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Some Work of Stefan Banach and the Mathematics It Has Generated*

1. Introduction

This year, 2012, marks the 120th anniversary of the birth of Stefan Banach, the father of modern functional analysis. I do not need to remind the reader about his many contributions – just think of the concepts and theorems that bear his name: Banach spaces, the Hahn–Banach Theorem, the Banach–Steinhaus Theorem, the Banach–Alaoglu Theorem, and the Banach–Tarski paradox, just to name a few. (I have deliberately omitted Banach algebras since, while they are named after Banach, the basic theory of them was really invented by Gelfand and Naimark.)

Banach is important for the history of analysis not just because of *what he did*, but because of *how he wrote it up*. Banach's works [4–6] are all models of clarity and conciseness. Unlike much of early 20th century mathematics, Banach's work has never really gone out of date, either in content or in style. It is still a pleasure to pick up almost any of his papers and to read it as if it were just composed recently.

The editors of this volume asked me to take on the daunting task to "focus on some results of Banach." So it seemed to me that an interesting exercise would be to take some of Banach's lesser-known papers, not the ones that have become household names, and to see what modern mathematics grew out of them. For this purpose I have chosen three of Banach's papers, one on measurable additive maps [1, 4], to be discussed in Section 2, one on

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metric groups [2, 5], to be discussed in Section 3, and one on continuous selection [3, 4], to be discussed in Section 4.

2. Measurable Homomorphisms and Cocycles

One of Banach's earliest papers, [1], reprinted in [4], deals with measurable solutions to the functional equation $f(x+y) = f(x) + f(y)$ for $f\colon \mathbb{R} \to \mathbb{R}$.

Theorem 2.1 (Banach). *If $f\colon \mathbb{R} \to \mathbb{R}$ is measurable and satisfies the functional equation $f(x+y) = f(x) + f(y)$, then f is a continuous homomorphism* (*for the additive group structure*).

Banach's proof takes only about a page and is based on Lusin's Theorem; the basic idea is that since (on an interval (a, b)) f agrees except on a set of small measure with a continuous function, one can use this fact together with the functional equation to show that f is continuous.

A generalization of this theorem appears in Banach's classic functional analysis book [6, Théorème 3, p. 23]. Translated into more modern terminology, this says the following:

Theorem 2.2 (Banach). *If G_1 and G_2 are Polish groups* (*complete separable metrizable groups – see* [12, Proposition 1, p. 3]) *and $f\colon G_1 \to G_2$ is a Borel-measurable homomorphism, then f is continuous.*

This "automatic continuity" theorem can easily be extended (see [11, p. 45]) from homomorphisms to "crossed homomorphisms" or 1-cocycles, in the situation where G_2 is a Polish G_1-module as in [12, 13].

Theorem 2.2 has led to many important developments in representation theory. First of all, it suggests the development of a theory of "Borel groups" as worked out later by Mackey [10]. This proved to be crucial for certain approaches to the decomposition theory of representations [8, 9]. Secondly, it suggests the development of a cohomology theory for locally compact groups, in which the cochains are only required to be Borel-measurable, not continuous. (In general, there are not enough continuous cochains to give a useful theory.) Such a theory was developed by Moore [11–13], and was shown by Wigner and Tu [16, 17] to coincide with other cohomology theories constructed by methods of homological algebra. It is now recognized that Moore's theory is the "right" cohomology theory for problems involving locally compact groups.

3. Metric Groups

The paper [2] of Banach proves another automatic continuity theorem about Polish groups, of a somewhat different nature than Theorem 2.2.

Theorem 3.1 (Banach). *If G_1 and G_2 are Polish groups and $f\colon G_1 \to G_2$ is a bijective continuous homomorphism, then f^{-1} is also continuous* (*i.e.*, *f is an isomorphism of topological groups*).

As Banach points out, the separability hypothesis is crucial here, since for example the identity map from $\mathbb{R}$ with the discrete topology to $\mathbb{R}$ with the usual topology is a bijective continuous homomorphism, but it is certainly not a homeomorphism. Theorem 3.1 is closely related to the more familiar "Open Mapping Theorem" for Banach spaces, also proved by Banach. Other variants of this theorem are possible. For example, suppose a topological group G acts simply transitively on a topological space X, if one fixes a basepoint $x_0 \in X$, then $g \mapsto g \cdot x_0$ defines a continuous bijection of G onto X, and one often wants to know if the inverse map $g \cdot x_0 \mapsto g$ is continuous. For the same reasons as before, the answer is "not always". However, by a variant of the proof of Theorem 3.1, the answer is "yes" if G is a Polish group and X is a complete separable metric space. This is a forerunner of a huge literature on "slices" for group actions, exemplified by [14].

4. Multivalued Functions

The paper [3] of Banach and Mazur deals with k-fold *multivalued functions*, and the question of when they "split into branches." In more modern language, we can express the problem as follows. Let Y be a topological space, Y^k the k-fold product of Y with itself, and $Y^k \xrightarrow{q} S^kY$ the quotient map from Y^k to S^kY, the k-th symmetric product of Y with itself. The space S^kY is simply the quotient of Y^k by the action of the symmetric group Σ_k, acting by permutations of the factors. Inside S^kY is the open subset $S^k_{\text{gen}}Y$ of "generic" points, the image under q of the points in Y^k with $y_i \neq y_j$ for all $i \neq j$, or in other words the set of points where all k coordinates are distinct. We can also identify $S^k_{\text{gen}}Y$ with the set of all subsets of Y with cardinality precisely k. Let X be another topological space. A (continuous) *k-fold multivalued function* from X to Y is a continuous map $f\colon X \to S^k_{\text{gen}}Y$; it is said to *admit a branch* (*Zweig* in the language of [3]) if there is a continuous map $b\colon X \to Y$ with $b(x) \in f(x)$ for all x, and to *split into branches* (*zerfallen in stetige Zweige* in the language of [3]) if it factors through a continuous map $\widetilde{f}\colon X \to Y^k$. This is not always possible. In fact, the map q, restricted to the inverse image $\widetilde{S^k_{\text{gen}}Y}$ of $S^k_{\text{gen}}Y$, is a Galois covering map with covering group Σ_k, so if the spaces X and Y are reasonable, the question of when f will admit a splitting into branches is reduced to covering space theory. For example, if $Y = S^2 \cong \mathbb{CP}^1$, $S^kY \cong \mathbb{CP}^k$, and if $k = 2$, then $\widetilde{S^k_{\text{gen}}Y}$ is the complement of the diagonal copy Δ of S^2 in $S^2 \times S^2$. In terms of homogeneous coordinates, when $k = 2$, the map q can be identified with

$$([z_0, z_1], [w_0, w_1]) \mapsto [z_0w_1 + z_1w_0, z_0w_0, z_1w_1] : (\mathbb{CP}^1)^2 \to \mathbb{CP}^2.$$

Note that the restriction of q to Δ is of degree 2, i.e., sends the generator of $H_2(S^2)$ to twice the generator of $H_2(\mathbb{CP}^2)$. From this it follows by duality that $H^3(\mathbb{CP}^2 \backslash q(\Delta)) \cong \mathbb{Z}/2$, and by Poincaré duality, $H_1(\mathbb{CP}^2 \backslash q(\Delta)) \cong \mathbb{Z}/2$. Since the normal bundle of Δ in $\mathbb{CP}^1 \times \mathbb{CP}^1$ is trivial, the complement of Δ is homeomorphic to $S^2 \times \mathbb{R}^2$, while $S^k_{\text{gen}}Y$ is the complement of $q(\Delta) \cong \mathbb{CP}^1$ in $\mathbb{CP}^2$, which has fundamental group $\mathbb{Z}/2$ and is homeomorphic to a vector bundle over $\mathbb{RP}^2$. Thus the covering map $\widetilde{S^k_{\text{gen}}Y} \to S^k_{\text{gen}}Y$ is non-trivial in this case.

Now let us state the theorem of Banach and Mazur.

Theorem 4.1 (Banach and Mazur). *Suppose X and Y are metric spaces and*

$$f\colon X \to S^kY$$

is a k-fold multivalued function. Assume that X satisfies:

1° *X is locally arcwise connected*;

2° *X is simply connected.*

Then f splits into branches.

From the analysis of the problem above, this theorem is clearly an early version of the theory of path lifting in covering spaces. Indeed, if X is simply connected, then pulling the covering $\widetilde{S^k_{\text{gen}}Y} \to S^k_{\text{gen}}Y$ back to X via f, we get a trivial covering $X \amalg X \amalg \cdots \amalg X \to X$, so the fact that f splits into branches is obvious. So the modern reader might wonder why this theorem needed to be stated at all; but remember (see [7, Chapter on *Fundamental Group and Covering Spaces*, pp. 293–310]) that the modern theory of covering spaces really only dates from the publication of [15], which was not available at the time Banach and Mazur did their work. And Seifert and Threlfall only dealt with the case of locally finite simplicial complexes, whereas the work of Banach and Mazur is more general. Thus while we tend to think of Banach as the father of functional analysis, this paper proves that Banach was also one of the pioneers of topology and its applications.

Conclusion

The examples we have cited illustrate only a few of the ways in which Banach's work connects with modern mathematics, not only in functional analysis, but also in representation theory, topology, and geometry. What makes this even more remarkable is the short length of Banach's career – his first paper was only published in 1919, and he was unable to do much mathematics past 1941 because of the Nazi occupation of Lwów. He still serves as an inspiration to mathematicians everywhere.

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