the quotient of G_f under the $\{\pm 1\}$-action given by $g \mapsto g^{-1}$. The conjugation action of G on G_f induces an action on $G_f/\{\pm 1\}$. The projection $G \to \mathrm{con}_{\mathbb{R}}(G)$ induces a bijection $G\backslash(G/\{\pm 1\}) \xrightarrow{\cong} \mathrm{con}_{\mathbb{R}}(G)$ and the isotropy group of $g \cdot \{\pm 1\} \in G_f/\{\pm 1\}$ is $C_{\mathbb{R}}(g)$. The set $\mathrm{con}_{\mathbb{Q}}(G)$ is the same as the set $\{(L) \in \mathrm{consub}(G) \mid L \text{ finite cyclic}\}$. For $g \in G_f$ the group $C_{\mathbb{Q}}(g)$ agrees with $N\langle g\rangle$ and is the isotropy group of $\langle g\rangle$ in $\{L \subseteq G \mid L \text{ finite cyclic}\}$ under the conjugation action of G. The quotient of $\{L \subseteq G \mid L \text{ finite cyclic}\}$ under the conjugation action of G is by definition $\mathrm{con}_{\mathbb{Q}}(G) = \{(L) \in \mathrm{consub}(G) \mid L \text{ finite cyclic}\}$.

Consider $(g)_F \in \mathrm{con}_F(G)$. For the sequel we fix a representative $g \in (g)_F$. Consider an object of $\Pi_0(G,X)$ of the special form $y\colon G/\langle g\rangle \to X$. Let $x\colon G/H \to X$ be any object of $\Pi_0(G,X)$. Recall that $W\langle g\rangle_y$ and thus the subgroup $Z_F(g)_y$ act on $\mathrm{mor}(y,x)$. Define

$$\alpha_F(y,x)\colon\; Z_F(g)_y\backslash \mathrm{mor}(y,x) \to \mathrm{con}_F(H)$$

by sending sending $Z_F(g)_y \cdot \sigma$ for a morphism $\sigma\colon y \to x$, which given by a G-map $\sigma\colon G/\langle g\rangle \to G/H$, to $(\sigma(1\langle g\rangle)^{-1} g \sigma(1\langle g\rangle))_F$. We obtain a map of sets

$$\begin{aligned} \alpha_F(x) = \coprod_{(g)_F \in \mathrm{con}(G)_{\mathbb{R}}} \; \coprod_{\substack{Z_F(g)\cdot C \in \\ Z_F(g)\backslash \pi_0(X^{\langle g\rangle})}} a(y(C),x)\colon & \\ \coprod_{(g)_F \in \mathrm{con}(G)_F} \; \coprod_{\substack{Z_F(g)\cdot C \in \\ Z_F(g)\backslash \pi_0(X^{\langle g\rangle})}} Z_F(g)_{y(C)}\backslash \mathrm{mor}(y(C),x) \xrightarrow{\cong} \mathrm{con}_F(H), & \quad (3.16) \end{aligned}$$

where we fix for each $Z_F(g) \cdot C \in Z_F(g)\backslash \pi_0(X^{\langle g\rangle})$ a representative $C \in \pi_0(X^{\langle g\rangle})$ and $y(C)$ is a fixed morphism $y(C)\colon G/\langle g\rangle \to X$ such that $X^{\langle g\rangle}(y) = C$ in $\pi_0(X^{\langle g\rangle})$. The map $\alpha_F(x)$ is bijective by the following argument.

Consider $(h)_F \in \mathrm{con}_F(H)$. Let $g \in G$ be the representative of the class $(g)_F$ for which $(g)_F = (h)_F$ holds in $\mathrm{con}_F(G)$. Choose $g_0 \in G$ with $g_0^{-1}gg_0 \in H$ and $(g_0^{-1}gg_0)_F = (h)_F$ in $\mathrm{con}_F(H)$. We get a G-map $R_{g_0}\colon G/\langle g\rangle \to G/H$ by mapping $g'\langle g\rangle$ to $g'g_0H$. Let $y = y(C)\colon G/\langle g\rangle \to X$ be the object chosen above for the fixed representative C of $Z_F(g)\cdot X^{\langle g\rangle}(x\circ R_{g_0}) \in Z_F(g)_F\backslash\pi_0(X^{\langle g\rangle})$. Choose $g_1 \in Z_F(g)$ such that the G-map $R_{g_1}\colon G/\langle g\rangle \to G/\langle g\rangle$ sending $g'\langle g\rangle$ to $g'g_1\langle g\rangle$ defines a morphism $R_{g_1}\colon y \to x\circ R_{g_0}$ in $\Pi_0(G,X)$. Then $\sigma := R_{g_0}\circ R_{g_1}\colon y \to x$ is a morphism such that

$$(\sigma(1\langle g\rangle))^{-1}g\sigma(1\langle g\rangle)_F = (h)_F$$

holds in $\mathrm{con}_F(H)$. This shows that $\alpha(x)$ is surjective.

Consider for $i = 0,1$ elements $(g_i) \in \mathrm{con}_F(G)$, $Z_F(g_i)\cdot C_i \in Z_F(g_i)\backslash\pi_0(X^{\langle g_i\rangle})$ and $Z_F(g_i)_{y_i}\cdot\sigma_i \in Z_F(g_i)_{y_i}\backslash\mathrm{mor}(y_i,x)$ for $y_i = y(C_i)$ such that

$$(\sigma_0(1\langle g_0\rangle))^{-1}g_0\sigma_0(1\langle g_0\rangle))_F = (\sigma_1(1\langle g_1\rangle))^{-1}g_1\sigma_1(1\langle g_1\rangle))_F$$

holds in $\mathrm{con}_F(H)$. So we get two elements in the source of $\alpha_F(x)$ which are mapped to the same element under $\alpha_F(x)$. We have to show that these elements in the source agree. Choose $g_i' \in G$ such that σ_i is given by sending $g''\langle g_i\rangle$ to $g''g_i'H$. Then $((g_0')^{-1}g_0g_0')_F$ and $((g_1')^{-1}g_1g_1')_F$ agree in $\mathrm{con}_F(H)$. This implies $(g_0)_F = (g_1)_F$ in $\mathrm{con}(G)_F$ and hence $g_0 = g_1$. In the sequel we write $g = g_0 = g_1$. Since $((g_0')^{-1}gg_0')_F$ and $((g_1')^{-1}gg_1')_F$ agree in $\mathrm{con}_F(H)$, there exists $h \in H$ with

$$h^{-1}(g_0')^{-1}gg_0'h \begin{cases} \in & \langle (g_1')^{-1}gg_1'\rangle & \text{if } F = \mathbb{Q};\\ \in & \{(g_1')^{-1}gg_1', (g_1')^{-1}g^{-1}g_1'\} & \text{if } F = \mathbb{R};\\ = & (g_1')^{-1}gg_1' & \text{if } F = \mathbb{C}.\end{cases}$$

We can assume without loss of generality that $h = 1$, otherwise replace g_0' by $g_0'h$. Put $g_2 := g_0'(g_1')^{-1}$. Then g_2 is an element in $Z_F(g)$. Let $\sigma_2\colon G/\langle g\rangle \to G/\langle g\rangle$ be the G-map which sends $g''\langle g\rangle$ to $g''g_2\langle g\rangle$. We get the equality of G-maps $\sigma_0 = \sigma_1\circ\sigma_2$. Since σ_i is a morphism $y_i \to x$ for $i = 0,1$, we conclude $g_i'\cdot x(1H) \in X^{\langle g\rangle}(y_i)$ for $i = 0,1$. This implies that $g_2\cdot X^{\langle g\rangle}(y_1) = X^{\langle g\rangle}(y_0)$ in $\pi_0(X^{\langle g\rangle})$. This shows $Z_F(g)\cdot X^{\langle g\rangle}(y_0) = Z_F(g)\cdot X^{\langle g\rangle}(y_1)$ in $Z_F(g)\backslash\pi_0(X^{\langle g\rangle})$ and hence $y_0 = y_1$. Write in the sequel $y = y_0 = y_1$. The G-map σ_2 defines a morphism $\sigma_2\colon y \to y$. We obtain an equality $\sigma_0 = \sigma_1\circ\sigma_2$ of morphisms $y \to x$. We conclude $Z_F(g)_y\cdot\sigma_0 = Z_F(g)_y\cdot\sigma_1$ in $Z_F(g)_y\backslash\pi_0(X^{\langle g\rangle})$. This finishes the proof that $\alpha(x)$ is bijective.

Let $\underline{\mathrm{con}_F}$ be the covariant functor from $\mathsf{Or}(G;\mathcal{F}\mathsf{in})$ to the category of finite sets which sends an object G/H to $\mathrm{con}_F(H)$. The map $\alpha_F(x)$ is natural in

$x\colon G/H \to X$, in other words, we get a natural equivalence of functors from $\Pi_0(G,X)$ to the category of finite sets. We obtain a bijection of sets

$$\alpha_F\colon \varinjlim_{x\colon G/H\to X\in\Pi_0(G,X)} \coprod_{(g)_F\in\operatorname{con}_F(G)} \coprod_{\substack{Z_F\langle g\rangle C\in\\ Z_F\langle g\rangle\backslash\pi_0(X^{\langle g\rangle})}} Z_F\langle g\rangle_y\backslash\operatorname{mor}(y,x) \xrightarrow{\cong} \varinjlim_{\Pi_0(G,X)} Q^*\underline{\operatorname{con}_F}.$$

One easily checks that $\varinjlim_{x\colon G/H\to X\in\Pi_0(G,X)} Z_F\langle g\rangle_y\backslash\operatorname{mor}(y,x)$ consists of one element, namely, the one represented by $Z_F\langle g\rangle_y\cdot\mathrm{id}_y\in Z_F\langle g\rangle_y\backslash\operatorname{mor}(y,y)$. Thus we obtain a bijection

$$\alpha_F^G(X)\colon \coprod_{(g)_F\in\operatorname{con}_F(G)} Z_F\langle g\rangle\backslash\pi_0(X^{\langle g\rangle}) \xrightarrow{\cong} \varinjlim_{\Pi_0(G,X)} Q^*\underline{\operatorname{con}_F},$$

which sends an element $Z_F\langle g\rangle\cdot C$ in $Z_F\langle g\rangle\backslash\pi_0(X^{\langle g\rangle})$ to the class in the colimit represented by $Z_F\langle g\rangle_y\cdot\mathrm{id}_y$ in $Z_F\langle g\rangle_y\backslash\operatorname{mor}(y,y)$ for any object $y\colon G/\langle g\rangle\to X$ for which $Z_F(g)\cdot X^{\langle g\rangle}(y) = Z_F(g)\cdot C$ holds in $Z_F(g)\backslash\pi_0(X^{\langle g\rangle})$. It yields an isomorphism of F-vector spaces denoted in the same way

$$\alpha_F^G(X)\colon \bigoplus_{(g)_F\in\operatorname{con}_F(G)} \bigoplus_{Z_F(g)\backslash\pi_0(X^{\langle g\rangle})} F \xrightarrow{\cong} \varinjlim_{\Pi_0(G,X)} Q^*\underline{\operatorname{class}_F}. \tag{3.17}$$

Let us consider in particular the case $F=\mathbb{Q}$. Recall that $\operatorname{consub}(G)$ is the set of conjugacy classes (H) of subgroups of G. Then $\operatorname{con}_{\mathbb{Q}}(G)$ is the same as the set $\{(L)\in\operatorname{consub}(G)\mid L \text{ finite cyclic}\}$ and $Z_{\mathbb{Q}}(g)$ agrees with $W\langle g\rangle$. Thus (3.17) becomes an isomorphism of $\mathbb{Q}$-vector spaces

$$\alpha_{\mathbb{Q}}^G(X)\colon \bigoplus_{\substack{(L)\in\operatorname{consub}(G)\\ L \text{ finite cyclic}}} \bigoplus_{WL\backslash\pi_0(X^L)} \mathbb{Q} \xrightarrow{\cong} \varinjlim_{\Pi_0(G,X)} Q^*\underline{\operatorname{class}_{\mathbb{Q}}}. \tag{3.18}$$

Denote by

$$\operatorname{pr}\colon \bigoplus_{(H)\in\operatorname{consub}(G)} \bigoplus_{WH\backslash\pi_0(X^H)} \mathbb{Q} \to \bigoplus_{\substack{(L)\in\operatorname{consub}(G)\\ L \text{ finite cyclic}}} \bigoplus_{WL\backslash\pi_0(X^L)} \mathbb{Q}$$

the obvious projection. Let

$$D^G(X)\colon \bigoplus_{\substack{(L)\in\operatorname{consub}(G)\\ L \text{ finite cyclic}}} \bigoplus_{WL\backslash\pi_0(X^L)} \mathbb{Q} \to \bigoplus_{\substack{(L)\in\operatorname{consub}(G)\\ L \text{ finite cyclic}}} \bigoplus_{WL\backslash\pi_0(X^L)} \mathbb{Q} \tag{3.19}$$

be the automorphism $\bigoplus_{\substack{(L)\in\operatorname{consub}(G)\\ L \text{ finite cyclic}}} \frac{|\operatorname{Gen}(L)|}{|L|}\cdot\mathrm{id}$, where $\operatorname{Gen}(L)$ is the set of generators of L.

Recall the isomorphisms $\mathrm{ch}^G(X)$, $\beta^G_{\mathbb{Q}}(X)$, $\mathrm{HS}^G_{\mathbb{Q}}(X)^{-1}$, $\alpha^G_{\mathbb{Q}}(X)$ and $D^G(X)$ from (2.6), (3.2), (3.13) (3.18) and (3.19). We define

$$\gamma^G_{\mathbb{Q}}(X)\colon \bigoplus_{\substack{(L)\in\mathrm{consub}(G)\\ L \text{ finite cyclic}}} \bigoplus_{WL\backslash\pi_0(X^L)} \mathbb{Q} \xrightarrow{\cong} \mathbb{Q}\otimes_{\mathbb{Z}} H_0^{\mathsf{Or}(G)}(X;\underline{R_{\mathbb{Q}}}) \quad (3.20)$$

to be the composition $\gamma^G_{\mathbb{Q}}(X) := \beta^G_{\mathbb{Q}}(X)\circ \mathrm{HS}^G_{\mathbb{Q}}(X)^{-1}\circ \alpha^G_{\mathbb{Q}}(X)\circ D^G(X)$.

Theorem 3.21 *The following diagram commutes*

$$\begin{array}{ccc}
\mathbb{Q}\otimes_{\mathbb{Z}} U^G(X) & \xrightarrow{\mathrm{id}_{\mathbb{Q}}\otimes_{\mathbb{Z}} e_1^G(X)} & \mathbb{Q}\otimes_{\mathbb{Z}} H_0^{\mathsf{Or}(G)}(X;\underline{R_{\mathbb{Q}}}) \\
{\scriptstyle \mathrm{id}_{\mathbb{Q}}\otimes_{\mathbb{Z}}\mathrm{ch}^G(X)}\Big\downarrow\cong & & {\scriptstyle\gamma^G_{\mathbb{Q}}(X)}\Big\uparrow\cong \\
\bigoplus_{(H)\in\mathrm{consub}(G)} \bigoplus_{WH\backslash\pi_0(X^H)}\mathbb{Q} & \xrightarrow{\mathrm{pr}} & \bigoplus_{\substack{(L)\in\mathrm{consub}(G)\\ L \text{ finite cyclic}}} \bigoplus_{WL\backslash\pi_0(X^L)}\mathbb{Q}
\end{array}$$

and has isomorphisms as vertical arrows.

The element $\mathrm{id}_{\mathbb{Q}}\otimes_{\mathbb{Z}} e_1^G(X)(\chi^G(X))$ *agrees with the image under the isomorphism* $\gamma^G_{\mathbb{Q}}$ *of the element*

$$\{\chi^{\mathbb{Q}WL_C}(C) \mid (L)\in\mathrm{consub}(G), L \text{ finite cyclic}, WL\cdot C\in WL\backslash\pi_0(X^L)\}$$

given by the various orbifold Euler characteristics of the WL_C*-*CW*-complexes* C*, where* WL_C *is the isotropy group of* $C\in\pi_0(X^L)$ *under the* WL*-action.*

Proof It suffices to prove the commutativity of the diagram above, then the rest follows from Lemma 2.8.

Recall that $U^G(X)$ is the free abelian group generated by the set of isomorphism classes $[x]$ of objects $x\colon G/H\to X$. Hence it suffices to prove for any G-map $x\colon G/H\to X$

$$\begin{aligned}
&\left(\alpha^G_{\mathbb{Q}}(X)\circ D^G(X)\circ\mathrm{pr}\circ\mathrm{ch}^G(X)\right)([x]) \\
&\qquad = \mathrm{HS}^G(X)\circ\beta^G(X)^{-1}\circ e_1^G(X)([x]). \quad (3.22)
\end{aligned}$$

Given a finite cyclic subgroup $L\subseteq G$ and a component $C\in\pi_0(X^L)$ the element $\left(D^G(X)\circ\mathrm{pr}\circ\mathrm{ch}^G(X)\right)([x])$ has as entry in the summand belonging to (L) and $WL\cdot C\in WL\backslash\pi_0(X^C)$ the number

$$\sum_{\substack{WL_{y(C)}\cdot\sigma\in\\ WL_{y(C)}\backslash\mathrm{mor}(y(C),x)}} \frac{|\mathrm{Gen}(L)|}{|L|\cdot|(WC_{y(C)})_\sigma|},$$

where $y(C)\colon G/L \to X$ is some object in $\Pi_0(G,X)$ with $X^L(y) = C$ in $\pi_0(X^L)$.

Recall the bijection $\alpha_F(x)$ from (3.16). In the case $F = \mathbb{Q}$ it becomes the map

$$\alpha_{\mathbb{Q}}(x)\colon \sum_{\substack{(L)\in\operatorname{consub}(G)\\ L \text{ finite cyclic}}} \coprod_{WL\backslash\pi_0(X^L)} WL_{y(C)}\backslash \operatorname{mor}(y(C),x)$$
$$\xrightarrow{\cong} \operatorname{con}_{\mathbb{Q}}(H) = \{(K) \in \operatorname{consub}(H) \mid L \text{ cyclic}\}$$

which sends $WL\cdot\sigma \in WL\backslash\pi_0(X^L)$ to $(\sigma(1L)^{-1}L\sigma(1L))$. Let

$$u_{[x]} \in \operatorname{class}_{\mathbb{Q}}(H)$$

be the element which assigns to $(K) \in \operatorname{con}_{\mathbb{Q}}(H)$ the number $\frac{|\operatorname{Gen}(L)|}{|L|\cdot|(WC_{y(C)})_\sigma|}$ if $\sigma \in \operatorname{mor}(y(C),x)$ represents the preimage of (K) under the bijection $\alpha_{\mathbb{Q}}(x)$. We conclude that $\left(\alpha^G_{\mathbb{Q}}(X)\circ D^G(X)\circ\operatorname{pr}\circ\operatorname{ch}^G(X)\right)([x])$ is given by the image under the structure map associated to the object $x\colon G/H \to X$

$$\operatorname{class}_{\mathbb{Q}}(H) \to \varinjlim_{\Pi_0(G,X)} Q^*\underline{\operatorname{class}_{\mathbb{Q}}}$$

of the element $u_{[x]} \in \operatorname{class}_{\mathbb{Q}}(H)$ above.

Consider $(K) \in \operatorname{con}_{\mathbb{Q}}(H)$. Let $\sigma \in \operatorname{mor}(y(C),x)$ represent the preimage of (K) under the bijection $\alpha_{\mathbb{Q}}(x)$. Choose g' such that $\sigma\colon G/\langle g\rangle \to G/H$ is given by $g''\langle g\rangle \mapsto g''g'H$. Let N_HK be the normalizer in H and $W_HK := N_HK/K$ be the Weyl group of $K \subseteq H$. Define a bijection

$$f\colon (WL_{y(C)})_\sigma \xrightarrow{\cong} W_H((g')^{-1}Lg'), \quad g''L \mapsto (g')^{-1}g''g'\cdot(g')^{-1}Lg'.$$

The map is well-defined because of

$$(WL_{y(C)})_\sigma = \{g''L \in WL_{y(C)} \mid (g')^{-1}g''g' \in H\}$$

and the following calculation

$$\begin{aligned}\left((g')^{-1}g''g'\right)^{-1}(g')^{-1}Lg'(g')^{-1}g''g' \;&=\; (g')^{-1}(g'')^{-1}g'(g')^{-1}Lg'(g')^{-1}g''g'\\ &=\; (g')^{-1}(g'')^{-1}Lg''g' \;=\; (g')^{-1}Lg'.\end{aligned}$$

One easily checks that f is injective. Consider $h\cdot(g')^{-1}Lg'$ in $W_H((g')^{-1}Lg')$. Define $g_0 = g'h(g')^{-1}$. We have $g_0 \in WL$. Since $h\cdot x(1H) = x(1H)$, we get $g_0L \in WL_y$. Hence g_0L is a preimage of $h\cdot(g')^{-1}Lg'$ under f. Hence f is bijective. This shows

$$|(WL_{y(C)})_\sigma| \;=\; |W_H((g')^{-1}Lg')|.$$

We conclude that $u_{[x]}$ is the element

$$\operatorname{con}_{\mathbb{Q}}(H) = \{(K) \in \operatorname{consub}(H) \mid K \text{ finite cyclic}\} \to \mathbb{Q}, \quad (K) \mapsto \frac{|\operatorname{Gen}(K)|}{|K| \cdot |W_H K|}.$$

Since right multiplication with $|H|^{-1} \cdot \sum_{h \in H} h$ induces an idempotent $\mathbb{Q}H$-linear map $\mathbb{Q}H \to \mathbb{Q}H$ whose image is $\mathbb{Q}$ with the trivial H-action, the element $\mathrm{HS}_{\mathbb{Q},H}([\mathbb{Q}]) \in \operatorname{class}_{\mathbb{Q}}(H)$ is given by

$$\operatorname{con}_{\mathbb{Q}}(H) \to \mathbb{Q}, \quad (K) \mapsto \frac{1}{|H|} \cdot |\{h \in H \mid \langle h \rangle \in (K)\}|.$$

From

$$\begin{aligned}
|\{h \in H \mid \langle h \rangle \in (K)\}| &= \left| \coprod_{K' \subseteq H, K' \in (K)} \operatorname{Gen}(K') \right| \\
&= \left| \{K' \subseteq H, K' \in (K)\} \right| \cdot |\operatorname{Gen}(K)| \\
&= \frac{|H|}{|N_H K|} \cdot |\operatorname{Gen}(K)| \\
&= \frac{|H| \cdot |\operatorname{Gen}(K)|}{|K| \cdot |W_H K|}
\end{aligned}$$

we conclude

$$u_{[x]} = \mathrm{HS}_{\mathbb{Q},H}([\mathbb{Q}]) \quad \in \operatorname{class}_{\mathbb{Q}}(H).$$

Now (3.22) and hence Theorem 3.21 follow. □

4 Examples

In this section we discuss some examples. Recall that we have described the non-equivariant case in the introduction.

4.1 Finite groups and connected non-empty fixed point sets

Next we consider the case where G is a finite group, M is a closed compact G-manifold, and M^H is connected and non-empty for all subgroups $H \subseteq G$. Let $i \colon \{*\} \to M$ be the G-map given by the inclusion of a point into M^G. Since we have assumed that M^G is connected, i is unique up to G-homotopy.

Let $A(G)$ be the Burnside ring of formal differences of finite G-sets (Definition 2.9). We have the following commutative diagram:

$$\begin{tikzcd}
U^G(M) \arrow[d, "e_1^G(M)"'] & U^G(\{*\}) \arrow[l, "U^G(i)"', "\cong"] \arrow[d, "e_1^G(\{*\})"'] & A(G) \arrow[l, "f_1"', "\cong"] \arrow[d, "j_1"'] \\
H_0^{\mathsf{Or}(G)}(M;\underline{R_{\mathbb{Q}}}) \arrow[d, "e_2^G(M)"'] & H_0^{\mathsf{Or}(G)}(\{*\};\underline{R_{\mathbb{Q}}}) \arrow[l, "H_0^{\mathsf{Or}(G)}(i;\underline{R_{\mathbb{Q}}})"', "\cong"] \arrow[d, "e_2^G(\{*\})"'] & R_{\mathbb{Q}}(G) \arrow[l, "f_2"', "\cong"] \arrow[d, "j_2"'] \\
H_0^{\mathsf{Or}(G)}(M;\underline{R_{\mathbb{R}}}) \arrow[d, "e_3^G(M)"'] & H_0^{\mathsf{Or}(G)}(\{*\};\underline{R_{\mathbb{R}}}) \arrow[l, "H_0^{\mathsf{Or}(G)}(i;\underline{R_{\mathbb{R}}})"', "\cong"] \arrow[d, "e_3^G(\{*\})"', "\cong"] & R_{\mathbb{R}}(G) \arrow[l, "f_3"', "\cong"] \arrow[d, "j_3"', "\cong"] \\
KO_0^G(M) & KO_0^G(\{*\}) \arrow[l, "KO_0^G(i)"'] & KO_0^G(\{*\}), \arrow[l, "\mathrm{id}"', "\cong"]
\end{tikzcd}$$

where j_1 sends the class of a G-set S in the Burnside ring $A(G)$ to the class of the rational G-representation $\mathbb{Q}[S]$ and j_2 is the change-of-coefficients homomorphism. The homomorphism $KO_0^G(i)\colon KO_0^G(\{*\}) \to KO_0^G(M)$ is split injective, with a splitting given by the map $KO_0^G(\mathrm{pr})$ induced by $\mathrm{pr}\colon M \to \{*\}$. The map $e_3^G(\{*\})$ is bijective since the category $\Pi_0(G,\{*\})$ has $G \to \{*\}$ as terminal object. This implies that

$$e_3^G(M)\colon H_0^{\mathsf{Or}(G)}(M;\underline{R_{\mathbb{R}}}) \to KO_0^G(M)$$

is split injective. We have already explained in the Section 3 that the map $j_2\colon R_{\mathbb{Q}}(G) \to R_{\mathbb{R}}(G)$ is rationally injective. Since $R_{\mathbb{Q}}(G)$ is a torsion-free finitely generated abelian group, j_2 is injective. Hence

$$e_2^G(M)\colon H_0^{\mathsf{Or}(G)}(M;\underline{R_{\mathbb{Q}}}) \to H_0^{\mathsf{Or}(G)}(M;\underline{R_{\mathbb{R}}})$$

is injective. The upshot of this discussion is, that $e_1^G(\chi^G(M))$ carries (integrally) the same information as $\mathrm{Eul}^G(M)$ because it is sent to $\mathrm{Eul}^G(M)$ by the injective map $e_3^G(M) \circ e_2^G(M)$.

Analyzing the difference between $e_1^G(\chi^G(M))$ and $\chi^G(M)$ is equivalent to analyzing the map $j_1\colon A(G) \to \mathrm{Rep}_{\mathbb{Q}}(G)$, which sends $\chi^G(M)$ to the element given by $\sum_{p\geq 0}(-1)^p \cdot [H_p(M;\mathbb{Q})]$. Recall that $\chi^G(M) \in A(G)$ is given by

$$\chi^G(M) \;=\; \sum_{(H)\in\mathrm{consub}(G)} \chi(WH\backslash M^H, WH\backslash M^{>H})\cdot G/H] \;=\; \sum_{p\geq 0}(-1)^p\cdot\sharp_p(G/H),$$

where $\chi(WH\backslash M^H, WH\backslash M^{>H}))$ is the non-equivariant Euler characteristic and $\sharp_p(G/H)$ is the number of equivariant cells of the type $G/H \times D^p$ appearing in some G-CW-complex structure on M. The following diagram commutes (see Theorem 3.21)

$$\begin{array}{ccc} A(G) & \xrightarrow{j_1} & R_{\mathbb{Q}}(G) \\ {\scriptstyle \mathrm{ch}^G}\downarrow & & \downarrow{\scriptstyle \mathrm{HS}_{\mathbb{Q},G}} \\ \prod_{(H)\in \mathrm{consub}(H)} \mathbb{Q} & \xrightarrow[\mathrm{pr}]{} & \prod_{\substack{(H)\,\mathrm{consub}(G) \\ H \text{ cyclic}}} \mathbb{Q} \end{array}$$

where pr is the obvious projection, $\mathrm{ch}^G(S)$ has as entry for $(H) \subseteq \mathrm{consub}(G)$ the number $\frac{1}{|WH|} \cdot |S^H|$ and $\mathrm{HS}_{\mathbb{Q},G}(V)$ has as entry at $(H) \in \mathrm{consub}(G)$ for cyclic $H \subseteq G$ the number $\frac{\mathrm{tr}_{\mathbb{Q}}(l_h)}{|WH|}$, where $\mathrm{tr}(l_h) \in \mathbb{Q}$ is the trace of the endomorphism of the rational vector space V given by multiplication with h for some generator $h \in H$. The vertical arrows ch^G and $\mathrm{HS}_{\mathbb{Q},G}$ are rationally bijective and $\chi^G(M)(\chi^G(M))$ has as component belonging to $(H) \in \mathrm{consub}(G)$ the number $\chi^{\mathbb{Q}WH}(M^H) = \frac{\chi(M^H)}{|WH|}$ (see Lemma 2.8 and Lemma 3.12). This implies that ch^G and $\mathrm{HS}_{\mathbb{Q},G}$ are injective because their sources are torsion-free finitely generated abelian groups. Moreover, $\chi^G(M) \in A(G)$ carries integrally the same information as all the collection of Euler characteristics $\{\chi(M^H) \mid (H) \in \mathrm{consub}(H)\}$, whereas $j_1(\chi^G(M)) = \sum_{p\geq 0}(-1)^p \cdot [H_p(M;\mathbb{Q})]$ carries integrally the same information as the collection of the Euler characteristics $\{\chi(M^H) \mid (H) \in \mathrm{consub}(H), H \text{ cyclic}\}$. In particular j_1 is injective if and only if G is cyclic. But any element u in $A(G)$ occurs as $\chi^G(M)$ for a closed smooth G-manifold M for which M^H is connected and non-empty for all $H \subseteq G$ (see [20, Section 7]). Hence, given a finite group G, the elements $\chi^G(M)$ and $j_1(\chi^G(M))$ carry the same information for all such G-manifolds M if and only if G is cyclic.

From this discussion we conclude that $\mathrm{Eul}^G(M)$ does not carry torsion information in the case where G is finite and M^H is connected and non-empty for all $H \subseteq G$, since $R_{\mathbb{R}}(G)$ is a torsion-free finitely generated abelian group. This is different from the case where one allows non-connected fixed point sets, as the following example shows.

4.2 The equivariant Euler class carries torsion information

Let S_3 be the symmetric group on 3 letters. It has the presentation

$$S_3 = \langle s, t \mid s^2 = 1, t^3 = 1, sts = t^{-1}\rangle.$$

Let $\mathbb{R}$ be the trivial 1-dimensional real representation of S_3. Denote by $\mathbb{R}^-$ the one-dimensional real representation on which t acts trivially and s acts by $-\mathrm{id}$. Denote by V the 2-dimensional irreducible real representation of S_3; we can take $\mathbb{R}^2$ as the underlying real vector space of V, with $s \cdot (r_1, r_2) = (r_2, r_1)$ and $t \cdot (r_1, r_2) = (-r_2, r_1 - r_2)$. Then $\mathbb{R}$, $\mathbb{R}^-$ and V are the irreducible real representations of S_3, and $\mathbb{R}S_3$ is as an $\mathbb{R}S_3$-module isomorphic to $\mathbb{R} \oplus \mathbb{R}^- \oplus V \oplus V$. Let L_2 be the cyclic group of order two generated by s and let L_3 be the cyclic group of order three generated by t. Any finite subgroup of S_3 is conjugate to precisely one of the subgroups $L_1 = \{1\}$, L_2, L_3 or S_3. One easily checks that $V^{L_2} \cong \mathbb{R}$, $V^{L_3} \cong 0$, $(\mathbb{R}^-)^{L_2} = 0$ and $(\mathbb{R}^-)^{L_3} \cong \mathbb{R}$ as real vector spaces. Put $W = \mathbb{R} \oplus \mathbb{R}^- \oplus V$. Then $W^{S_3} \cong \mathbb{R}$, $W^{L_2} \cong W^{L_3} \cong \mathbb{R}^2$ and $W \cong \mathbb{R}^4$ as real vector spaces. Let M be the closed 3-dimensional S_3-manifold SW. Then

$$\begin{array}{rcl} M & \cong & S^3; \\ M^{L_2} & \cong & S^1; \\ M^{L_3} & \cong & S^1; \\ M^{S_3} & \cong & S^0. \end{array}$$

Since $\chi(M^{S_3}) \neq 0$, $\chi^G(M) \in U^G(M)$ cannot vanish. But since the fixed sets for all cyclic subgroups have vanishing Euler characteristic, Theorem 0.9 implies that $\mathrm{Eul}^G(M)$ is a torsion element in $KO_0^G(M)$. We want to show that it has order precisely two.

Let $x_i\colon S_3/L_i \to X$ for $i = 1,2,3$ be a G-map. Let $x_-\colon S_3/S_3 \to X$ and $x_+\colon S_3/S_3 \to X$ be the two different G-maps for which M^{S_3} is the union of the images of x_- and x_+. Then x_1, x_2, x_3, x_- and x_+ form a complete set of representatives for the isomorphism classes of objects in $\Pi_0(S_3, M)$. Notice for $i = 1,2,3$ that $\mathrm{mor}(x_i, x_-)$ and $\mathrm{mor}(x_i, x_+)$ consist of precisely one element. Therefore we get an exact sequence

$$R_{\mathbb{R}}(\mathbb{Z}/2) \oplus R_{\mathbb{R}}(\mathbb{Z}/3) \xrightarrow{\begin{pmatrix} i_2 & i_3 \\ -i_2 & -i_3 \end{pmatrix}} R_{\mathbb{R}}(G) \oplus R_{\mathbb{R}}(G)$$
$$\xrightarrow{s_- + s_+} \varinjlim_{\Pi_0(S_3,M)} Q^* \underline{R_{\mathbb{R}}} \to 0, \quad (4.1)$$

where $i_2\colon R_{\mathbb{R}}(\mathbb{Z}/2) \to R_{\mathbb{R}}(S_3)$ and $i_3\colon R_{\mathbb{R}}(\mathbb{Z}/3) \to R_{\mathbb{R}}(S_3)$ are the induction homomorphisms associated to any injective group homomorphism from $\mathbb{Z}/2$ and $\mathbb{Z}/3$ into S_3 and s_- and s_+ are the structure maps of the colimit belonging to the objects x_- and x_+. Define a map

$$\delta\colon R_{\mathbb{R}}(G) \to \mathbb{Z}/2, \quad \lambda_{\mathbb{R}} \cdot [\mathbb{R}] + \lambda_{\mathbb{R}^-} \cdot [\mathbb{R}^-] + \lambda_V \cdot [V] \mapsto \overline{\lambda_{\mathbb{R}}} + \overline{\lambda_{\mathbb{R}^-}} + \overline{\lambda_V}.$$

If $\mathbb{R}$ denotes the trivial and $\mathbb{R}^-$ denotes the non-trivial one-dimensional real $\mathbb{Z}/2$-representation, then

$$\begin{aligned}
&i_2\colon\ R_{\mathbb{R}}(\mathbb{Z}/2) \to R_{\mathbb{R}}(S_3),\\
&\mu_{\mathbb{R}} \cdot [\mathbb{R}] + \mu_{\mathbb{R}^-} \cdot [\mathbb{R}^-] \ \mapsto \mu_{\mathbb{R}} \cdot [\mathbb{R}] + \mu_{\mathbb{R}^-} \cdot [\mathbb{R}^-] + (\mu_{\mathbb{R}} + \mu_{\mathbb{R}^-}) \cdot [V].
\end{aligned}$$

If $\mathbb{R}$ denotes the trivial one-dimensional and W the 2-dimensional irreducible real $\mathbb{Z}/3$-representation, then

$$i_3\colon\ R_{\mathbb{R}}(\mathbb{Z}/3) \to R_{\mathbb{R}}(S_3) \qquad \mu_{\mathbb{R}} \cdot [\mathbb{R}] + \mu_W \cdot [W] \ \mapsto \mu_{\mathbb{R}} \cdot [\mathbb{R}] + \mu_{\mathbb{R}} \cdot [\mathbb{R}^-] + 2\mu_W \cdot [V].$$

This implies that the following sequence is exact

$$R_{\mathbb{R}}(\mathbb{Z}/2) \oplus R_{\mathbb{R}}(\mathbb{Z}/3) \xrightarrow{i_2+i_3} R_{\mathbb{R}}(G) \xrightarrow{\delta} \mathbb{Z}/2 \to 0. \tag{4.2}$$

We conclude from the exact sequences (4.1) and (4.2) above that the epimorphism

$$s_- + s_+\colon\ R_{\mathbb{R}}(G) \oplus R_{\mathbb{R}}(G) \to \varinjlim_{\Pi_0(S_3,M)} Q^* \underline{R_{\mathbb{R}}}$$

factorizes through the map

$$u := \begin{pmatrix} 1 & 1 \\ 0 & \delta \end{pmatrix}\colon\ R_{\mathbb{R}}(G) \oplus R_{\mathbb{R}}(G) \to R_{\mathbb{R}}(G) \oplus \mathbb{Z}/2$$

to an isomorphism

$$v\colon\ R_{\mathbb{R}}(G) \oplus \mathbb{Z}/2 \quad \xrightarrow{\cong} \quad \varinjlim_{\Pi_0(S_3,M)} Q^* \underline{R_{\mathbb{R}}}. \tag{4.3}$$

Define a map

$$f\colon\ U^{S_3}(M) \to R_{\mathbb{R}}(G) \oplus \mathbb{Z}/2$$

by

$$\begin{aligned}
f([x_1]) &:= &&= ([\mathbb{R}[S_3]], 0);\\
f([x_2]) &:= &&= ([\mathbb{R}[S_3/L_2]], 0);\\
f([x_3]) &:= &&= ([\mathbb{R}[S_3/L_3]], 0);\\
f([x_-]) &:= &&= ([\mathbb{R}], 0);\\
f([x_+]) &:= &&= ([\mathbb{R}], 1).
\end{aligned}$$

The reader may wonder why f does not look symmetric in x_- and x_+. This comes from the choice of u which affects the isomorphism v. The composition

$$U^G(M) \xrightarrow{e_1^G(M)} H_0^{\mathsf{Or}(G)}(M; \underline{R_{\mathbb{Q}}}) \xrightarrow{e_2^G(M)} H_0^{\mathsf{Or}(G)}(M; \underline{R_{\mathbb{R}}})$$

agrees with the composition

$$U^G(M) \ \xrightarrow{f}\ R_{\mathbb{R}}(G) \oplus \mathbb{Z}/2 \ \xrightarrow{v}\ \varinjlim_{\Pi_0(S_3,M)} Q^* \underline{R_{\mathbb{R}}} \ \xrightarrow{\beta_{\mathbb{R}}^G(M)}\ H_0^{\mathsf{Or}(G)}(M; \underline{R_{\mathbb{R}}}).$$

We get

$$\chi^G(M) \quad = \quad [x_+] + [x_-] - 2\cdot[x_2] - [x_3] + [x_1] \qquad \in U^G(M) \tag{4.4}$$

since the image of the element on the right side and the image of $\chi^G(M)$ under the injective character map $\mathrm{ch}^G(M)$ (see (2.6) and Lemma 2.8) agree by the following calculation

$$\begin{aligned}
\mathrm{ch}^G(\chi^G(M)) &= \frac{\chi(M^{S_3}(x_-))}{|(WS_3)_{x_-}|}\cdot[x_-] + \frac{\chi(M^{S_3}(x_+))}{|(WS_3)_{x_+}|}\cdot[x_+] + \frac{\chi(M^{L_2})}{|WL_2|}\cdot[x_2] \\
&\qquad + \frac{\chi(M^{L_3})}{WL_3}\cdot[x_3] + \frac{\chi(M)}{WL_1}\cdot[x_1] \\
&= [x_-] + [x_+] \\
&= \mathrm{ch}^G(M)([x_+] + [x_-] - 2\cdot[x_2] - [x_3] + [x_1]).
\end{aligned}$$

Now one easily checks

$$f(\chi^G(M)) \quad = \quad (0,1) \qquad \in R_{\mathbb{R}}(S_3)\oplus\mathbb{Z}/2. \tag{4.5}$$

Since $R_{\mathbb{R}}(G)\oplus\mathbb{Z}/2 \xrightarrow{v} \varinjlim_{\Pi_0(S_3,M)} Q^*\underline{R_{\mathbb{R}}} \xrightarrow{\beta^G_{\mathbb{R}}(M)} H_0^{\mathsf{Or}(G)}(M;\underline{R_{\mathbb{R}}})$ is a composition of isomorphisms, we conclude that $e_2^G(M)\circ e_1^G(M)(\chi^G(M))$ is an element of order two in $H_0^{\mathsf{Or}(G)}(M;\underline{R_{\mathbb{R}}})$. Since $\dim(M)\le 4$ and $\dim(M^{>1})\le 2$, we conclude from Theorem 3.6 (d) that $e_3^G(M)\colon H_0^{\mathsf{Or}(G)}(M;\underline{R_{\mathbb{R}}}) \to KO_0^G(M)$ is injective. We conclude from Theorem 0.3 and Theorem 3.6 (a) that $\mathrm{Eul}^G(M)\in KO_0^G(M)$ is an element of order two as promised in Theorem 0.10.

4.3 The equivariant Euler class is independent of the stable equivariant Euler characteristic

In this subsection we will give examples to show that $\mathrm{Eul}^G(M)$ is independent of the stable equivariant Euler characteristic with values in the Burnside ring $A(G)$, in the sense that it is possible for either one of these invariants to vanish while the other does not vanish.

For the first example, take $G=\mathbb{Z}/p$ cyclic of prime order, so that G has only two subgroups (the trivial subgroup and G itself) and $A(G)$ has rank 2. We will see that it is possible for $\mathrm{Eul}^G(M)$ to be non-zero, even rationally, while $\chi_s^G(M)=0$ in $A(G)$ (see Definition 2.10). To see this, we will construct a closed 4-dimensional G-manifold M with $\chi(M)=0$ and such that M^G has two components of dimension 2, one of which is S^2 and the other of which is a surface N^2 of genus 2, so that $\chi(N)=-2$. Then

$$\chi(M^G) = \chi(S^2 \amalg N^2) = \chi(S^2)+\chi(N^2) = 2-2 = 0,$$

while also $\chi(M^{\{1\}}) = \chi(M) = 0$, so that $\chi_s^G(M) = 0$ in $A(G)$ and hence also $\chi_a^G(M) = 0$.

For the construction, simply choose any bordism W^3 between S^2 and N^2, and let

$$M^4 = \left(S^2 \times D^2\right) \cup_{S^2 \times S^1} \left(W^3 \times S^1\right) \cup_{N^2 \times S^1} \left(N^2 \times D^2\right).$$

We give this the G-action which is trivial on the S^2, W^3, and N^2 factors, and which is rotation by $2\pi/p$ on the D^2 and S^1 factors. Then the action of G is free except for M^G, which consists of $S^2 \times \{0\}$ and of $N^2 \times \{0\}$. Furthermore, we have

$$\begin{aligned}
\chi(M) &= \chi\left(S^2 \times D^2\right) - \chi\left(S^2 \times S^1\right) + \chi\left(W^3 \times S^1\right) \\
&\quad -\chi\left(N^2 \times S^1\right) + \chi\left(N^2 \times D^2\right) \\
&= 2 - 0 + 0 - 0 - 2 = 2 - 2 = 0.
\end{aligned}$$

Thus $\chi(M) = \chi(M^G) = 0$ and $\chi_s^G(M) = 0$ in $A(G)$. On the other hand, $\mathrm{Eul}^G(M)$ is non-zero, even rationally, since from it (by Theorem 0.9) we can recover the two (non-zero) Euler characteristics of the two components of M^G.

For the second example, take $G = S_3$ and retain the notation of Subsection 4.2. By [20, Theorem 7.6], there is a closed G-manifold M with M^H connected for each subgroup H of G, with $\chi(M^H) = 0$ for H cyclic, and with $\chi(M^G) \neq 0$. (Note that G is the only noncyclic subgroup of G.) In fact, we can write down such an example explicitly; simply let $Q = W' \oplus \mathbb{R} \oplus \mathbb{R}$, the S_3-representation $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}^- \oplus V$, and let $M = SW'$ be the unit ball in W'. Then each fixed set in M is a sphere of dimension bigger by 2 than in the example of 4.2, so $M \cong S^5$, $M^{L_2} \cong S^3$, $M^{L_3} \cong S^3$, and $M^G \cong S^2$. Since the fixed sets are all connected and each fixed set of a cyclic subgroup has vanishing Euler characteristic, it follows by Subsection 4.1 that $\mathrm{Eul}^G(M) = 0$. On the other hand, since $\chi(M^G) = 2$, $\chi_s^G(M) \neq 0$ in $A(G)$.

4.4 The image of the equivariant Euler class under the assembly maps

Now let us consider an infinite (discrete) group G. Let $\underline{E}G$ be a model for the *classifying space for proper G-actions*, i.e., a G-CW-complex $\underline{E}G$ such that $\underline{E}G^H$ is contractible (and in particular non-empty) for finite $H \subseteq G$ and $\underline{E}G^H$ is empty for infinite $H \subseteq G$. It has the universal property that for any proper G-CW-complex X there is up to G-homotopy precisely one G-map $X \to \underline{E}G$. This implies that all models for $\underline{E}G$ are G-homotopy equivalent. If

G is a word-hyperbolic group, its Rips complex is a model for $\underline{E}G$ [21]. If G is a discrete subgroup of the Lie group L with finitely many path components, then L/K for a maximal compact subgroup K with the (left) G-action is a model for $\underline{E}G$ (see [1, Corollary 4.14]). If G is finite, $\{*\}$ is a model for $\underline{E}G$.

Consider a proper smooth G-manifold M. Let $f\colon M \to \underline{E}G$ be a G-map. We obtain a commutative diagram

$$
\begin{tikzcd}
U^G(M) \arrow[r, "U^G(f)"] \arrow[d, "e_1^G(M)"'] & U^G(\underline{E}G) \arrow[r, "\mathrm{id}"] \arrow[d, "e_1^G(\underline{E}G)"'] & U^G(\underline{E}G) \arrow[d, "j_1"'] \\
H_0^{\mathsf{Or}(G)}(M;\underline{R_{\mathbb{Q}}}) \arrow[r, "H_0^{\mathsf{Or}(G)}(f;\underline{R_{\mathbb{Q}}})"] \arrow[d, "e_2^G(M)"'] & H_0^{\mathsf{Or}(G)}(\underline{E}G;\underline{R_{\mathbb{Q}}}) \arrow[r, "\mathrm{asmb}_1"] \arrow[d, "e_2^G(\underline{E}G)"'] & K_0(\mathbb{Q}G) \arrow[d, "j_2"'] \\
H_0^{\mathsf{Or}(G)}(M;\underline{R_{\mathbb{R}}}) \arrow[r, "H_0^{\mathsf{Or}(G)}(f;\underline{R_{\mathbb{R}}})"] \arrow[d, "e_3^G(M)"'] & H_0^{\mathsf{Or}(G)}(\underline{E}G;\underline{R_{\mathbb{R}}}) \arrow[r, "\mathrm{asmb}_2"] \arrow[d, "e_3^G(\underline{E}G)"'] & K_0(\mathbb{R}G) \arrow[d, "j_3"'] \\
KO_0^G(M) \arrow[r, "KO_0^G(f)"] & KO_0^G(\underline{E}G) \arrow[r, "\mathrm{asmb}_3"] & KO_0(C_r^*(G;\mathbb{R}))
\end{tikzcd}
$$

Here asmb_i for $i = 1, 2, 3$ are the assembly maps appearing in the Farrell-Jones Isomorphism Conjecture and the Baum-Connes Conjecture. The maps asmb_i for $i = 1, 2$ are the obvious maps $\varinjlim_{\mathsf{Or}(G;\mathcal{F}\mathsf{in})} \underline{R_F} \to K_0(FG)$ for $F = \mathbb{Q}, \mathbb{R}$ under the identifications $R_F(H) = K_0(FH)$ for finite $H \subseteq G$ and $\beta_F^G(M)$ of (3.2). The Baum-Connes assembly map is given by the index with values in the reduced real group C^*-algebra $C_r^*(G;\mathbb{R})$. The Farrell-Jones Isomorphism Conjecture and the Baum-Connes Conjecture predict that asmb_i for $i = 1, 2, 3$ are bijective. The abelian group $U^G(\underline{E}G)$ is the free abelian group with $\{(H) \in \mathrm{consub}(G) \mid |H| < \infty\}$ as basis and the map j_1 sends the basis element (H) to the class of the finitely generated projective $\mathbb{Q}G$-module $\mathbb{Q}[G/H]$. The maps j_2 and j_3 are change-of-rings homomorphisms. The maps $e_2^G(M)$ and $e_3^G(M)$ are rationally injective (see Theorem 3.6 (b) and (c)). If M^H is connected and non-empty for all finite subgroups $H \subseteq G$, then the horizontal arrows $U^G(f)$, $H_0^{\mathsf{Or}(G)}(f;\underline{R_{\mathbb{Q}}})$ and $H_0^{\mathsf{Or}(G)}(f;\underline{R_{\mathbb{R}}})$ are bijective.

In contrast to the case where G is finite, the groups $K_0(\mathbb{Q}G)$, $K_0(\mathbb{R}G)$ and $KO_0(C_r^*(G;\mathbb{R}))$ may not be torsion free. The problem whether asmb_i is bijective for $i = 1, 2$ or 3 is a difficult and in general unsolved problem. Moreover, $\underline{E}G$ is a complicated G-CW-complex for infinite G, whereas for a finite group G we can take $\{*\}$ as a model for $\underline{E}G$.

References

[1] **H Abels**, *A universal proper G-space*, Math. Z. 159 (1978) 143–158

[2] **M F Atiyah**, **I M Singer**, *The index of elliptic operators, III*, Ann. of Math. (2) 87 (1968) 546–604

[3] **S Baaj**, **P Julg**, Théorie bivariante de Kasparov et opérateurs non bornés dans les C^*-modules hilbertiens, C. R. Acad. Sci. Paris Sér. I Math. 296 (1983) no. 21, 875–878

[4] **B Blackadar**, *K-Theory for Operator Algebras*, Math. Sci. Res. Inst. Publ., 5, Springer-Verlag, New York, Berlin (1986)

[5] **J Davis**, **W Lück**, *Spaces over a Category, Assembly Maps in Isomorphism Conjecture in K-and L-Theory*, K-Theory 15 (1998) 201–252

[6] **T tom Dieck**, *Transformation groups*, Studies in Math. 8, de Gruyter (1987)

[7] **J Dixmier**, *C^*-Algebras*, North-Holland Mathematical Library, 15, North-Holland, Amsterdam and New York (1977)

[8] **M P Gaffney**, *A special Stokes's theorem for complete Riemannian manifolds*, Ann. of Math. (2) 60 (1954) 140–145

[9] **N Higson**, A primer on KK-theory, *Operator theory: operator algebras and applications, Part 1* (Durham, NH, 1988), W. Arveson and R. Douglas, eds., Proc. Sympos. Pure Math. 51, part 1, Amer. Math. Soc., Providence, RI (1990) 239–283

[10] **D S Kahn**, **J Kaminker**, **C Schochet**, *Generalized homology theories on compact metric spaces*, Michigan Math. J. 24 (1977) no. 2, 203–224

[11] **J Kaminker**, **J G Miller**, *Homotopy invariance of the analytic index of signature operators over C^*-algebras*, J. Operator Theory 14 (1985) no. 1, 113–127

[12] **G Kasparov**, *The operator K-functor and extensions of C^*-algebras*, Izv. Akad. Nauk SSSR, Ser. Mat. 44 (1980) 571–636; English transl. in Math. USSR–Izv. 16 (1981) 513–572

[13] **G Kasparov**, *Equivariant KK-theory and the Novikov conjecture*, Invent. Math. 91 (1988) no. 1, 147–201

[14] **T Kato**, *Perturbation Theory for Linear Operators*, 2nd ed., corrected printing, Grundlehren der Mathematischen Wissenschaften, 132, Springer-Verlag, Berlin (1980); reprinted in Classics in Mathematics, Springer-Verlag, Berlin (1995)

[15] **H B Lawson, Jr.**, **M-L Michelsohn**, *Spin Geometry*, Princeton Mathematical Ser., 38, Princeton Univ. Press, Princeton, NJ (1989)

[16] **W Lück**, *Transformation groups and algebraic K-theory*, Lecture Notes in Math. 1408, Springer (1989)

[17] **W Lück**, *L^2-Invariants: Theory and Applications to Geometry and K-Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete 44, Springer (2002)

[18] **W Lück**, *Chern characters for proper equivariant homology theories and applications to K- and L-theory*, Journal für reine und angewandte Mathematik 543 (2002) 193–234

[19] **W Lück**, *The relation between the Baum-Connes Conjecture and the Trace Conjecture*, Inventiones Math. 149 (2002) 123–152

[20] **W Lück**, **J Rosenberg**, *The equivariant Lefschetz fixed point theorem for proper cocompact G-manifolds*, preprint (2002), available at `http://www.math.umd.edu/~jmr/jmr_pub.html`

[21] **D Meintrup**, *On the Type of the Universal Space for a Family of Subgroups*, Ph. D. thesis, Münster (2000)

[22] **J Rosenberg**, Analytic Novikov for topologists, *Novikov Conjectures, Index Theorems and Rigidity*, vol. 1, S. Ferry, A. Ranicki, and J. Rosenberg, eds., London Math. Soc. Lecture Notes, 226, Cambridge Univ. Press, Cambridge (1995), 338–372

[23] **J Rosenberg**, *The K-homology class of the Euler characteristic operator is trivial*, Proc. Amer. Math. Soc. 127 (1999) 3467–3474

[24] **J Rosenberg**, *The G-signature theorem revisited*, *Tel Aviv Topology Conference: Rothenberg Festschrift*, international conference on topology, June 1-5, 1998, Tel Aviv, Contemporary Mathematics 231, 251–264 (1999)

[25] **J Rosenberg**, *The K-homology class of the equivariant Euler characteristic operator*, unpublished preprint, available at `http://www.math.umd.edu/~jmr/jmr_pub.html`

[26] **J Rosenberg**, **S Weinberger**, *Higher G-signatures for Lipschitz manifolds*, K-Theory 7 (1993) 101–132

[27] **J Rosenberg**, **S Weinberger**, *The signature operator at 2*, in preparation

[28] **J-P Serre**, *Linear representations of finite groups*, Springer-Verlag (1977)

[29] **S Waner**, **Y Wu**, *The local structure of tangent G-vector fields*, Topology and its Applications 23 (1986) 129–143

[30] **S Waner**, **Y Wu**, *Equivariant SKK and vector field bordism*, Topology and its Appl. 28 (1988) 29–44